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Strongly regular points of mappings

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Dedicated to the memory of Jonathan Borwein, in recognition of his deep contributions to mathematics and of his lasting friendship. He was one of the pioneers of metric regularity.

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Abstract

In this paper, we use a *robust lower directional derivative* and provide some sufficient conditions to ensure the *strong regularity* of a given mapping at a certain point. Then, we discuss the Hoffman estimation and achieve some results for the estimate of the distance to the set of solutions to a system of *linear equalities*. The advantage of our estimate is that it allows one to calculate the coefficient of the error bound.

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1 Introduction

Let f be a mapping acting between the normed spaces \mathbb{X} and \mathbb{Y} , whose norms are denoted by the same symbol $\|\cdot\|$. To estimate the approximate solutions of the equation $y = f(x)$, we seek an *error bound*

$$\text{dist}(x, f^{-1}(y)) \leq \kappa \|y - f(x)\|,$$

locally, for all (x, y) near $(\bar{x}, \bar{y} = f(\bar{x}))$, or globally, for all x and y , where κ is some positive constant. The infimum of such κ is called the *modulus of regularity* of f . For instance, when $f: \mathbb{R} \rightarrow \mathbb{R}$ is smooth and verifies $f'(\bar{x}) \neq 0$, it is easily observed that the modulus of regularity of f at \bar{x} is exactly $|f'(\bar{x})|^{-1}$. A first approach to the concept of regularity goes back to a celebrated fundamental result proved in 1934 by Lyusternik [1]:

Theorem 1.1 (Lyusternik, [1]) *Let f be a mapping from a Banach space \mathbb{X} to a Banach space \mathbb{Y} . Suppose that f is Fréchet differentiable in a neighborhood of \bar{x} and that its derivative $f'(x)$ is continuous at \bar{x} and $f'(\bar{x})$ is surjective. Then, for every $\varepsilon > 0$, there exists $r > 0$ such that*

$$\text{dist}(x, f^{-1}(0)) \leq \varepsilon \|x - \bar{x}\|,$$

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whenever

$$\|x - \bar{x}\| \leq r \quad \text{and} \quad f'(\bar{x})(x - \bar{x}) = 0.$$

In other words, the tangent manifold to $f^{-1}(0)$ is equal to $\bar{x} + \text{Ker} f'(\bar{x})$, where $\text{Ker} f'(\bar{x})$ is the set of those x such that $f'(\bar{x})(x) = 0$.

We refer to Dontchev [2] for a nice overview on the Lyusternik theorem and to the fact that the Lyusternik theorem can be easily obtained from the Graves theorem. We also refer to the forthcoming book by Thibault [3].

Theorem 1.2 (Graves, [4]) *Let \mathbb{X} and \mathbb{Y} be Banach spaces, $\bar{x} \in \mathbb{X}$, and $f : \mathbb{X} \rightarrow \mathbb{Y}$ be a C^1 -mapping whose derivative $f'(\bar{x})$ is onto. Then, there exist a neighborhood \mathbb{U} of \bar{x} and a constant $c > 0$ such that for every $x \in \mathbb{U}$ and $\tau > 0$ with $\mathbb{B}(\bar{x}, \tau) \subset \mathbb{U}$,*

$$\mathbb{B}(\bar{x}, c\tau) \subset f(\mathbb{B}(\bar{x}, \tau)) \quad (\text{partial openness property with linear rate}).$$

Ioffe and Tihomirov showed in [5] that the original Lyusternik proof may lead to a stronger result and proved that if $f'(\bar{x})$ is surjective, then there are $\kappa > 0$ and $\delta > 0$ such that

$$\text{dist}(x, f^{-1}(\bar{y})) \leq \kappa \|f(x) - f(\bar{x})\| \quad \text{whenever } \|x - \bar{x}\| < \delta. \quad (1.1)$$

Ioffe's remark leads to a standard definition:

Definition 1.1 Point $\bar{x} \in \mathbb{X}$ is said to be a *regular point* of a mapping $f : \mathbb{X} \rightarrow \mathbb{Y}$ if the relation (1.1) is satisfied.

In this note, we will call \bar{x} a *strongly regular point of f* if the inequality

$$\|x - \bar{x}\| \leq \kappa \|f(x) - f(\bar{x})\|, \quad (1.2)$$

holds locally, for all x belonging to a neighborhood of \bar{x} , where $\kappa > 0$ is a positive constant. Next, we will provide sufficient conditions for \bar{x} to be a strongly regular point. Our results allow us to estimate the constant κ in (1.2). Then, we apply our results to the Hoffman estimate and obtain some results for the estimate of the distance to the set of solutions to a system of linear equalities. The advantage of our estimate is that it allows one to calculate the upper limit of the error. In particular, for a finite-dimensional space \mathbb{X} and a linear (continuous) mapping $A : \mathbb{X} \rightarrow \mathbb{X}$, we prove that the estimate

$$\text{dist}(x, \text{Ker} A) \leq \frac{\|A(x)\|}{\inf\{\|A(u)\| : u \in \mathbb{X}, \text{dist}(u, \text{Ker} A) = 1\}},$$

holds for all $x \in \mathbb{X}$ (Corollary 3.2 below). We can easily see that this estimate is *sharp* for injective linear mappings, in the sense that, if A is an injective linear mapping and

$$\inf\{\|A(u)\| : u \in \mathbb{X}, \text{dist}(u, \text{Ker} A) = 1\} < \mu,$$

then there exists some $x \in \mathbb{X}$ such that $\|x\| = \text{dist}(x, \text{Ker} A) > \mu^{-1} \|A(x)\|$.

Our work is outlined as follows. In Sect. 1, we recall the famous Lyusternik theorem and survey briefly its relationship with the concept of metric regularity. In Sect. 2, we first introduce the notion of *homogeneous continuity* of mappings. Then, using an appropriate notion of lower directional derivative, we achieve some results ensuring in finite dimension that for a given mapping a point is strongly regular. Finally, in Sect. 3, we focus our attention on Hoffman's estimate of approximate solutions of finite systems of linear inequalities and prove some similar estimates.

2 Sufficient conditions of regularity via generalized derivative

Throughout the paper, we use standard notations. For a normed space \mathbb{X} , we denote its norm by $\|\cdot\|$ and by \mathbb{X}^* its (continuous) dual. The symbol \mathbb{S} stands for the unit sphere, that is, the set of all points of \mathbb{X} of norm one, while $\mathbb{B}(x, r)$ and $\overline{\mathbb{B}}(x, r)$ denote, respectively, the open and closed balls centered at x with radius r . Some other notations are introduced as and when needed.

2.1 Homogeneous continuity

We begin with the following definition.

Definition 2.1 Let \mathbb{X} and \mathbb{Y} be normed spaces and $\mathbb{E} \subset \mathbb{X}$. The mapping $f : \mathbb{X} \rightarrow \mathbb{Y}$ is said to be *homogeneously continuous at $\bar{x} \in \mathbb{X}$ on \mathbb{E}* if for every $\epsilon > 0$ there exist $\delta > 0$ and $0 < \beta \leq 1$ such that

$$\|x - y\| < \delta \implies \|f(\bar{x} + tx) - f(\bar{x} + ty)\| < t\epsilon,$$

for all $0 < t \leq \beta$ and all $x, y \in \mathbb{E}$.

We are going to provide some sufficient conditions under which a mapping f is homogeneously continuous. Let us recall that a mapping $f : \mathbb{X} \rightarrow \mathbb{Y}$ is said to be *locally Lipschitz* around $\bar{x} \in \mathbb{X}$ if there exist a neighborhood \mathbb{O} of \bar{x} and a real number $\lambda > 0$ such that

$$\|f(x) - f(y)\| \leq \lambda \|x - y\|, \quad (2.1)$$

for all $x, y \in \mathbb{O}$.

Lemma 2.1 Suppose that \mathbb{X} and \mathbb{Y} are normed spaces. If $f : \mathbb{X} \rightarrow \mathbb{Y}$ is locally Lipschitz around $\bar{x} \in \mathbb{X}$, then f is homogeneously continuous at \bar{x} on some closed ball $\overline{\mathbb{B}}(0, r)$.

Proof By hypothesis, there exist a constant $\lambda > 0$ and a neighborhood \mathbb{O} of \bar{x} in \mathbb{X} such that (2.1) holds for all $x, y \in \mathbb{O}$. Choose $r > 0$ such that $\overline{\mathbb{B}}(\bar{x}, r) \subset \mathbb{O}$. It follows that

$$\|f(\bar{x} + tx) - f(\bar{x} + ty)\| \leq \lambda \|\bar{x} + tx - (\bar{x} + ty)\| = t\lambda \|x - y\|,$$

for all $x, y \in \overline{\mathbb{B}}(0, r)$ and all $0 \leq t \leq 1$. Now for each $\epsilon > 0$ take $0 < \delta < \epsilon\lambda^{-1}$. It follows that

$$\|x - y\| < \delta \implies \|f(\bar{x} + tx) - f(\bar{x} + ty)\| < t\epsilon,$$

for all $x, y \in \overline{\mathbb{B}}(0, r)$ and all $0 < t \leq 1$. This completes the proof. \square

Proposition 2.1 Let \mathbb{X} and \mathbb{Y} be normed spaces, $f : \mathbb{X} \rightarrow \mathbb{Y}$ be a mapping, \mathbb{E} be a subset of \mathbb{X} equipped with the topology induced by the norm and $\bar{x} \in \mathbb{X}$. If the bifunction $f_{\mathbb{E}} : \mathbb{E} \times (0, 1] \rightarrow \mathbb{Y}$, defined by

$$f_{\mathbb{E}}(x, t) := \frac{f(\bar{x} + tx) - f(\bar{x})}{t},$$

is uniformly continuous ($\mathbb{E} \times (0, 1]$ equipped with the product topology with the usual linear operations of vector addition and scalar multiplication), then f is homogeneously continuous at \bar{x} on \mathbb{E} .

Proof Let $\epsilon > 0$. By hypothesis, there exist $\delta, \beta > 0$ such that for all $x, y \in \mathbb{E}$ with $\|x - y\| < \delta$ and all $s, h \in (0, 1]$ with $|s - h| < \beta$ we have $\|f_{\mathbb{E}}(x, s) - f_{\mathbb{E}}(y, h)\| < \epsilon$. It follows that

$$\left\| \frac{f(\bar{x} + tx) - f(\bar{x})}{t} - \frac{f(\bar{x} + ty) - f(\bar{x})}{t} \right\| < \epsilon,$$

for all $x, y \in \mathbb{E}$ with $\|x - y\| < \delta$ and all $0 < t \leq 1$. Thus

$$\|f(\bar{x} + tx) - f(\bar{x} + ty)\| < t\epsilon,$$

for all $x, y \in \mathbb{E}$ with $\|x - y\| < \delta$ and all $0 < t \leq 1$. This completes the proof. \square

2.2 Generalized derivatives

We recall the definitions of the Hadamard and Gateaux derivatives: The *Hadamard directional derivative* $f'_H(\bar{x})(v)$ of f at \bar{x} in direction v is defined as

$$f'_H(\bar{x})(v) := \lim_{t \downarrow 0, \mu \rightarrow v} \frac{f(\bar{x} + t\mu) - f(\bar{x})}{t} = \lim_{n \rightarrow +\infty} \frac{f(\bar{x} + t_n v_n) - f(\bar{x})}{t_n},$$

where (v_n) and (t_n) are any sequences such that $v_n \rightarrow v$ and $t_n \rightarrow 0^+$.

The *Gateaux directional derivative* $f'_G(\bar{x})(v)$ of f at \bar{x} in direction v is defined by

$$f'_G(\bar{x})(v) := \lim_{t \downarrow 0} \frac{f(\bar{x} + tv) - f(\bar{x})}{t}.$$

The following facts are well known:

- Hadamard differentiability is a *stronger* notion than Gateaux differentiability, see, e.g., [6]; when f is Hadamard differentiable at \bar{x} , it is Gateaux (directional) differentiable at \bar{x} and, moreover, $f'_G(\bar{x})$ is continuous;
- For locally Lipschitz mappings in normed spaces, Hadamard and Gateaux directional derivatives coincide.

The following corollary uses Hadamard differentiability and provides another sufficient condition for a mapping f to be homogeneously continuous.

Corollary 2.1 Let \mathbb{X} and \mathbb{Y} be normed spaces, $f : \mathbb{X} \rightarrow \mathbb{Y}$ be a continuous mapping, \mathbb{E} be a compact subset of \mathbb{X} (equipped with the topology induced by the norm) and $\bar{x} \in \mathbb{X}$. If the Hadamard directional derivative of f at \bar{x} in every direction $v \in \mathbb{E}$ exists, then f is homogeneously continuous at \bar{x} on \mathbb{E} .

Proof Define the bifunction $\tilde{f}_{\mathbb{E}} : \mathbb{E} \times [0, 1] \rightarrow \mathbb{Y}$ as

$$\tilde{f}_{\mathbb{E}}(v, t) := \begin{cases} \frac{f(\bar{x} + tv) - f(\bar{x})}{t} & \text{if } 0 < t \leq 1, \\ f'_H(\bar{x})(v) & \text{if } t = 0. \end{cases}$$

Since f is continuous and the Hadamard directional derivative of f at \bar{x} in every direction $v \in \mathbb{E}$ exists, the bifunction $\tilde{f}_{\mathbb{E}}$ is continuous. Since $\mathbb{E} \times [0, 1]$ is compact, $\tilde{f}_{\mathbb{E}}$ is uniformly continuous. It follows that the bifunction $f_{\mathbb{E}} : \mathbb{E} \times (0, 1] \rightarrow \mathbb{Y}$, defined by

$$f_{\mathbb{E}}(x, t) := \frac{f(\bar{x} + tx) - f(\bar{x})}{t},$$

is uniformly continuous. Now apply Proposition 2.1. \square

The following proposition illustrates our main motivation for introducing the homogeneously continuous mappings.

Proposition 2.2 *Let \mathbb{X} and \mathbb{Y} be normed vector spaces, $f : \mathbb{X} \rightarrow \mathbb{Y}$ be a mapping, \mathbb{E} be a subset of \mathbb{X} and $\bar{x} \in \mathbb{X}$. If f is homogeneously continuous at \bar{x} on \mathbb{E} , then there exist $\delta > 0$ and $\beta > 0$ such that*

$$\left\| \frac{f(\bar{x} + tx) - f(\bar{x})}{t} - \frac{f(\bar{x} + ty) - f(\bar{x})}{t} \right\| < \epsilon,$$

for all $x, y \in \mathbb{E}$ with $\|x - y\| < \delta$ and all $0 < t \leq \beta$.

Proof The proof is obvious; we therefore omit it. \square

For a mapping $f : \mathbb{X} \rightarrow \mathbb{Y}$, we consider the following notions of *lower directional derivatives* which are crucial to our approach:

$$f'_l(\bar{x})(v) := \liminf_{t \downarrow 0} \frac{\|f(\bar{x} + tv) - f(\bar{x})\|}{t},$$

$$f'_0(\bar{x})(v) := \liminf_{t \downarrow 0, \mu \rightarrow v} \frac{\|f(\bar{x} + t\mu) - f(\bar{x})\|}{t}.$$

Note that we have

$$0 \leq f'_0(\bar{x})(v) \leq f'_l(\bar{x})(v), \quad (2.2)$$

for every $v \in \mathbb{X}$. We shall observe that if $\inf_{v \in \mathbb{S}} f'_l(\bar{x})(v) > 0$ and f is homogeneously continuous at \bar{x} on \mathbb{S} , then f satisfies the property (1.2) above.

2.3 Main results

Throughout the remaining part of the discussion, unless specified otherwise, we assume that \mathbb{X} is a *finite-dimensional space* and \mathbb{Y} is an arbitrary normed space. We now are completely ready to state the main theorem of the paper. For a positive scalar $\alpha \in \mathbb{R}$, let

$$\mathbb{S}_{\alpha} := \{x \in \mathbb{X} : \|x\| = \alpha\} = \alpha\mathbb{S}.$$

Theorem 2.1 Let $f : \mathbb{X} \rightarrow \mathbb{Y}$ be homogeneously continuous at $\bar{x} \in \mathbb{X}$ on \mathbb{S}_α for some positive scalar α . If there exists some $\kappa > 0$ such that $\inf_{v \in \mathbb{S}_\alpha} f'_l(\bar{x})(v) > \kappa$, then there exists $\delta > 0$ such that

$$\|x - \bar{x}\| \leq \frac{\alpha}{\kappa} \|f(x) - f(\bar{x})\|,$$

for all $x \in \mathbb{B}(\bar{x}, \delta)$. In other words, \bar{x} is a strongly regular point of f .

Proof Let $\kappa < \gamma < \inf_{v \in \mathbb{S}_\alpha} f'_l(\bar{x})(v)$ and $\epsilon := \gamma - \kappa$. Hence, for all $v \in \mathbb{S}_\alpha$ there exists $0 < r_v \leq 1$ such that

$$\inf_{0 < h \leq r_v} \frac{\|f(\bar{x} + hv) - f(\bar{x})\|}{h} > \gamma. \quad (2.3)$$

Since f is homogeneously continuous at \bar{x} on \mathbb{S}_α , there exist $\theta > 0$ and $\beta > 0$ such that

$$\|v - \mu\| < \theta \implies \left\| \frac{f(\bar{x} + tv) - f(\bar{x})}{t} - \frac{f(\bar{x} + t\mu) - f(\bar{x})}{t} \right\| < \epsilon, \quad (2.4)$$

for all $v, \mu \in \mathbb{S}_\alpha$ and all $0 < t \leq \beta$, by Proposition 2.2. Let $\hat{r}_v := \min\{\theta, \beta, r_v\}$ for all $v \in \mathbb{S}_\alpha$. Clearly, $\mathbb{S}_\alpha \subset \bigcup_{v \in \mathbb{S}_\alpha} \mathbb{B}(v, \hat{r}_v)$. The compactness of \mathbb{S}_α implies that there exist $v_1, v_2, \dots, v_m \in \mathbb{S}_\alpha$ such that $\mathbb{S}_\alpha \subset \bigcup_{k=1}^m \mathbb{B}(v_k, \hat{r}_{v_k})$. Now let $x \in \mathbb{B}(\bar{x}, \alpha\hat{\delta}) \setminus \{\bar{x}\}$ and $v := \frac{\alpha}{\|x - \bar{x}\|}(x - \bar{x})$, where $\hat{\delta} := \min\{\hat{r}_{v_k} : 1 \leq k \leq m\}$. Then, $v \in \mathbb{S}_\alpha$ and therefore $v \in \mathbb{B}(v_s, \hat{r}_{v_s})$ for some $1 \leq s \leq m$. It follows that $\|v - v_s\| < \theta$ and $\alpha^{-1}\|x - \bar{x}\| < \beta$. By (2.4), we deduce that

$$\left\| \frac{f(\bar{x} + \alpha^{-1}\|x - \bar{x}\|v_s) - f(\bar{x})}{\alpha^{-1}\|x - \bar{x}\|} - \frac{f(\bar{x} + \alpha^{-1}\|x - \bar{x}\|v) - f(\bar{x})}{\alpha^{-1}\|x - \bar{x}\|} \right\| < \epsilon.$$

Hence

$$\begin{aligned} \frac{\|f(x) - f(\bar{x})\|}{\alpha^{-1}\|x - \bar{x}\|} &= \frac{\|f(\bar{x} + \alpha^{-1}\|x - \bar{x}\|v) - f(\bar{x})\|}{\alpha^{-1}\|x - \bar{x}\|} > \frac{\|f(\bar{x} + \alpha^{-1}\|x - \bar{x}\|v_s) - f(\bar{x})\|}{\alpha^{-1}\|x - \bar{x}\|} - \epsilon \\ &> \gamma - \epsilon = \kappa \quad \text{by (2.3),} \end{aligned}$$

since $\alpha^{-1}\|x - \bar{x}\| < r_{v_s}$. It follows that

$$\|x - \bar{x}\| \leq \frac{\alpha}{\kappa} \|f(x) - f(\bar{x})\|,$$

for all $x \in \mathbb{B}(\bar{x}, \alpha\hat{\delta})$. Letting $\delta := \alpha\hat{\delta}$ completes the proof. \square

Corollary 2.2 Let $f : \mathbb{X} \rightarrow \mathbb{Y}$ be homogeneously continuous at $\bar{x} \in \mathbb{X}$ on \mathbb{S} . If there exists some $\kappa > 0$ such that $\inf_{v \in \mathbb{S}} f'_l(\bar{x})(v) > \kappa$, then there exists $\delta > 0$ such that

$$\|x - \bar{x}\| \leq \frac{1}{\kappa} \|f(x) - f(\bar{x})\|, \quad (2.5)$$

for all $x \in \mathbb{B}(\bar{x}, \delta)$.

Proof Apply Theorem 2.1 and (2.2). \square

Corollary 2.3 Suppose that $f : \mathbb{X} \rightarrow \mathbb{Y}$ is locally Lipschitz around $\bar{x} \in \mathbb{X}$. If there exists some $\kappa > 0$ such that $\inf_{v \in \mathbb{S}} f'_l(\bar{x})(v) > \kappa$, then there exists $\delta > 0$ such that (2.5) holds for all $x \in \mathbb{B}(\bar{x}, \delta)$.

Proof By Lemma 2.1, f is homogeneously continuous at \bar{x} on some closed ball $\overline{\mathbb{B}}(0, r)$. It follows that f is homogeneously continuous at \bar{x} on \mathbb{S}_r (since $\mathbb{S}_r \subset \overline{\mathbb{B}}(0, r)$). The condition $\inf_{v \in \mathbb{S}} f'_l(\bar{x})(v) > \kappa$ implies that $\inf_{v \in \mathbb{S}_r} f'_l(\bar{x})(v) > r\kappa$. Now apply Theorem 2.1. \square

Corollary 2.4 Let $f : \mathbb{X} \rightarrow \mathbb{Y}$ be a continuous mapping and $\bar{x} \in \mathbb{X}$. Assume that the Hadamard directional derivative of f at \bar{x} in every direction $v \in \mathbb{S}$ exists. If there exists some $\kappa > 0$ such that $\inf_{v \in \mathbb{S}} \|f'_H(\bar{x})(v)\| > \kappa$, then there exists $\delta > 0$ such that (2.5) holds for all $x \in \mathbb{B}(\bar{x}, \delta)$.

Proof By hypothesis, $f'_H(\bar{x})(v)$ exists for every $v \in \mathbb{S}$. By continuity of f and $\|\cdot\|$, it follows that

$$\|f'_H(\bar{x})(v)\| = \|f'_G(\bar{x})(v)\| = \lim_{t \downarrow 0} \frac{\|f(\bar{x} + tv) - f(\bar{x})\|}{t} = f'_l(\bar{x})(v),$$

for every $v \in \mathbb{S}$. It follows that $\inf_{v \in \mathbb{S}} f'_l(\bar{x})(v) > \kappa$. Since \mathbb{X} is finite dimensional, \mathbb{S} is compact. Hence, f is homogeneously continuous at $\bar{x} \in \mathbb{X}$ on \mathbb{S} , by Corollary 2.1. Now apply Theorem 2.1. \square

The following example has been considered in [7] (Example 2.1). We shall prove that the origin is a regular point of the involved mapping f once again by Theorem 2.1.

Example 2.1 Consider the mapping $f : \mathbb{R} \rightarrow \mathbb{R}$ defined as

$$f(x) := \begin{cases} |x|(\frac{2}{\pi} - x \sin(\frac{1}{x})), & x \neq 0; \\ 0, & x = 0. \end{cases}$$

We have $\mathbb{S} = \{\pm 1\}$ and therefore f is homogeneously continuous at 0 on \mathbb{S} . One may easily verify that

$$f'_l(0)(\pm 1) = \liminf_{t \downarrow 0} \frac{|\pm t(\frac{2}{\pi} - (\pm t) \sin(\frac{1}{\pm t}))|}{t} = \frac{2}{\pi}.$$

It follows that $\inf_{s \in \mathbb{S}} f'_l(0)(s) = \frac{2}{\pi} > 0$. Hence, if $0 < \kappa < \frac{2}{\pi}$, then there exists $\delta > 0$ such that

$$|x - 0| = |x| \leq \frac{1}{\kappa} |f(x) - f(0)| = \frac{1}{\kappa} |f(x)|,$$

for all $|x| < \delta$, by Theorem 2.1. Hence, 0 is a strongly regular point of f . Since f is continuous, thus the subset $f^{-1}(0)$ is closed and therefore the distance function $\text{dist}(\cdot, f^{-1}(0))$ is Lipschitz around 0 (see [8, p. 11]). Hence, 0 is a regular point of f .

3 Hoffman's estimate for the distance to the set of solutions to a system of linear inequalities

Theorem 3.1 (Hoffman, 1952, [9, 10]) *Let $x_i^*, i = 1, 2, \dots, k$ be a finite family of linear forms on \mathbb{X}^* . Set*

$$\mathbb{C}_{\leq} := \{x \in \mathbb{X} \text{ such that } \langle x_i^*, x \rangle \leq 0, i = 1, 2, \dots, k\}. \quad (3.1)$$

Then, there exists $\kappa > 0$ such that

$$\text{dist}(x, \mathbb{C}_{\leq}) \leq \kappa [\Phi(x)]_+, \quad (3.2)$$

where $\Phi(x) := \max\{\langle x_i^, x \rangle, i = 1, 2, \dots, k\}$ and $[\Phi(x)]_+ := \max(\Phi(x), 0)$.*

Hoffman's result is considered as the starting point of the theory of error bounds, theory that has been extended over the years to different contexts. We refer to [3, 11–13] and the references therein for the discussion of the fundamental role played by Hoffman bounds and more generally by error bounds in mathematical programming. As described, for example, in [14], they are used, for instance, in convergence properties of algorithms, in sensitivity analysis, in designing solution methods for nonconvex quadratic problems. When $\mathbb{C} := \{x \in \mathbb{X} : A(x) = 0, \langle x_i^*, x \rangle \leq 0, i = 1, 2, \dots, k\}$ where $x_i^* \in X^*, i = 1, 2, \dots, k$, are some given functionals and $A : \mathbb{X} \rightarrow \mathbb{Y}$ is a linear (continuous) mapping, we have the following result.

Theorem 3.2 (Ioffe, 1979, [15]) *There exists some $\kappa > 0$ such that*

$$\text{dist}(x, \mathbb{C}) \leq \kappa \left(\|A(x)\| + \sum_{i=1}^k [\langle x_i^*, x \rangle]_+ \right), \quad (3.3)$$

for all $x \in \mathbb{X}$.

Now let $\mathbb{G} := \text{Ker } A \cap (\bigcap_{i=1}^k \text{Ker } x_i^*)$. Then, Theorem 3.2 yields the following result.

Corollary 3.1 *There exists some $\kappa' > 0$ such that*

$$\text{dist}(x, \mathbb{G}) \leq \kappa' \left(\|A(x)\| + \sum_{i=1}^k |\langle x_i^*, x \rangle| \right), \quad (3.4)$$

for all $x \in \mathbb{X}$.

In this section, we apply Theorem 2.1 and establish similar estimates. We prove that there exists $\bar{\kappa} > 0$ such that

$$\text{dist}(x, \mathbb{G}) \leq \bar{\kappa} \left(\|L(x)\| + \sum_{i=1}^k [\langle x_i^*, x \rangle]_+ \right),$$

for all $x \in \mathbb{X}$, where $L : \mathbb{X} \rightarrow \mathbb{X}$ is a linear mapping with $\text{Ker } L = \mathbb{G}$. Our results also allow us to evaluate the constant $\bar{\kappa}$. The details are as follows.

Proposition 3.1 *Let $A : \mathbb{X} \rightarrow \mathbb{Y}$ be a linear mapping and $x_i^* \in \mathbb{X}^*$, $i = 1, 2, \dots, k$ be given. Suppose that $L : \mathbb{X} \rightarrow \mathbb{X}$ is a linear mapping such that $\text{Ker } L = \mathbb{G}$. Then*

$$\text{dist}(x, \mathbb{G}) \leq \frac{1}{\gamma} \left(\|L(x)\| + \sum_{i=1}^k [\langle x_i^*, x \rangle]_+ \right), \quad (3.5)$$

where γ is a positive real number given by

$$\gamma := \inf \left\{ \|L(v)\| + \sum_{i=1}^n [\langle x_i^*, v \rangle]_+ : v \in \mathbb{X}, \text{dist}(v, \mathbb{G}) = 1 \right\}. \quad (3.6)$$

Proof Let us consider the finite-dimensional quotient space $\mathcal{M} := \frac{\mathbb{X}}{\mathbb{G}}$, and denote by $[x]$ the equivalence class containing x in \mathcal{M} , that is, $[x] := x + \mathbb{G}$. We note $\|[x]\| := \inf\{\|x + y\| : y \in \mathbb{G}\}$. Denote by $\mathbb{S}_{\mathcal{M}}$ the unit sphere of \mathcal{M} (i.e., the elements of \mathcal{M} of norm one). Obviously,

$$\mathbb{S}_{\mathcal{M}} = \{x \in \mathbb{X} : \inf\{\|x + y\| : y \in \mathbb{G}\} = 1\} = \{x \in \mathbb{X} : \text{dist}(x, \mathbb{G}) = 1\}. \quad (3.7)$$

Consider the continuous linear mapping $\bar{L} : \mathcal{M} \rightarrow \mathbb{X}$ defined as $\bar{L}([x]) := L(x)$ for all $[x] \in \mathcal{M}$. Also for each $1 \leq i \leq k$ define $\langle [x_i]^*, [x] \rangle := \langle x_i^*, x \rangle$ for all $[x] \in \mathcal{M}$. Obviously, each $[x_i]^*$ belongs to \mathbb{G}^\perp (the orthogonal complement of \mathbb{G}), and hence belongs to the dual of \mathcal{M} (which is isometrically isomorphic to \mathbb{G}^\perp [16]). Set

$$\bar{\mathbb{C}} := \{[x] \in \mathcal{M} : \bar{L}([x]) = 0, \langle [x_i]^*, [x] \rangle \leq 0, i = 1, 2, \dots, k\}.$$

We have $\text{Ker } \bar{L} = \text{Ker } L = \mathbb{G}$ and therefore $\bar{\mathbb{C}} = \{[0]\}$. Now define the mapping $f : \mathcal{M} \rightarrow \mathbb{R}$ as

$$f([x]) := \|\bar{L}([x])\| + \sum_{i=1}^k [\langle [x_i]^*, [x] \rangle]_+.$$

We show that the conditions of Theorem 2.1 for f at $[\bar{x}] = [0]$ are all satisfied. For all $[v] \in \mathbb{S}_{\mathcal{M}}$, one has

$$\lim_{t \downarrow 0, \mu \rightarrow v} \frac{f([t\mu]) - f([0])}{t} = \|\bar{L}([v])\| + \sum_{i=1}^k [\langle [x_i]^*, [v] \rangle]_+.$$

Hence, f is homogeneously continuous at $[0]$ on $\mathbb{S}_{\mathcal{M}}$, by Corollary 2.1. We also have

$$f'([0])([v]) = \|\bar{L}([v])\| + \sum_{i=1}^k [\langle [x_i]^*, [v] \rangle]_+ = \|L(v)\| + \sum_{i=1}^n [\langle x_i^*, v \rangle]_+.$$

The continuity of f implies that the mapping $f|_{\mathbb{S}_{\mathcal{M}}}$ (the restriction of f to $\mathbb{S}_{\mathcal{M}}$) attains its minimum at some $[v_0] \in \mathbb{S}_{\mathcal{M}}$. Then, $[v_0] \notin \bar{\mathbb{C}}$ (note that $\bar{\mathbb{C}} = \{[0]\}$) and therefore

$$\|\bar{L}([v_0])\| + \sum_{i=1}^k [\langle [x_i]^*, [v_0] \rangle]_+ > 0.$$

It follows that $\inf_{[v] \in \mathbb{S}_{\mathcal{M}}} f'_l([0])([v]) > 0$. Using (3.7), we obtain

$$\inf_{[v] \in \mathbb{S}_{\mathcal{M}}} f'_l([0])([v]) = \inf \left\{ \|L(v)\| + \sum_{i=1}^n [(x_i^*, v)]_+ : v \in \mathbb{X}, \text{dist}(v, \mathbb{G}) = 1 \right\} = \gamma.$$

Thus $\gamma > 0$. Now let $0 < \kappa < \gamma$. Theorem 2.1 implies that there exists some $\delta > 0$ such that

$$\|[x] - [0]\| = \|[x]\| \leq \frac{1}{\kappa} \left(\|\bar{L}([x])\| + \sum_{i=1}^k [([x_i]^*, [x])]_+ \right),$$

for all $[x] \in \mathbb{B}_{\mathcal{M}}([0], \delta)$. Since f is sublinear,

$$\|[x]\| \leq \frac{1}{\kappa} \left(\|\bar{L}([x])\| + \sum_{i=1}^k [([x_i]^*, [x])]_+ \right),$$

for all $[x] \in \mathcal{M}$. It follows that

$$\text{dist}(x, \mathbb{G}) \leq \frac{1}{\kappa} \left(\|L(x)\| + \sum_{i=1}^k [(x_i^*, x)]_+ \right). \quad (3.8)$$

For all $x \in \mathbb{X}$. Letting $\kappa \rightarrow \gamma$ in (3.8), we obtain the desired result. \square

Remark 3.1 The existence of the linear mapping $L : \mathbb{X} \rightarrow \mathbb{X}$ discussed in Proposition 3.1 is straightforward. Indeed, \mathbb{G} is a closed subspace of \mathbb{X} and \mathbb{X} is separable, thus there exists a (continuous) linear mapping $L : \mathbb{X} \rightarrow \mathbb{X}$ with $\text{Ker } L = \mathbb{G}$ (see [17]). Of course, one can easily define L directly (without using [17]). To see this, suppose that $\dim \mathbb{X} = n$ and let $\{e_1, \dots, e_j\}$ be a linearly independent basis for the vector space \mathbb{G} . By linear algebra, we can extend $\{e_1, \dots, e_j\}$ to get a linearly independent basis for \mathbb{X} (since \mathbb{G} is a subspace of \mathbb{X}). Let us denote this basis by $\{e_1, \dots, e_j, e_{j+1}, \dots, e_n\}$. Now for every $x := x_1 e_1 + \dots + x_n e_n \in \mathbb{X}$, define the mapping $L : \mathbb{X} \rightarrow \mathbb{X}$ as

$$L(x) := (\underbrace{0, 0, \dots, 0}_j, \underbrace{x_{j+1}, \dots, x_n}_{n-j}).$$

One can easily check that L is well-defined, linear, and $\text{Ker } L = \mathbb{G}$.

Corollary 3.2 *Let $A : \mathbb{X} \rightarrow \mathbb{X}$ be a linear mapping. Then*

$$\text{dist}(x, \text{Ker } A) \leq \frac{\|A(x)\|}{\inf\{\|A(v)\| : v \in \mathbb{X}, \text{dist}(v, \text{Ker } A) = 1\}},$$

for all $x \in \mathbb{X}$.

Proof Let $\mathbb{X} = \mathbb{Y}$, and $x_i^* \equiv 0$ for all $1 \leq i \leq k$. Then, $\mathbb{G} = \text{Ker } A$. Letting $L := A$ in Proposition 3.1 yields the result. \square

Corollary 3.3 *Let $A : \mathbb{X} \rightarrow \mathbb{Y}$ and $L : \mathbb{X} \rightarrow \mathbb{X}$ be linear mappings with $\text{Ker } L = \mathbb{G}$ and $x_i^* \in \mathbb{X}^*$, $i = 1, 2, \dots, k$, be some given functionals. Then*

$$\text{dist}(x, \mathbb{G}) \leq \frac{\|L(x)\|}{\inf\{\|L(v)\| + \sum_{i=1}^n [\langle x_i^*, v \rangle]_+ : v \in \mathbb{X}, \text{dist}(v, \mathbb{G}) = 1\}}, \quad (3.9)$$

for every $x \in \mathbb{C}_\leq$ (see (3.1) above).

Proof Proposition 3.1 implies that

$$\text{dist}(x, \mathbb{G}) \leq \frac{1}{\gamma} \left(\|L(x)\| + \sum_{i=1}^k [\langle x_i^*, x \rangle]_+ \right),$$

where

$$\gamma := \inf \left\{ \|L(v)\| + \sum_{i=1}^n [\langle x_i^*, v \rangle]_+ : v \in \mathbb{X}, \text{dist}(v, \mathbb{G}) = 1 \right\}.$$

Now let $x \in \mathbb{C}_\leq$. Thus $\langle x_i^*, x \rangle \leq 0$ for every $1 \leq i \leq k$. Hence $[\langle x_i^*, x \rangle]_+ = 0$ for every $1 \leq i \leq k$. Then, the above inequality yields

$$\text{dist}(x, \mathbb{G}) \leq \frac{\|L(x)\|}{\inf\{\|L(v)\| + \sum_{i=1}^n [\langle x_i^*, v \rangle]_+ : v \in \mathbb{X}, \text{dist}(v, \mathbb{G}) = 1\}},$$

for every $x \in \mathbb{C}_\leq$. This completes the proof. \square

Finally, let us make a comparison between the two estimations (3.5) in Proposition 3.1 and (3.4) in Corollary 3.1 described above. First, note that an application of Corollary 3.1 (or a direct application of Theorem 3.2) with L (described in Proposition 3.1) in place of A and no inequalities immediately produces the following estimate:

$$\text{dist}(x, \mathbb{G}) \leq \kappa_0 \|L(x)\|, \quad (3.10)$$

for all $x \in \mathbb{X}$, where κ_0 is a constant. On the other hand, doing the same replacements in Proposition 3.1 (i.e., applying Proposition 3.1 with L in place of A and L in place of itself without the inequalities) yields

$$\text{dist}(x, \mathbb{G}) \leq \frac{1}{\gamma_0} \|L(x)\|, \quad (3.11)$$

where

$$\gamma_0 := \inf\{\|L(v)\| : v \in \mathbb{X}, \text{dist}(v, \mathbb{G}) = 1\} > 0.$$

The question is: which of the above estimates (3.10) and (3.11) is better? To answer this question, we need to know the relationship between the coefficients κ_0 and γ_0 stated above. As long as the value of the constant κ_0 in (3.10) is not known, we can't say which of the estimates (3.10) and (3.11) produces a better result. We can just say that the estimate (3.11)

technically is better, since it also allows us to estimate the unknown constant κ_0 in (3.10). Indeed, $\kappa_0 \leq \frac{1}{\gamma_0}$.

Another question which may arise is: with the simple estimate (3.11) in hand, what is the necessity of using the estimate (3.5) in Proposition 3.1 (regarding the inequalities)? To answer this question, let's take a closer look at the estimate (3.5). Indeed, by Proposition 3.1, we have

$$\text{dist}(x, \mathbb{G}) \leq \frac{1}{\gamma} \left(\|L(x)\| + \sum_{i=1}^k [\langle x_i^*, x \rangle]_+ \right), \quad (3.12)$$

where

$$\gamma := \inf \left\{ \|L(v)\| + \sum_{i=1}^n [\langle x_i^*, v \rangle]_+ : v \in \mathbb{X}, \text{dist}(v, \mathbb{G}) = 1 \right\} > 0.$$

We observe that, on the one hand, $\|L(x)\| \leq \|L(x)\| + \sum_{i=1}^k [\langle x_i^*, x \rangle]_+$ and, on the other hand, $\frac{1}{\gamma} \leq \frac{1}{\gamma_0}$. As a result, we cannot generally compare the right-hand sides of the estimates (3.11) and (3.12) to determine which is better. Corollary 3.3 says that when $x \in \mathbb{C}_{\leq}$, it would be better to use (3.12).

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Authors' contributions

All authors contributed equally in writing this article. All authors read and approved the final manuscript.

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