Chaira et al. *Fixed Point Theory Algorithms Sci Eng* (2021) 2021:4
https://doi.org/10.1186/s13663-021-00690-8

Fixed Point Theory and Algorithms for Sciences and Engineering

RESEARCH

Open Access

Common fixed point for some generalized contractive mappings in a modular metric space with a graph

Karim Chaira[1,2], Abderrahim Eladraoui[2*], Mustapha Kabil[1] and Abdessamad Kamouss[1]

*Correspondence: a.adraoui@live.fr
[2]Laboratory of Algebra, Analysis and Applications, Faculty of Sciences Ben M'sik, University Hassan II of Casablanca, Casablanca, Morocco
Full list of author information is available at the end of the article

Abstract

In this paper, we investigate the existence and the uniqueness of a common fixed point of a pair of self-mappings satisfying new contractive type conditions on a modular metric space endowed with a reflexive digraph. An application is given to show the use of our main result.

MSC: 46A80; 47H10

Keywords: Common fixed point; Weak contraction; Digraph; Modular metric space

1 Introduction and preliminaries

More generalized contractive type conditions are considered in the study of the existence and uniqueness of the fixed point. Alber and Guerre-Delabriere in [2] introduced a class of weakly contractive maps on closed convex sets of Hilbert spaces. In [9], Rhoades extended a part of this study to an arbitrary Banach space. The notion of weak contraction has been studied by other authors in the setting of metric spaces (see [8, 12] and the references therein). In [13], Zhang gave some new generalized contractive type conditions for a pair of mappings in a metric space and proved some common fixed point results for these mappings. Let $F : [0, +\infty[\longrightarrow \mathbb{R}$ be a function satisfying the three conditions:

(i) $F(0) = 0$ and $F(t) > 0$ for all $t > 0$;
(ii) F is nondecreasing on $[0, +\infty[$;
(iii) F is continuous on $[0, +\infty[$.

Consider the function $\phi : [0, +\infty[\longrightarrow [0, +\infty[$ such that

(i) $\phi(t) < t$ for all $t > 0$;
(ii) ϕ is nondecreasing and right upper semicontinuous on $[0, +\infty[$;
(iii) $\lim_{n\to+\infty} \phi^n(t) = 0$ for all $t > 0$.

In this paper, motivated by some works as [10], we extend the following theorem to the setting of the modular metric space endowed with a reflexive digraph.

Theorem ([13]) *Let X be a complete metric space, and let $T,S : X \longrightarrow X$ be two self-mappings satisfying*

$$F\big(d(Tx,Sy)\big) \leq \phi\big(F\big(M(x,y)\big) \quad \text{for each } x,y \in X,$$

where

$$M(x,y) = \max\left\{d(x,y), d(Tx,x), d(Sy,y), \frac{d(Tx,y)+d(Sy,x)}{2}\right\}.$$

Then T and S have a unique common fixed point in X. Moreover, for each $x_0 \in X$, the iterative sequence $\{x_n\}$ with $x_{2n+1} = Tx_{2n}$ and $x_{2n+2} = Sx_{2n+1}$ converges to the common fixed point of T and S.

In the sequel, we recall some basic notions: Let X be a nonempty set. For a function $]0,+\infty[\times X \times X \to [0,+\infty]$, we will use the notation

$$\_\lambda(x,y) = (\lambda,x,y) \quad \text{for all } \lambda > 0 \text{ and } x,y \in X.$$

Definition 1.1 ([7]) A function $\omega :]0,+\infty[\times X \times X \to [0,+\infty]$ is said to be modular metric on X if it satisfies the following conditions:

(i) Given $x,y \in X$, $x = y$ if and only if $\omega_\lambda(x,y) = 0$ for all $\lambda > 0$;
(ii) For all $x,y \in X$, for all $\lambda > 0$, $\omega_\lambda(x,y) = \omega_\lambda(y,x)$;
(iii) For all $x,y,z \in X$ and for all $\lambda,\mu > 0$, $\omega_{\lambda+\mu}(x,y) \leq \omega_\lambda(x,z) + \omega_\mu(z,y)$.

In this case, (X,ω) is called modular metric space.

The modular ω is said to be regular if condition (i) holds for some $\lambda > 0$.

The modular ω is said to be convex if, for all $\lambda,\mu > 0$ and $x,y,z \in X$, we have

$$\omega_{\lambda+\mu}(x,y) \leq \frac{\lambda}{\lambda+\mu}\omega_\lambda(x,z) + \frac{\mu}{\lambda+\mu}\omega_\mu(z,y).$$

Let (X,ω) be a modular metric space. Fix $x_0 \in X$. Set

$$X_\omega = X_\omega(x_0) = \big\{x \in X : \omega_\lambda(x,x_0) \longrightarrow 0 \text{ as } \lambda \longrightarrow \infty\big\}$$

and

$$X^*_\omega = X^*_\omega(x_0) = \big\{x \in X : \exists\lambda > 0, \omega_\lambda(x,x_0) < \infty\big\}.$$

The two linear spaces X_ω and X^*_ω are said to be modular spaces (around x_0). It is clear that $X_\omega \subseteq X^*_\omega$.

Definition 1.2 ([7]) We say that ω satisfies the Δ_2-type condition if, for every $\alpha > 0$, there exists a constant $K_\alpha > 0$ such that

$$\omega_{\frac{\lambda}{\alpha}}(x,y) \leq K_\alpha \omega_\lambda(x,y)$$

for all $x,y \in X_\omega$ and any $\lambda > 0$.

Remark 1.3 If ω satisfies the Δ_2-type condition, then ω is regular and $X_\omega = X_\omega^* = X$.

A condition weaker than the Δ_2-type condition is often used in the literature:

Definition 1.4 We say that ω satisfies the Δ_2-condition if $\lim_{n\to+\infty} \omega_\lambda(x_n, x) = 0$ for some $\lambda > 0$ implies that $\lim_{n\to+\infty} \omega_\lambda(x_n, x) = 0$ for all $\lambda > 0$.

It is clear that if ω satisfies the Δ_2-type condition, then ω satisfies the Δ_2-condition, and that the converse is not true. Throughout this paper, we consider the modular metrics satisfying the Δ_2-type condition, and we adopt the definitions of some topological notions as stated in [11].

Definition 1.5 Let ω be a modular metric on X.

1. We say that a sequence $\{x_n\} \subset X_\omega$ is ω-convergent to some $x \in X_\omega$ if $\lim_{n\to+\infty} \omega_\lambda(x_n, x) = 0$ for some $\lambda > 0$. We will call x the ω-limit of $\{x_n\}$.
   If ω satisfies the Δ_2-type condition, then $\lim_{n\to+\infty} \omega_\lambda(x_n, x) = 0$ for all $\lambda > 0$.
2. We say that a sequence $\{x_n\} \subset X_\omega$ is ω-Cauchy if, for some $\lambda > 0$,
   $$\lim_{n,m\to+\infty} \omega_\lambda(x_n, x_m) = 0.$$
   If ω satisfies the Δ_2-type condition, then $\{x_n\}$ is ω-Cauchy if $\lim_{n,m\to+\infty} \omega_\lambda(x_n, x_m) = 0$ for all $\lambda > 0$.
3. We say that $M \subset X_\omega$ is ω-closed if the ω-limit of any ω-convergent sequence of M is in M.
4. We say that $M \subset X_\omega$ is ω-complete if any ω-Cauchy sequence in M is ω-convergent and its ω-limit belongs to M.
5. We say that ω satisfies the Fatou property if, for some $\lambda > 0$, we have
   $$\omega_\lambda(x, y) \le \liminf_{n\to+\infty} \omega_\lambda(x_n, y)$$
   for any sequence $\{x_n\} \subset X_\omega$ which is ω-convergent to x and for any $y \in X_\omega$.

Let V be an arbitrary set. A directed graph, or digraph, is a pair $G = (V, E)$ where E is a subset of the Cartesian product $V \times V$. The elements of V are called vertices or nodes of G, and the elements of E are the edges also called oriented edges or arcs of G. An edge of the form (v, v) is a loop on v. Another way to express that E is a subset of $V \times V$ is to say that E is a binary relation over V. Given a digraph G, the set of vertices (respectively of edges) of G is denoted by $V(G)$ (respectively $E(G)$). A digraph $G' = (V', E')$ is said to be an induced subgraph of a digraph $G = (V, E)$ on V' if $V' \subseteq V$ and $E' = E \cap (V' \times V')$. We denote G' by $G[V']$.

The digraph $G = (V, E)$ is said to be

(i) transitive if whenever $(x, y) \in E$ and $(y, z) \in E$, then $(x, z) \in E$.
(ii) reflexive if $\Delta := \{(v, v) : v \in V\}$ is a subset of E.

A vertex x is said to be

(i) a start point of G if there exists no vertex y such that $(y, x) \in E$.
(ii) isolated if, for each vertex $y \neq x$, we have neither $(x, y) \in E$ nor $(y, x) \in E$.

Given two vertices $x, y \in V$. A path in G, from (or joining) x to y is a sequence of vertices $p = \{a_i\}_{0\leq i\leq n}$, $n \in \mathbb{N}^*$ such that $a_0 = x$, $a_n = y$ and $(a_i, a_{i+1}) \in E$ for all $i \in \{0, 1, \dots, n-1\}$. The integer n is the length of the path p. If $x = y$ and $n > 1$, the path p is called a directed cycle. An acyclic digraph is a digraph which has no directed cycle.

We denote by $y \in [x]_G$ the fact that there is a directed path in G joining x to y.

A sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to be G-nondecreasing if $x_{n+1} \in [x_n]_G$ for all $n \in \mathbb{N}$.

A modular metric space (X, ω) endowed with a digraph G such that $V(G) = X$ is denoted by (X, ω, G). In recent years, there has been a great interest in the study of the fixed point property in modular metric spaces endowed with a partial order, see [5] and the references therein.

In this work, we investigate the existence and uniqueness of the common fixed point of a pair of mappings satisfying a generalized contractive condition in the setting of a modular metric space with a reflexive digraph. The main result is illustrated by an example and is used to show the existence of a solution of a system of Fredholm integral equations.

As in [6], we use the property (OSC) defined as follows.

Definition 1.6 Let (X, ω, G) be a modular metric space endowed with a digraph. We say that X satisfies the property (OSC) if, for any G-nondecreasing sequence $\{x_n\} \subseteq X$ which is ω-convergent to $x \in X$, we have $x \in [x_n]_G$ for all $n \in \mathbb{N}$.

2 Main result

The following technical lemmas borrowed from [5] are useful in the sequel and highlight the use of the Δ_2-type condition to establish the main result.

Lemma 2.1 *If ω satisfies the Δ_2-type condition, then*

$$\omega_\lambda(x, y) < \infty \quad \text{for all } \lambda > 0 \text{ and for all } (x, y) \in X_\omega^2.$$

Lemma 2.2 *Let $s, t \in \mathbb{N}^*$. If ω satisfies the Δ_2-type condition and $\{x_n\}$ is not ω-Cauchy, then there exist $\varepsilon > 0$ and two subsequences of integers $\{n_k\}$ and $\{m_k\}$ such that $n_k > m_k \geq k$, $\omega_{2^s}(x_{n_k}, x_{m_k}) \geq \varepsilon$, and $\omega_{\frac{1}{2^t}}(x_{n_k-1}, x_{m_k}) < \varepsilon$.*

From now on, we mean 1 instead of λ for the same reason Abdou and Khamsi used in [1]. One can see that the proof of the main result remains even if we replace 1 with any $\lambda > 0$.

Let $\psi : [0, +\infty[\longrightarrow [0, +\infty[$ be a function satisfying the two conditions:

(i) $\psi(t) < t$ for all $t > 0$;

(ii) ψ is right upper semicontinuous on $[0, +\infty[$.

Let

$$M(x, y) = \max\left\{\omega_1(x, y), \omega_1(x, Sx), \omega_1(y, Ty), \frac{\omega_2(x, Ty) + \omega_2(y, Sx)}{2}\right\}$$

and

$$\mathcal{O}_{x_0}(S, T) = \left\{(TS)^n(x_0), S(TS)^n(x_0) : n \in \mathbb{N}\right\}.$$

Theorem 2.1 *Let (X,ω,G) be a modular metric space endowed with a reflexive digraph G where ω satisfies the Δ_2-type condition and the Fatou property. Let C be an ω-complete nonempty subset of X_ω and $T,S : C \to C$ be two self-mappings. If the following conditions are satisfied*:

(i) *for all $x,y \in C$,*

$$\bigl(y \in [x]_G \text{ or } x \in [y]_G\bigr) \quad\Longrightarrow\quad F\bigl(\omega_1(Sx,Ty)\bigr) \le \psi\bigl(F\bigl(M(x,y)\bigr)\bigr); \tag{1}$$

(ii) *there exists an element $x_0 \in C$ such that the induced subgraph $G[\mathcal{O}_{x_0}(S,T)]$ is a directed path with a unique starting point x_0;*

(iii) *ω satisfies the property (OSC),*

then S and T have a common fixed point in C.

Proof Let x_0 be an element of C such that $G[\mathcal{O}_{x_0}(S,T)]$ is a directed path. Consider the sequence $\{x_n\}$ defined by

$$x_{2n+1} = Sx_{2n} \quad \text{and} \quad x_{2n+2} = Tx_{2n+1} \quad \text{for all } n \in \mathbb{N}.$$

Condition (ii) insures that $\{x_n\}$ is G-nondecreasing. If there exists an integer n such that

$$x_{2n} = x_{2n+1} = x_{2n+2},$$

then x_{2n} is a common fixed point of S and T. Otherwise, suppose that

$$x_{2n} \neq x_{2n+1} \quad \text{or} \quad x_{2n} \neq x_{2n+2} \quad \text{for all } n \in \mathbb{N}.$$

Let $n \in \mathbb{N}$. From $x_{2n+1} \in [x_{2n}]_G$ and applying (1) for $x = x_{2n}$ and $y = x_{2n+1}$, we obtain

$$F\bigl(\omega_1(x_{2n+1},x_{2n+2})\bigr) \le \psi\bigl(F\bigl(M(x_{2n},x_{2n+1})\bigr)\bigr). \tag{2}$$

From

$$M(x_{2n},x_{2n+1}) = \max\left\{\omega_1(x_{2n},x_{2n+1}), \omega_1(x_{2n+1},x_{2n+2}), \frac{\omega_2(x_{2n},x_{2n+2})}{2}\right\}$$

and

$$\frac{\omega_2(x_{2n},x_{2n+2})}{2} \le \frac{\omega_1(x_{2n},x_{2n+1}) + \omega_1(x_{2n+1},x_{2n+2})}{2},$$

it follows that

$$M(x_{2n},x_{2n+1}) = \max\bigl\{\omega_1(x_{2n},x_{2n+1}), \omega_1(x_{2n+1},x_{2n+2})\bigr\}.$$

If we suppose that there exists an integer n such that

$$\omega_1(x_{2n},x_{2n+1}) \le \omega_1(x_{2n+1},x_{2n+2}),$$

then

$$M(x_{2n},x_{2n+1}) = \omega_1(x_{2n+1},x_{2n+2}).$$

Thus

$$F\bigl(\omega_1(x_{2n+1},x_{2n+2})\bigr) \le \psi\bigl(F\bigl(\omega_1(x_{2n+1},x_{2n+2})\bigr)\bigr),$$

which implies that $F(\omega_1(x_{2n+1},x_{2n+2})) = 0$. Hence, $x_{2n+1} = x_{2n+2}$ and, from (2), $x_{2n} = x_{2n+1}$, a contradiction. Hence, for each integer n, we have

$$\omega_1(x_{2n+1},x_{2n+2}) \le \omega_1(x_{2n},x_{2n+1}).$$

By the same argument, if we take, in inequality (1), $x = x_{2n-1}$ and $y = x_{2n}$, we obtain

$$\omega_1(x_{2n},x_{2n+1}) < \omega_1(x_{2n-1},x_{2n}) \quad \text{for all } n \in \mathbb{N}^*.$$

Then $\omega_1(x_{n+1},x_{n+2}) < \omega_1(x_n,x_{n+1})$ for all $n \in \mathbb{N}$. Thus, the sequence $\{\omega_1(x_n,x_{n+1})\}$ is decreasing and bounded below. Therefore it is ω-convergent to some $r \ge 0$. Since

$$\lim_{n\to+\infty} M(x_{2n},x_{2n+1}) = \lim_{n\to+\infty} \max\bigl\{\omega_1(x_{2n},x_{2n+1}),\omega_1(x_{2n+1},x_{2n+2})\bigr\} = r,$$

by letting to limit superior in inequality (2), we obtain

$$F(r) \le \limsup_n \psi\bigl(F\bigl(M(x_{2n},x_{2n+1})\bigr)\bigr) \le \psi\bigl(F(r)\bigr),$$

which implies that $r = 0$. Thus, $\lim_{n\to+\infty} \omega_1(x_n,x_{n+1}) = 0$.

Let us prove that the sequence $\{x_n\}$ is ω-Cauchy. For this, it is sufficient to show that the subsequence $\{x_{2n}\}$ is ω-Cauchy. Assume the contrary. Then, according to Lemma 2.2, there exists $\varepsilon > 0$ such that we can find two subsequences $\{m_k\}$ and $\{n_k\}$ of positive integers satisfying $n_k > m_k \ge k$ such that the following inequalities hold:

$$\omega_8(x_{2n_k},x_{2m_k}) \ge \varepsilon \quad \text{and} \quad \omega_{\frac{1}{4}}(x_{2n_k-1},x_{2m_k}) < \varepsilon.$$

If we take $x = x_{2n_k}$ and $y = x_{2m_k-1}$, then $y \in [x]_G$ and inequality (1) becomes

$$\psi\bigl(F\bigl(\omega_1(x_{2n_k+1},x_{2m_k})\bigr)\bigr) \le F\bigl(M(x_{2n_k},x_{2m_k-1})\bigr),$$

where

$$M(x_{2n_k},x_{2m_k-1}) = \max\biggl\{\omega_1(x_{2n_k},x_{2m_k-1}),\omega_1(x_{2n_k},x_{2n_k+1}),\omega_1(x_{2m_k-1},x_{2m_k}), \frac{\omega_2(x_{2n_k},x_{2m_k}) + \omega_2(x_{2m_k-1},x_{2n_k+1})}{2}\biggr\}.$$

Since

$$\varepsilon \le \omega_8(x_{2n_k},x_{2m_k}) \le \omega_2(x_{2n_k},x_{2m_k})$$

$$\begin{aligned}
&\le \omega_1(x_{2n_k}, x_{2m_k}) \\
&\le \omega_{\frac{1}{2}}(x_{2n_k-1}, x_{2m_k}) + \omega_{\frac{1}{2}}(x_{2n_k-1}, x_{2n_k}) \\
&\le \omega_{\frac{1}{4}}(x_{2n_k-1}, x_{2m_k}) + \omega_{\frac{1}{2}}(x_{2n_k-1}, x_{2n_k}) \\
&\le \varepsilon + \omega_{\frac{1}{2}}(x_{2n_k-1}, x_{2n_k}),
\end{aligned}$$

it follows that $\lim_{k\to+\infty} \omega_2(x_{2n_k}, x_{2m_k}) = \lim_{k\to+\infty} \omega_1(x_{2n_k}, x_{2m_k}) = \varepsilon$.

From

$$\varepsilon \le \omega_2(x_{2n_k}, x_{2m_k}) \le \omega_1(x_{2n_k}, x_{2n_k+1}) + \omega_1(x_{2n_k+1}, x_{2m_k}),$$

we get

$$\begin{aligned}
\varepsilon - \omega_1(x_{2n_k}, x_{2n_k+1}) &\le \omega_1(x_{2n_k+1}, x_{2m_k}) \\
&\le \omega_{\frac{1}{2}}(x_{2n_k-1}, x_{2m_k}) + \omega_{\frac{1}{4}}(x_{2n_k-1}, x_{2n_k}) \\
&\quad + \omega_{\frac{1}{4}}(x_{2n_k}, x_{2n_k+1}) \\
&\le \varepsilon + \omega_{\frac{1}{4}}(x_{2n_k-1}, x_{2n_k}) + \omega_{\frac{1}{4}}(x_{2n_k}, x_{2n_k+1}).
\end{aligned}$$

Thus

$$\lim_{k\to+\infty} \omega_1(x_{2n_k+1}, x_{2m_k}) = \varepsilon.$$

Similarly, using

$$\varepsilon \le \omega_2(x_{2n_k}, x_{2m_k}) \le \omega_1(x_{2n_k}, x_{2m_k-1}) + \omega_1(x_{2m_k-1}, x_{2m_k}),$$

we get

$$\begin{aligned}
\varepsilon - \omega_1(x_{2m_k-1}, x_{2m_k}) &\le \omega_1(x_{2n_k}, x_{2m_k-1}) \\
&\le \omega_{\frac{1}{2}}(x_{2n_k}, x_{2n_k-1}) + \omega_{\frac{1}{4}}(x_{2n_k-1}, x_{2m_k}) \\
&\quad + \omega_{\frac{1}{4}}(x_{2m_k}, x_{2m_k-1}) \\
&\le \omega_{\frac{1}{2}}(x_{2n_k}, x_{2n_k-1}) + \varepsilon + \omega_{\frac{1}{4}}(x_{2m_k}, x_{2m_k-1}).
\end{aligned}$$

Therefore $\lim_{k\to+\infty} \omega_1(x_{2n_k}, x_{2m_k-1}) = \varepsilon$.

From

$$\begin{aligned}
&\omega_8(x_{2n_k}, x_{2m_k}) - \omega_4(x_{2n_k}, x_{2n_k+1}) - \omega_2(x_{2m_k-1}, x_{2m_k}) \\
&\quad \le \omega_2(x_{2m_k-1}, x_{2n_k+1}) \\
&\quad \le \omega_1(x_{2m_k-1}, x_{2n_k}) + \omega_1(x_{2n_k}, x_{2n_k+1}),
\end{aligned}$$

we get $\lim_{k\to+\infty} \omega_2(x_{2m_k-1}, x_{2n_k+1}) = \varepsilon$. Since

$$\omega_2(x_{2m_k-1}, x_{2n_k+1}) \le \omega_1(x_{2m_k-1}, x_{2n_k+1})$$

$$\begin{aligned}
&\le \omega_{\frac{1}{2}}(x_{2m_k-1}, x_{2m_k}) + \omega_{\frac{1}{4}}(x_{2n_k-1}, x_{2m_k}) + \omega_{\frac{1}{8}}(x_{2n_k-1}, x_{2n_k}) \\
&\quad + \omega_{\frac{1}{8}}(x_{2n_k}, x_{2n_k+1}) \\
&\le \omega_{\frac{1}{2}}(x_{2m_k-1}, x_{2m_k}) + \varepsilon + \omega_{\frac{1}{8}}(x_{2n_k-1}, x_{2n_k}) + \omega_{\frac{1}{8}}(x_{2n_k}, x_{2n_k+1})
\end{aligned}$$

and by letting $k \to +\infty$, we obtain $\lim_{k\to+\infty} \omega_1(x_{2m_k-1}, x_{2n_k+1}) = \varepsilon$. Therefore

$$\lim_{k\to+\infty} M(x_{2n_k}, x_{2m_k-1}) = \varepsilon.$$

From the continuity of F and the upper semicontinuity of ψ, we have

$$F(\varepsilon) \le \psi\big(F(\varepsilon)\big),$$

a contradiction since $\epsilon > 0$. Therefore the sequence $\{x_n\}$ is ω-Cauchy. Using the ω-completeness of C, there exists $x^* \in C$ such that $\lim_{n\to+\infty} \omega_1(x_n, x^*) = 0$. The property (OSC) insures that $x^* \in [x_n]$ for all $n \in \mathbb{N}$. Then

$$F\big(\omega_1\big(Sx_{2n}, Tx^*\big)\big) \le \psi\big(F\big(M\big(x_{2n}, x^*\big)\big)\big), \tag{3}$$

where

$$\begin{aligned}
M\big(x_{2n}, x^*\big) = \max\bigg\{&\omega_1\big(x_{2n}, x^*\big), \omega_1(x_{2n}, x_{2n+1}), \omega_1\big(x^*, Tx^*\big), \\
&\frac{\omega_2(x_{2n}, Tx^*) + \omega_2(x^*, x_{2n+1})}{2}\bigg\}.
\end{aligned}$$

Since $\omega_2(x_{2n}, Tx^*) \le \omega_1(x_{2n}, x^*) + \omega_1(x^*, Tx^*)$, $\lim_n M(x_{2n}, x^*) = \omega_1(x^*, Tx^*)$.

Using the continuity of F and the upper continuity of ψ, we obtain

$$\limsup_n \psi\big(F\big(M\big(x_{2n}, x^*\big)\big)\big) \le \psi(F\big(\omega_1\big(x^*, Tx^*\big)\big).$$

By the Fatou property, we have

$$\omega_1\big(x^*, Tx^*\big) \le \liminf_n \omega_1\big(Sx_{2n}, Tx^*\big).$$

Since F is continuous and nondecreasing on $[0, +\infty[$, we have

$$\begin{aligned}
F(\omega_1\big(x^*, Tx^*\big) &\le F\Big(\liminf_n \omega_1\big(Sx_{2n}, Tx^*\big)\Big) \\
&\le F\Big(\liminf_n \omega_1\big(Sx_{2n}, Tx^*\big)\Big) \\
&\le \limsup_n F\big(\omega_1\big(Sx_{2n}, Tx^*\big)\big) \\
&\le \limsup_n \psi\big(F\big(M\big(x_{2n}, x^*\big)\big)\big) \\
&\le \psi(F\big(\omega_1\big(x^*, Tx^*\big)\big),
\end{aligned}$$

which implies that $\omega_1(x^*, Tx^*) = 0$, and according to the regularity of ω, we have $Tx^* = x^*$. Since $x^* \in [x^*]_G$, $F(\omega_1(Sx^*, Tx^*)) \leq \psi(F(M(x^*, x^*)))$ where

$$M(x^*, x^*) = \max\{\omega_1(x^*, Sx^*), \omega_2(x^*, Sx^*)\} = \omega_1(x^*, Sx^*),$$

which implies that $F(\omega_1(Sx^*, x^*)) \leq \psi(F(\omega_1(Sx^*, x^*)))$. Hence $\omega_1(Sx^*, x^*) = 0$ and the regularity of ω insures that $Sx^* = x^*$. □

The next example illustrates Theorem 2.1 and shows that the class of mappings satisfying our main result is a proper nonempty subset of the set of the mappings considered in [13].

Example 2.3 Consider the modular metric space (X, ω) where

$$X = [0,1] \quad \text{and} \quad \omega_\lambda(x,y) = \frac{|x-y|^2}{2\lambda} \quad \text{for all } \lambda \in]0,+\infty[\text{ and } x,y \in X.$$

Consider the reflexive digraph $G = (X, E)$ represented in Fig. 1, where

$$E = \Delta \cup \left\{ \left(\frac{1}{3^n}, 0\right), \left(\frac{1}{3^n}, \frac{1}{3^{n+1}}\right) : n \in \mathbb{N} \right\}.$$

Consider the two self-mapping S and T defined on X by

$$Tx = \frac{x}{3} \quad \text{and} \quad Sx = \frac{x}{9} \quad \text{for all } x \in X,$$

and the two functions F and ψ defined on $[0,+\infty[$ by

$$F(t) = \sqrt{t} \quad \text{and} \quad \psi(t) = \frac{t}{\sqrt{2}} \quad \text{for all } t \in [0,+\infty[.$$

We can see that

1. X is ω-complete;

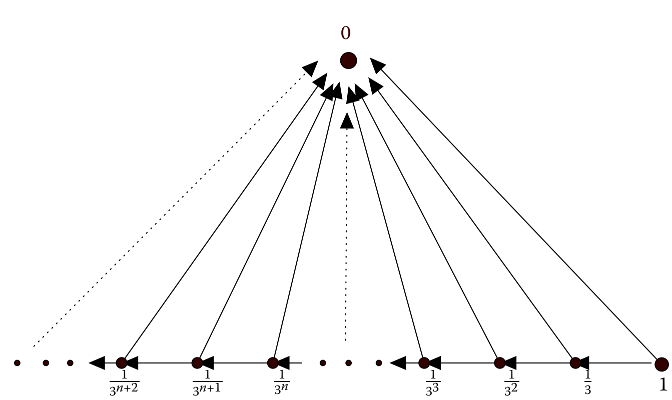

Figure 1 The digraph G (the loops and the isolated vertices are not represented)

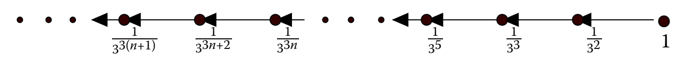

Figure 2 The digraph $G[\mathcal{O}_1(S,T)]$ (the loops are not represented)

2. ω satisfies the Δ_2-type condition and the Fatou property;
3. $G[\mathcal{O}_1(S,T)]$ is a directed path with a unique starting point x_0 (see Figure 2).

Let us show that, for all $x, y \in C$,

$$\big(y \in [x]_G \text{ or } x \in [y]_G\big) \quad \Longrightarrow \quad F\big(\omega_1(Sx,Ty)\big) \leq \psi\big(F\big(M(x,y)\big)\big).$$

For this, we proceed by disjunction of the cases:

- The case where $x = y = 0$ is avoided.
- If $x = \frac{1}{3^n}$ for $n \in \mathbb{N}$ and $y = 0$, then

$$F\big(\omega_1(Sx,Ty)\big) = \frac{1}{\sqrt{2.3^{n+2}}} \leq \frac{1}{2.3^n} = \psi\big(F\big(M(x,y)\big)\big).$$

- If $x = 0$ and $y = \frac{1}{3^n}$ for $n \in \mathbb{N}$, then

$$F\big(\omega_1(Sx,Ty)\big) = \frac{1}{\sqrt{2.3^{n+1}}} \leq \frac{1}{2.3^n} \leq \psi\big(F\big(M(x,y)\big)\big).$$

- If $x = y = \frac{1}{3^n}$ for $n \in \mathbb{N}$, then

$$F\big(\omega_1(Sx,Ty)\big) = \frac{\sqrt{2}}{3^{n+2}} \leq \frac{4}{3^{n+2}} \leq \psi\big(F\big(M(x,y)\big)\big).$$

- If $x = \frac{1}{3^n}$ and $y = \frac{1}{3^m}$ for $m, n \in \mathbb{N}$ such that $m > n$, then

$$F\big(\omega_1(Sx,Ty)\big) = \frac{1}{\sqrt{2}}\left(\frac{1}{3^{n+2}} - \frac{1}{3^{m+1}}\right) \leq \frac{4}{3^{n+2}} \leq \psi\big(F\big(M(x,y)\big)\big).$$

- If $x = \frac{1}{3^m}$ and $y = \frac{1}{3^n}$ for $m, n \in \mathbb{N}$ such that $m > n$, then

$$F\big(\omega_1(Sx,Ty)\big) = \frac{1}{\sqrt{2}}\left(\frac{1}{3^{m+2}} - \frac{1}{3^{n+1}}\right) \leq \frac{\sqrt{2}}{3^{n+1}} \leq \psi\big(F\big(M(x,y)\big)\big).$$

All assumptions of Theorem 2.1 are satisfied and S and T have a fixed point $x^* = 0$.

Remark 2.4 In Example 2.3, if we consider the function $\psi(t) = 0.8 \times \ln(1+t)$ for all $t \in [0, +\infty[$, we get

$$F\big(d(Sx,Ty)\big) = \frac{1}{2} > 0.8 \ln\left(1 + \frac{\sqrt{3}}{2}\right) = \psi\big(F\big(M'(x,y)\big) \quad \text{for } x = 0 \text{ and } y = \frac{3}{4},$$

where $d(x,y) = |x - y|$ and

$$M'(x,y) = \max\left\{d(x,y), d(Tx,x), d(Sy,y), \frac{d(Tx,y) + d(Sy,x)}{2}\right\}.$$

Theorem on page 2 is not applicable, but by Theorem 2.1, we obtain the existence of a common fixed point of S and T. Indeed, we have, for all $x, y \in X$,

$$\left(y \in [x]_G \text{ or } x \in [y]_G\right) \quad \Longrightarrow \quad F\left(\omega_1(Sx,Ty)\right) \leq \psi\left(F\left(M(x,y)\right)\right).$$

Corollary 2.2 *Let (X,ω,G) be a modular metric space endowed with a reflexive digraph G where ω satisfies the Δ_2-type condition and the Fatou property. Let C be an ω-complete nonempty subset of X_ω and $T, S : C \to C$ be two self-mappings. If the following conditions are satisfied*:

(i) *there exists $k \in [0,1[$ such that, for all $x, y \in C$,*

$$\left(y \in [x]_G \text{ or } x \in [y]_G\right) \quad \Longrightarrow \quad \omega_1(Sx,Ty) \leq \left(1 + \omega_1(x,y)\right)^k - 1; \tag{4}$$

(ii) *there exists an element $x_0 \in C$ such that $G[\mathcal{O}_{x_0}(S,T)]$ is a directed path with a unique starting point x_0*;

(iii) *ω satisfies the property (OSC)*,

then S and T have a common fixed point in C.

Proof If we consider the two functions F and ψ defined on $[0,+\infty[$ by

$$F(t) = \ln(1+t) \quad \text{and} \quad \psi(t) = kt,$$

then we can verify that the second part of implication (4) is equivalent to

$$F\left(\omega_1(Sx,Ty)\right) \leq \psi\left(F\left(\omega_1(x,y)\right)\right),$$

which implies that $F(\omega_1(Sx,Ty)) \leq \psi(F(M(x,y)))$, since F and ψ are nondecreasing on $[0,+\infty[$. By applying Theorem 2.1, we terminate the demonstration. □

In the sequel, we use the following lemma.

Lemma 2.5 ([5]) *Let (X,ω) be a modular space such that ω is convex and satisfies the Δ_2-condition. If $\{x_n\}$ is a sequence in X_ω such that $\lim_{n\to+\infty} \omega_1(x_n,x_{n+1}) = 0$, then $\{x_n\}$ is ω-Cauchy.*

Theorem 2.3 *Let (X,ω,G) be a modular metric space endowed with a reflexive digraph G where ω is convex and satisfies the Δ_2-type condition and the Fatou property. Let C be an ω-complete nonempty subset of X_ω and $T, S : C \to C$ be two self-mappings. If the following conditions are satisfied*:

(i) *for all $x, y \in C$,*

$$\left(y \in [x]_G \text{ or } x \in [y]_G\right) \quad \Longrightarrow \quad F\left(\omega_1(Sx,Ty)\right) \leq \psi\left(F\left(M(x,y)\right)\right), \tag{5}$$

where

$$M(x,y) = \max\{\omega_1(x,y), \omega_1(x,Sx), \omega_1(y,Ty), \omega_2(x,Ty) + \omega_2(y,Sx)\};$$

(ii) *there exists an element* $x_0 \in C$ *such that* $G[\mathcal{O}_{x_0}(S,T)]$ *is a directed path with a unique starting point* x_0;

(iii) ω *satisfies the property* (OSC),

then S *and* T *have a common fixed point in* C *and* $\mathfrak{F}(S,T) = \mathfrak{F}(S) = \mathfrak{F}(T)$, *where* $\mathfrak{F}(T)$ *is the set of fixed points of* T.

Proof Let x_0 an element of C such that $G[\mathcal{O}_{x_0}(S,T)]$ is a directed path. Consider the sequence $\{x_n\}$ defined by

$$x_{2n+1} = Sx_{2n} \quad \text{and} \quad x_{2n+2} = Tx_{2n+1} \quad \text{for all } n \in \mathbb{N}.$$

Condition (ii) insures that $\{x_n\}$ is G-nondecreasing. If there exists an integer n such that

$$x_{2n} = x_{2n+1} = x_{2n+2},$$

then x_{2n} is a common fixed point of S and T. Otherwise, suppose that

$$x_{2n} \neq x_{2n+1} \quad \text{or} \quad x_{2n} \neq x_{2n+2} \quad \text{for all } n \in \mathbb{N}.$$

Let $n \in \mathbb{N}$. From $x_{2n+1} \in [x_{2n}]_G$ and applying (5) for $x = x_{2n}$ and $y = x_{2n+1}$, we obtain

$$F\big(\omega_1(x_{2n+1}, x_{2n+2})\big) \le \psi\big(F\big(M(x_{2n}, x_{2n+1})\big)\big). \tag{6}$$

From

$$M(x_{2n}, x_{2n+1}) = \max\{\omega_1(x_{2n}, x_{2n+1}), \omega_1(x_{2n+1}, x_{2n+2}), \omega_2(x_{2n}, x_{2n+2})\},$$

since ω is convex,

$$\omega_2(x_{2n}, x_{2n+2}) \le \frac{\omega_1(x_{2n}, x_{2n+1}) + \omega_1(x_{2n+1}, x_{2n+2})}{2},$$

from which it follows that

$$M(x_{2n}, x_{2n+1}) = \max\{\omega_1(x_{2n}, x_{2n+1}), \omega_1(x_{2n+1}, x_{2n+2})\}.$$

By the same arguments as in the proof of Theorem 2.1, we prove that

$$\lim_{n\to+\infty} \omega_1(x_n, x_{n+1}) = 0.$$

According to Lemma 2.5, the sequence $\{x_n\}$ is ω-Cauchy, and since C is ω-complete, then $\{x_n\}$ is ω-convergent to an element $x^* \in C$. Again similar to the proof of Theorem 2.1, we prove that x^* is a common fixed point of S and T. □

3 Application

Consider the space $X = \mathcal{C}^1([0,1],\mathbb{R})$. Let $G = (X,E)$ be the digraph such that, for all $x,y \in X$,

$$(x,y) \in E \quad \Longleftrightarrow \quad x(t) \le y(t) \quad \text{for each } t \in [0,1].$$

Consider the function $\omega :]0,+\infty[\times X \times X \longrightarrow [0,+\infty]$ defined, for each $\lambda \in]0,+\infty[$ and $x,y \in X$, by

$$\omega(\lambda,x,y) = \omega_\lambda(x,y) = \frac{1}{\lambda}\|x-y\|_\infty^2 = \frac{1}{\lambda}\Big(\sup_{t\in[0,1]}\big|x(t)-y(t)\big|\Big)^2.$$

It is easy to check the following result.

Lemma 3.1 *The function ω is a modular metric satisfying the following*:

(i) *ω satisfies the Δ_2-type condition and the Fatou property*;

(ii) *$X_\omega = X$ is ω-complete*;

(iii) *ω satisfies the (OSC) property.*

Let us consider the following **integral equations system:**

$$(IES): \quad \begin{cases} x(t) = \int_0^1 f(t,y(s))\,\mathrm{d}s + a(t) & \forall t \in [0,1], \\ y(t) = \int_0^1 g(t,x(s))\,\mathrm{d}s + a(t) & \forall t \in [0,1], \end{cases}$$

where $a \in X$ and $f,g : [0,1] \times \mathbb{R} \to \mathbb{R}$ are two mappings such that f and g are of the class C^1 on $[0,1] \times \mathbb{R}$.

Let us consider the two mappings T and S defined in X as follows:

$$\begin{cases} Tx(t) = \int_0^1 f(t,x(s))\,\mathrm{d}s + a(t), \\ Sx(t) = \int_0^1 g(t,x(s))\,\mathrm{d}s + a(t), \end{cases} \quad t \in [0,1].$$

One can see that Tx and Sx are in X for all $x \in X$.

Theorem 3.2 *If the following two conditions are satisfied*:

(i) *for every $s,t \in [0,1]$ and for all comparable elements $x,y \in X$,*

$$\big|f\big(t,x(s)\big) - g\big(t,y(s)\big)\big| \le -1 + \sqrt{1 + \big|x(s)-y(s)\big|},$$

(ii) *there exists $x_0 \in X$ such that, for all $t \in [0,1]$, we have*

$$x_0(t) \preceq Sx_0(t) \preceq TSx_0(t) \preceq STSx_0(t) \preceq (TS)^2x_0(t) \preceq S(TS)^2x_0(t) \preceq \cdots,$$

then the system (IES) admits at least a solution which belongs to the diagonal of X^2.

Proof Let x and y be two comparable elements in X, that is, $x \in [y]_G$ or $y \in [x]_G$. Since, for each $t,s \in [0,1]$,

$$\big|f\big(t,x(s)\big) - g\big(t,y(s)\big)\big| \le -1 + \sqrt{1 + \big|x(s)-y(s)\big|} \le -1 + \sqrt{1 + \big\|x(s)-y(s)\big\|_\infty}$$

and

$$\|Tx - Sy\|_\infty = \sup_{t\in[0,1]} \left|Tx(t) - Sy(t)\right| = \sup_{t\in[0,1]} \int_0^1 \left|f\left(t, x(s)\right) - g\left(t, y(s)\right)\right| \mathrm{d}s,$$

we have

$$\|Tx - Sy\|_\infty \le -1 + \sqrt{1 + \left\|x(s) - y(s)\right\|_\infty}.$$

Since

$$\left(-1 + \sqrt{1 + \left\|x(s) - y(s)\right\|_\infty}\right)^2 \le -1 + \sqrt{1 + \left\|x(s) - y(s)\right\|_\infty^2},$$

we have

$$\omega_1(Tx, Sy) \le -1 + \left(1 + \omega_1(x,y)\right)^{\frac{1}{2}}.$$

Since, for all $t \in [0,1]$,

$$x_0(t) \preceq Sx_0(t) \preceq TSx_0(t) \preceq STSx_0(t) \preceq (TS)^2 x_0(t) \preceq S(TS)^2 x_0(t) \preceq \cdots,$$

the induced subgraph $G[\mathcal{O}_{x_0}(S,T)]$ is a directed path with the unique starting point x_0.

According to Corollary 2.2, T and S have a common fixed point in X, i.e., there exists an element $x^* \in X$ such that (x^*, x^*) verifies the system (IES). Then the system (IES) admits at least a solution in X^2 which belongs to $\Delta(X \times X) = \{(u,u)/u \in X\}$ the diagonal of X^2. □

Conclusion *Our results improve, extend, and generalize some classical results*:

(i) *In Theorem* 2.3, *if we take* $\omega_\lambda(x,y) = \frac{d(x,y)}{\lambda}$ *for all* $\lambda \in]0, +\infty[$, *we get an improved version of the main result of Zhang* [13, *Theorem* 1] *by removing condition* (*iii*) *verified by the function* ϕ *and the monotony of* ϕ.

(ii) *In Theorem* 2.1, *if the function* F *is the identity and the function* ψ *is nondecreasing, we obtain an analogue of* [4, *Theorem* 2] *but for a common fixed point in the setting of modular metric spaces with graph*.

(iii) *Theorem* 2.3 *generalizes and extends* [3, *Theorem* 2.1] *in the setting of a modular metric space with graph*.

(iv) *Corollary* 2.2 *generalizes and extends* [1, *Theorem* 3.1] *in the setting of modular metric spaces with graph, since*

$$\omega_1(Sx, Ty) \le k\omega_1(x,y) \quad \Longrightarrow \quad \omega_1(Sx, Ty) \le \left(1 + \omega_1(x,y)\right)^k - 1.$$

Acknowledgements
The authors would like to thank the reviewers for their constructive comments and suggestions.

Funding
We have no funding for this article.

Availability of data and materials
Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions

The authors declare that this work was realized in collaboration with the same responsibility. All authors read and approved the manuscript.

Author details

[1]Laboratory of Mathematics and Applications, Faculty of Sciences and Technologies, Mohammedia, University Hassan II of Casablanca, Casablanca, Morocco. [2]Laboratory of Algebra, Analysis and Applications, Faculty of Sciences Ben M'sik, University Hassan II of Casablanca, Casablanca, Morocco.

Publisher's Note

Received: 25 September 2019 Accepted: 15 January 2021 Published online: 08 February 2021

References

1. Abdou, A.A.N., Khamsi, M.A.: Fixed point results of pointwise contractions in modular metric spaces. Fixed Point Theory Appl. **2013**(1), 163 (2013)
2. Alber, Y.A.I., Guerre-Delabriere, S.: Principle of weakly contractive maps in Hilbert spaces. Oper. Theory, Adv. Appl. **98**, 7–22 (1997)
3. Bishta, R.K., Pant, R.P.: Contractive definitions and discontinuity at fixed point. Appl. Gen. Topol. **18**(1), 173–182 (2017)
4. Boyd, D.W., Wong, J.S.W.: On nonlinear contractions. Proc. Am. Math. Soc. **20**(2), 458–464 (1969)
5. Chaira, K., Eladraoui, A., Kabil, M.: Some fixed point theorems of generalized $(\psi_1, \psi_2, \varphi, \phi)$-contraction in partially ordered modular metric spaces. Int. J. Pure Appl. Math. **119**(4), 573–592 (2018)
6. Chaira, K., Eladraoui, A., Kabil, M., Lazaiz, S.: Extension of Kirk–Saliga fixed point theorem in a metric space with a reflexive digraph. Int. J. Math. Math. Sci. **2018**, 1471256 (2018)
7. Chistyakov, V.V.: Metric Modular Spaces: Theory and Applications. Springer, Cham (2015). ISBN 978-3-319-25283-4
8. Khan, L.: Fixed point theorem for weakly contractive maps in metrically convex spaces under C-class function. Nonlinear Funct. Anal. Appl. **25**(1), 153–160 (2020)
9. Rhoades, B.E.: Some theorems on weakly contractive maps. Nonlinear Anal. **47**, 2683–2693 (2001)
10. Suantai, S., Chaipornjareansri, S.: Best proximity points of $\alpha - \beta - \psi$-proximal contractive mappings in complete metric spaces endowed with graphs. Nonlinear Funct. Anal. Appl. **24**(4), 759–773 (2019)
11. Turkoglu, D., Manav, N.: Fixed point theorems in a new type of modular metric spaces. Fixed Point Theory Appl. **2018**, 25 (2018)
12. Xue, Z.: The convergence theorems of fixed points for generalized φ-weakly contractive mappings. Nonlinear Funct. Anal. Appl. **22**(4), 803–809 (2017)
13. Zhang, X.: Common fixed point theorems for some new generalized contractive type mappings. J. Math. Anal. Appl. **333**, 780–786 (2007)