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Common fixed points of monotone ρ-nonexpansive semigroup in modular spaces

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Abstract

In this paper, we consider the class of monotone ρ-nonexpansive semigroups and give existence and convergence results for common fixed points. First, we prove that the set of common fixed points is nonempty in uniformly convex modular spaces and modular spaces. Then we introduce an iteration algorithm to approximate a common fixed point for the same class of semigroups.

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1 Introduction

We prove the existence and convergence to a common fixed point of monotone ρ-nonexpansive semigroups in modular spaces. Recall that a family $\mathcal{S} = \{T_t : t \geq 0\}$ is called a semigroup on a subset C of a modular space X_ρ if

(i) $T_0(x) = x$ for all $x \in C$.
(ii) $T_{s+t} = T_s \circ T_t$ for all positive s, t.

The theory of semigroups is very interesting in mathematics and applications. As a situation, in the theory of dynamical systems the space X_ρ on which the semigroup $\mathcal{S}$ is defined represents the states space, and the mapping

$$\mathbb{R}_+ \times C \longrightarrow C,$$
$$(t,s) \longmapsto T_t(x)$$

represents the evolution function of the dynamical system (see [9, 11]).

The problem of the existence of common fixed points for semigroup is still in its infancy. Kozlowski [9] has demonstrated the existence of common fixed points for semigroups of monotone contractions and monotone ρ-nonexpansive mappings in Banach spaces. Afterward, Bashar et al. [3] generalized Kozlowski's work in Banach spaces. In the case of monotone nonexpansive semigroups, they proved the following theorem.

Theorem 1.1 ([3]) *Let $(X, \|\cdot\|)$ be a Banach space uniformly convex in every direction. Let C be a weakly compact convex nonempty subset of X, and let $\mathcal{S} = \{T_t : t \geq 0\}$ be a monotone nonexpansive semigroup defined on C. Assume that there exists $x_0 \in C$ such that $x_0 \leq T_t(x_0)$ (resp., $T_t(x_0) \leq x_0$) for all $t \geq 0$. Then there exists a common fixed point $z \in Fix(\mathcal{S})$ such that $x_0 \leq z$ (resp., $z \geq x_0$).*

Under the frame of modular function spaces, Kozlowski [7] has shown that the set of common fixed points of any ρ-nonexpansive semigroups, acting on a ρ-closed convex and ρ-bounded subset of a uniformly convex modular function space L_ρ, is nonempty ρ-closed and convex (see Theorem 6.5 in [8]).

For finding a common fixed point of a nonexpansive mapping, Halpern [5] has introduced in Hilbert spaces H the following explicit iteration scheme for elements $u \in H$ and $x_0 \in H$:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)T(x_n) \quad \text{for all } n \geq 0, \tag{1}$$

where $(\alpha_n)_n$ is a sequence in $(0, 1)$. Subsequently, many mathematicians paid their attention to studying the convergence of Halpern iteration for semigroups of various nonlinear mappings in different spaces and under different conditions.

2 Methods

In 2002, Xu [15] showed, under certain assumptions on semigroups, the strong convergence of the modified Halpern iteration given by

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)T_{t_n}(x_n) \tag{2}$$

for all $t_n > 0$ (see also Wangkeeree et al. [14] for the asymptotically nonexpansive semigroups, Yao et al. [16] for the nonexpansive semigroups, and Song et al. [12] for ρ-nonexpansive semigroups in Hilbert spaces).

Motivated by the results cited, we begin by generalizing Theorem 1.1 for the monotone ρ-nonexpansive semigroups in uniformly convex and uniformly convex in every direction modular spaces. Next, we define a new iteration algorithm for monotone ρ-nonexpansive semigroups as follows:

$$\begin{cases} x_{n+1} = (1 - \alpha_n)T_{t_n}(x_n) + \alpha_n T_{t_n}(y_n), \\ y_n = (1 - \beta_n)x_n + \beta_n T_{t_n}(x_n), \end{cases}$$

where the sequences $(t_n)_n \subset \mathbb{R}_+$ and $(\alpha_n)_n, (\beta_n)_n \subset (0, 1)$ satisfy some conditions. This process generalizes the work of [2]. Later, we show under some assumptions that the sequence $(x_n)_n$ ρ-converges to a common fixed point of a monotone ρ-nonexpansive semigroup.

3 Results and discussion

Throughout this work, X stands for a real vector space.

Definition 3.1 ([1]) A function $\rho : X \longrightarrow [0, +\infty]$ is called a modular if the following holds:

(1) $\rho(x) = 0$ if and only if $x = 0$;
(2) $\rho(-x) = \rho(x)$;
(3) $\rho(\alpha x + (1-\alpha)y) \leq \rho(x) + \rho(y)$ for all $\alpha \in [0,1]$ and $x, y \in X$.

If (3) is replaced by

$$\rho\big(\alpha x + (1-\alpha)y\big) \leq \alpha\rho(x) + (1-\alpha)\rho(y)$$

for all $\alpha \in [0,1]$ and $x, y \in X$, then ρ is called a convex modular.

A modular ρ defines the corresponding modular space, that is, the vector space

$$X_\rho = \Big\{x \in X : \lim_{\lambda \to 0} \rho(\lambda x) = 0\Big\}.$$

Let ρ be a convex modular. Then the modular space X_ρ is equipped with a norm called *the Luxemburg norm*, defined by

$$\|x\|_\rho = \inf\left\{\lambda > 0 : \rho\left(\frac{x}{\lambda}\right) \leq 1\right\}.$$

We now give the basic definitions.

Definition 3.2 ([1]) Let ρ be a modular defined on a vector space X.

(1) We say that a sequence $(x_n)_{n\in\mathbb{N}} \subset X_\rho$ is ρ-convergent to $x \in X_\rho$ if and only if $\rho(x_n - x)$ converges to 0 as n goes to infinity. Note that the limit is unique.
(2) A sequence $(x_n)_n \subset X_\rho$ is called ρ-Cauchy if $\rho(x_n - x_m) \longrightarrow 0$ as $n, m \longrightarrow +\infty$.
(3) We say that X_ρ is ρ-complete if any ρ-Cauchy sequence is ρ-convergent.
(4) A subset C of X_ρ is said ρ-closed if the ρ-limit of a ρ-convergent sequence of C always belongs to C.
(5) A subset C of X_ρ is said to be ρ-bounded if

$$\delta_\rho(C) = \sup\big\{\rho(x-y) : x, y \in C\big\} < \infty.$$

(6) A subset K of X_ρ is said to be ρ-compact if any sequence $(x_n)_n$ of C has a subsequence that ρ-converges to a point $x \in C$.
(7) We say that ρ satisfies the Fatou property if

$$\rho(x-y) \leq \lim_{n\to+\infty} \rho(x - y_n)$$

for any x whenever $(y_n)_n$ ρ-converges to y in X_ρ.

Note that the ρ-convergence does not imply the ρ-Cauchy condition. Also, $x_n \xrightarrow{\rho} x$ does not imply in general that $\lambda x_n \xrightarrow{\rho} \lambda x$ for every $\lambda > 1$.

An important property associated with a modular, which plays a powerful role in modular spaces, is the Δ_2-condition and the Δ_2-type condition.

Definition 3.3 ([1, 10]) Let ρ be a modular defined on a vector space X. We say that ρ satisfies

(i) the Δ_2-condition if $\rho(2x_n) \longrightarrow 0$ whenever $\rho(x_n) \longrightarrow 0$ as $n \to +\infty$;
(ii) the Δ_2-type condition, if there exists $K > 0$ such that $\rho(2x) \leq K\rho(x)$.

Definition 3.4 Let ρ be a modular, and let C be a nonempty subset of the modular space X_ρ. A mapping $T : C \longrightarrow C$ is said to be

(a) monotone if $T(x) \leq T(y)$ for all $x, y \in C$ such that $x \leq y$;
(b) monotone ρ-nonexpansive if T is monotone such that

$$\rho\bigl(T(x) - T(y)\bigr) \leq \rho(x - y)$$

for all $x, y \in X_\rho$ such that $x \leq y$.

Recall that $T : C \longrightarrow C$ is said to be ρ-continuous if $(T(x_n))_n$ ρ-converges to $T(x)$ whenever $(x_n)_n$ ρ-converges to x. It is not true that a monotone ρ-nonexpansive mapping is ρ-continuous since this result is not true in general when ρ is a norm.

We further assume that ρ is a convex modular.

Definition 3.5 ([1]) Let ρ be a modular, and let $r > 0$ and $\varepsilon > 0$. Define, for $i \in \{1, 2\}$,

$$D_i(r,\varepsilon) = \left\{(x,y) \in X_\rho \times X_\rho : \rho(x) \leq r, \rho(y) \leq r, \rho\left(\frac{x-y}{i}\right) \geq r\varepsilon\right\}.$$

If $D_i(r,\varepsilon) \neq \emptyset$, then let

$$\delta_i(r,\varepsilon) = \inf\left\{1 - \frac{1}{r}\rho\left(\frac{x+y}{2}\right) : (x,y) \in D_i\right\}.$$

If $D_i(r,\varepsilon) = \emptyset$, then we set $\delta_i(r,\varepsilon) = 1$.

(i) We say that ρ satisfies uniform convexity (UCi) if for all $r > 0$ and $\varepsilon > 0$, we have $\delta_i(r,\varepsilon) > 0$.
(ii) We say that ρ satisfies unique uniform convexity (UUCi) if for all $s \geq 0$ and $\varepsilon > 0$, there exists $\eta(s,\varepsilon) > 0$ such that

$$\delta_i(r,\varepsilon) > \eta(s,\varepsilon) \quad \text{for } r > s.$$

(iii) We say that ρ is strictly convex (SC) if for all $x, y \in X_\rho$ such that $\rho(x) = \rho(y)$ and

$$\rho\left(\frac{x+y}{2}\right) = \frac{\rho(x) + \rho(y)}{2},$$

we have $x = y$.

The following proposition characterizes the relationship between the above notions.

Proposition 3.6 ([1])
(a) *(UUCi) implies (UCi) for $i = 1, 2$;*
(b) *$\delta_1(r,\varepsilon) \leq \delta_2(r,\varepsilon)$ for $r > 0$ and $\varepsilon > 0$;*
(c) *(UC1) implies (UC2);*
(d) *(UC2) implies (SC);*

(e) $(UUC1)$ *implies* $(UUC2)$.

In the following definition, we introduce the uniform convexity in every direction (UCED) of a modular.

Definition 3.7 Let ρ be a modular. We say that ρ is uniformly convex in every direction (UCED) if for any $r > 0$ and nonzero $z \in X_\rho$, we have

$$\delta(r,z) = \inf\left\{1 - \frac{1}{r}\rho\left(x + \frac{z}{2}\right) : \rho(x) \le r, \rho(x+z) \le r\right\} > 0.$$

We say that ρ satisfies unique uniform convexity in every direction (UUCED) if there exists $\eta(s,z) > 0$ for $s \ge 0$ and nonzero $z \in X_\rho$ such that

$$\delta(r,z) > \eta(s,z) \quad \text{for } r > s.$$

Proposition 3.8
(a) (UCi) (*resp.*, $(UUCi)$) *implies* $(UCED)$ (*resp.*, $(UUCED)$) *for* $i = 1, 2$;
(b) $(UUCED)$ *implies* $(UCED)$;
(c) $(UCED)$ *implies* (SC).

Proof It is quite easy to show (a) and (b). To prove (c), let $x, y \in X_\rho$ be such that $x \neq y$.

First, if $\rho(x) \neq \rho(y)$, then there is nothing to prove. Otherwise, we assume that $\rho(x) = \rho(y) = r > 0$ and consider $z = y - x (\neq 0)$. Hence $\rho(x+z) = \rho(y) = r$. Since ρ is $(UCED)$, $\delta(r,z) > 0$, which implies

$$1 - \frac{1}{r}\rho\left(x + \frac{z}{2}\right) \ge \delta(r,z) > 0.$$

Thus

$$\rho\left(\frac{x+y}{2}\right) = \rho\left(x + \frac{y-x}{2}\right) \le \bigl(1 - \delta(r,z)\bigr)r < r,$$

that is, $\rho(\frac{x+y}{2}) < r = \frac{\rho(x)+\rho(y)}{2}$ □

The following property plays a similar role as the reflexivity in Banach spaces for modular spaces.

Definition 3.9 ([10]) Let ρ be a modular. We say that the modular space X_ρ satisfies property (R) if for every decreasing sequence $(C_n)_{n\in\mathbb{N}}$ of nonempty ρ-closed convex and ρ-bounded subsets of X_ρ, we have

$$\bigcap_{n\in\mathbb{N}} C_n \neq \emptyset.$$

Lemma 3.10 ([1]) *Let* ρ *be a convex modular satisfying the Fatou property. Assume that* X_ρ *is* ρ*-complete and* ρ *is* $(UUC2)$. *Then* X_ρ *satisfies property* (R).

Proposition 3.11 ([1]) *Let ρ be a convex modular. Assume that X_ρ is ρ-complete and ρ is (UUC2). Let C be a ρ-closed convex and ρ-bounded nonempty subset of X_ρ. Let $(C_i)_{i\in I}$ be a family of ρ-closed convex nonempty subsets of C such that $\bigcap_{i\in F} C_i$ is nonempty for any finite subset F of I. Then*

$$\bigcap_{i\in I} C_i \neq \emptyset.$$

The ρ-type function is a powerful technical tool to prove the existence of a fixed point.

Definition 3.12 ([6]) Let $(x_n)_n$ be a sequence in X_ρ, and let K be a nonempty subset of X_ρ.

The function $\tau : K \longrightarrow [0,\infty]$ defined by

$$\tau(x) = \limsup_{n\to\infty} \rho(x_n - x)$$

is called a ρ-type function.

The next definition is an adaptation of the definition of ρ-type functions to a one-parameter family of mappings.

Definition 3.13 ([6]) Let $C \subset X_\rho$ be convex ρ-bounded. A function $\tau : C \longrightarrow \overline{\mathbb{R}_+}$ is called a ρ-type function (or shortly a type) if there exists a one-parameter family $\{T_t : t \geq 0\}$ of elements of a nonempty subset K of X_ρ such that for all $x \in K$,

$$\tau(y) = \limsup_{t\to\infty} \rho\bigl(T_t(x) - y\bigr)$$

for all $y \in K$.

A sequence $(c_n)_n \subset K$ is called a minimizing sequence of τ if

$$\lim_{n\to+\infty} \tau(c_n) = \inf_{x\in K} \tau(x).$$

Note that the ρ-type function τ is convex since ρ is convex.

Recall the definition of the uniform continuity of a modular.

Definition 3.14 A modular ρ is said to be uniformly continuous if for any $\varepsilon > 0$ and $R > 0$, there exists $\eta > 0$ such that

$$\bigl|\rho(y) - \rho(x+y)\bigr| \leq \varepsilon$$

whenever $\rho(x) \leq \eta$ and $\rho(y) \leq R$.

The following lemma plays an important role in the proof of the next fixed point theorem. To prove it, we use the ideas of the proof of Lemma 3.5 in [3].

Lemma 3.15 *Let ρ be a convex modular uniformly continuous and (UUCED). Assume that the modular space X_ρ satisfies property (R). Let C be a ρ-closed ρ-bounded convex nonempty subset of X_ρ. Let K be a nonempty ρ-closed convex subset of C. Let $(x_k)_{k\in\mathbb{N}}$ be a sequence in C and consider the ρ-type function $\tau : K \longrightarrow [0,+\infty]$ defined by*

$$\tau(y) = \limsup_{k\to+\infty} \rho(x_k - y). \tag{3}$$

Then τ has a unique minimum point in K.

A subset $P \subset X_\rho$ is called a pointed ρ-closed convex cone if P is a nonempty ρ-closed subset of X_ρ satisfying the following properties:

(i) $P + P \subset P$,

(ii) $\lambda P \subset P$ for all $\lambda \in \mathbb{R}_+$,

(iii) $P \cap (-P) = \{0\}$.

Using P, we define an ordering on X_ρ by

$$x \le y \quad \text{if and only if} \quad y - x \in P.$$

We further suppose that the modular space X_ρ is equipped with the partial order defined by P.

3.1 Common fixed point results for a monotone ρ-nonexpansive semigroup

Before we state our main results, let us recall the definition of a monotone ρ-Lipschitz semigroup.

Definition 3.16 Let C be a nonempty subset of a modular space X_ρ. A one-parameter family $\mathcal{S} = \{T_t : t \ge 0\}$ of mappings from C into C is said to be a monotone semigroup on C if it satisfies the following conditions:

(i) $T_0(x) = x$ for all $x \in C$,

(ii) $T_{s+t} = T_s \circ T_t$ for all $s, t \ge 0$,

(iii) T_t is monotone for all $t \ge 0$, that is, $T_t x \le T_t y$ for all $x, y \in C$ such that $x \le y$.

Definition 3.17 A semigroup $\mathcal{S}$ is said to be a monotone ρ-Lipschitz semigroup if $\mathcal{S}$ is monotone and there exists $k \ge 0$ such that

$$\rho\big(T_t(x) - T_t(y)\big) \le k\rho(x - y)$$

for all $x, y \in C$ such that $x \le y$ and all $t \ge 0$.

If $k < 1$, then $\mathcal{S}$ is said to be a monotone ρ-contraction semigroup. If $k = 1$, then $\mathcal{S}$ is said to be a monotone ρ-nonexpansive semigroup. The set of all common fixed points of $\mathcal{S}$ is defined by

$$Fix(\mathcal{S}) = \big\{x \in C : T_t(x) = x \text{ for all } t \ge 0\big\} = \bigcap_{t\ge 0} Fix(T_t).$$

The following lemma generalizes the minimizing sequence property for type functions generated by a sequence to the case of type functions defined by a one-parameter family $\{h_t : t \ge 0\}$. To prove it, we use the ideas of the proof of Lemma 7.11 of [6].

Lemma 3.18 *Let ρ be a convex modular satisfying the Fatou property and $(UUC1)$, and let X_ρ be a ρ-complete modular space. Let C be a nonempty ρ-closed convex subset of X_ρ. Let $\mathcal{S}$ be a monotone ρ-nonexpansive semigroup on C. Fix $x_0 \in C$ and consider the function $\varphi : C \longrightarrow \mathbb{R}_+$ given by*

$$\varphi(y) = \limsup_{t\to+\infty} \rho\bigl(T_t(x_0) - y\bigr) = \inf_{s\geq 0} \sup_{t\geq s} \rho\bigl(T_t(x_0) - y\bigr).$$

Then every minimizing sequence of φ ρ-converges to the same limit.

Theorem 3.19 *Let ρ be a convex modular satisfying the Fatou property and $(UUC1)$. Let C be a nonempty ρ-closed convex ρ-bounded subset of a ρ-complete modular space X_ρ. Let $\mathcal{S} = \{T_t : t \geq 0\}$ be a monotone ρ-nonexpansive semigroup such that T_t is ρ-continuous for any $t \geq 0$. Assume that there exists $x_0 \in C$ such that $x_0 \leq T_t(x_0)$ (resp., $T_t(x_0) \leq x_0$) for all $t \geq 0$. Then there exists a common fixed point $z \in Fix(\mathcal{S})$ such that $x_0 \leq z$ (resp., $z \leq x_0$).*

Proof Without loss of generality, we assume that $x_0 \leq T_t x_0$ for all $t > 0$. By the definition of the partial order and Proposition 3.11 we have that

$$K = \bigcap_{t\geq 0} \bigl[T_t(x_0), \rightarrow\bigr) \cap C$$

is nonempty. In fact, using Proposition 3.11, it suffices to prove that

$$\bigcap_{t\in F} \bigl[T_t(x_0), \rightarrow\bigr) \cap C$$

is nonempty for any finite subset $F = \{t_0, \dots, t_n\}$ of $\mathbb{R}_+$, where t_i are arbitrarily chosen in $\mathbb{R}_+$.

Let $x = T_{t_0+\cdots+t_n}(x_0) \in C$. Since $\mathcal{S}$ is a monotone semigroup and $x_0 \leq T_t(x_0)$ for all $t \geq 0$, we have $T_s(x_0) \leq T_{s+t}(x_0)$ for all $s, t \geq 0$. Hence

$$T_{t_i}(x_0) \leq x$$

for all $i \in \{1, \dots, n\}$, that is, $x \in [T_{t_i}(x_0), \rightarrow) \cap C$. Thus $\bigcap_{t_i\in F} [T_{t_i}(x_0), \rightarrow) \cap C$ is nonempty for all $n \geq 0$. Moreover, K is ρ-closed convex.

Furthermore, K is invariant by $\mathcal{S}$. Indeed, let $x \in K$ and $t, s \geq 0$. If $t \geq s$, then $t - s \geq 0$. Hence $T_{t-s}(x_0) \leq x$ implies $T_t(x_0) \leq T_s(x)$. If $t < s$, then $\varepsilon = s - t > 0$. Since $x_0 \leq x$, we have

$$x_0 \leq T_\varepsilon(x_0) \leq T_\varepsilon(x) \quad\Longrightarrow\quad T_t(x_0) \leq T_{t+\varepsilon}(x) = T_s(x).$$

Thus $T_t(x_0) \leq T_s(x)$ for all $t, s \geq 0$. Then $T_s(x) \in K$ for all $s \geq 0$. Therefore $\mathcal{S}(K) \subset K$.

Consider the function $\varphi : K \longrightarrow [0, +\infty[$ defined by

$$\varphi(y) = \limsup_{t\to+\infty} \rho\bigl(T_t(x_0) - y\bigr) = \inf_{s\geq 0} \sup_{t\geq s} \rho\bigl(T_t(x_0) - y\bigr).$$

Since K is ρ-bounded, $\varphi_0 = \inf_{y\in C} \varphi(y) < \infty$. Thus for any $n \geq 1$, there exists $z_n \in K$ such that

$$\varphi_0 \leq \varphi(z_n) \leq \varphi_0 + \frac{1}{n}.$$

Then $(z_n)_n$ is a minimizing sequence of φ and ρ-converges to $z \in K$ by Lemma 3.18. To prove that $z \in Fix(\mathcal{S})$, it suffices to show that $(T_t(z_n))_n$ is also a minimizing sequence of φ for any $t \geq 0$.

Fix s, $\eta \geq 0$, and let $t \geq s + \eta$ and $y \in K$. Then $T_{t-s}(x_0) \leq y$. As $\mathcal{S}$ is a monotone ρ-nonexpansive semigroup, we have

$$\begin{aligned}\rho\bigl(T_s\bigl(T_{t-s}(x_0)\bigr) - T_s(y)\bigr) &= \rho\bigl(T_t(x_0) - T_s(y)\bigr)\\ &\leq \rho\bigl(T_{t-s}(x_0) - y\bigr)\\ &\leq \sup_{\bar{t}\geq\eta} \rho\bigl(T_{\bar{t}}(x_0) - y\bigr).\end{aligned}$$

Hence

$$\sup_{t\geq\eta} \rho\bigl(T_t(x_0) - T_s(z)\bigr) \leq \sup_{t\geq s+\eta} \rho\bigl(T_t(x_0) - T_s(z)\bigr) \leq \sup_{\bar{t}\geq\eta} \rho\bigl(T_{\bar{t}}(x_0) - z\bigr).$$

Taking the $\inf_{\eta\geq 0}$ in the previous inequality, we get

$$\inf_{\eta\geq 0}\sup_{t\geq\eta} \rho\bigl(T_t(x_0) - T_s(z)\bigr) \leq \inf_{\eta\geq 0}\sup_{\bar{t}\geq\eta} \rho\bigl(T_{\bar{t}}(x_0) - z\bigr),$$

which implies

$$\varphi\bigl(T_s(y)\bigr) \leq \inf_{\eta\geq 0}\sup_{\bar{t}\geq\eta} \rho\bigl(T_{\bar{t}}(x_0) - y\bigr).$$

Since η is arbitrary positive, we have

$$\varphi\bigl(T_s(y)\bigr) \leq \varphi(y)$$

for any $s \geq 0$. Therefore $(T_s(z_n))_n$ is also a minimizing sequence of φ for all $s \geq 0$.

By Lemma 3.18 we get that $(T_s(z_n))_n$ ρ-converges to z for all $s \geq 0$. Since T_s is ρ-continuous for all $s \geq 0$, $(T_s(z_n))_n$ ρ-converges to $T_s(z)$ for all $s \geq 0$. By the uniqueness of the limit we conclude that $z = T_s(z)$ for all $s \geq 0$. Then z is a common fixed point of the semigroup $\mathcal{S}$. □

Example 3.20 Let $(p_n)_{n\geq 1}$ be a sequence of real numbers such that $1 \leq p_n < \infty$ for all $n \geq 1$. Consider the vector space

$$\ell_{(p_n)} = \left\{(x_n)_n \in \mathbb{R}^{\mathbb{N}^*} : \sum_{n=1}^{+\infty} \frac{1}{p_n}|\lambda x_n|^{p_n} < \infty \text{ for some } \lambda > 0\right\},$$

where the modular ρ is given by $\rho((x_n)_n) = \sum_{n=0}^{+\infty} \frac{1}{p_n}|x_n|^{p_n}$ for all $(x_n)_n \in \ell_{(p_n)}$. Suppose that $1 < p^- = \inf_{n\geq 1} p_n \leq p_n \leq \sup_{n\geq 1} p_n = p^+ < \infty$ for all $n \geq 1$.

According to [13], the modular ρ is convex and satisfies (UUC1), and the space $\ell_{(p_n)}$ under the Luxemburg norm $\|\cdot\|_\rho$ endowed by the modular ρ is a Banach space. Moreover, ρ satisfies the Δ_2-type condition. In fact, let $(x_n)_n \in \ell_{(p_n)}$. Since $p_n \leq p^+$, we have

$\sum_{n=1}^{q} \frac{2^{p_n}}{p_n} |x_n|^{p_n} \le \sum_{n=1}^{q} \frac{2^{p^+}}{p_n} |x_n|^{p_n}$ for all $q \in \mathbb{N}^*$. Taking $\lim_{n\to\infty}$, we have

$$\rho(2x) = \sum_{n=1}^{+\infty} \frac{2^{p_n}}{p_n} |x_n|^{p_n} \le 2^{p^+} \sum_{n=1}^{+\infty} \frac{1}{p_n} |x_n|^{p_n} = 2^{p^+} \rho(x).$$

Recall that if ρ satisfies the Δ_2-type condition, then $\|\cdot\|_\rho$ convergence is equivalent to ρ-convergence (see [6]). Thus $\ell_{(p_n)}$ is a ρ-complete modular space. Moreover, ρ satisfies the Fatou property.

Consider the partial ordering $\preceq$ defined by

$$(x_n)_n \preceq (y_n)_n \iff x_n \le y_n, \quad \forall n \ge 1,$$

for all $(x_n)_n$ and $(y_n)_n$ in $\ell_{(p_n)}$.

Let $C = B_\rho(0,r)$ be the ρ-closed ball of $\ell_{(p_n)}$ centered at 0 with radius $r > 1$; it is ρ-bounded. Let the family $\mathcal{S} = \{T_t : t \ge 0\}$ of mappings be given by

$$\begin{aligned} &T_t : C \longrightarrow C, \\ &(x_n)_n \longmapsto T_t\big((x_n)_n\big) = \big(e^{-t}x_1, e^{-2t}x_2, \dots\big) \end{aligned}$$

for all $t \ge 0$. It easy to verify that $\mathcal{S}$ is a monotone ρ-nonexpansive semigroup and T_t is ρ-continuous for all $t \ge 0$. As an example, we consider $p_n = \frac{4n^2}{n^2+1}$ for $n \ge 1$. We have $p^- = 2$ and $p^+ = 4$. Let $x^0 = (x_n^0)_{n\ge1} = (\frac{1}{2^n})_{n\ge1}$. We have $x^0 \in C$ and $T_t(x^0) \preceq x^0$ for all $t \ge 0$. Then by Theorem 3.19 there exists a common fixed point $z = 0$ such that $z \preceq x_0$.

The next lemma is a generalization of Lemma 3.15 for ρ-type functions defined by a given one-parameter family of mappings.

Lemma 3.21 *Let ρ be a convex modular uniformly continuous and (UUCED), and let X_ρ be a modular space satisfying property (R). Let C be a nonempty ρ-closed convex ρ-bounded subset of X_ρ, let $\mathcal{S}$ be a monotone ρ-nonexpansive semigroup on C, and let K be a ρ-closed convex subset of C. Fix $x_0 \in C$ and consider the function $\varphi : C \longrightarrow \mathbb{R}_+$ given by*

$$\varphi(y) = \limsup_{t\to+\infty} \rho\big(T_t(x_0) - y\big) = \inf_{s\ge0} \sup_{t\ge s} \rho\big(T_t(x_0) - y\big).$$

Then there exists a unique $z \in K$ such that $\varphi(z) = \inf_{y\in K} \varphi(y)$.

Proof Fix $x_0 \in C$. Since C is ρ-bounded, $\varphi_0 = \inf_{y\in K} \varphi(y) < \infty$. First, assume that $\varphi_0 > 0$. Let $\varepsilon > 0$. There exists $y \in K$ such that $\varphi(y) \le \varphi_0 + \varepsilon$. Then, for $\varepsilon = \frac{1}{n}$ with $n \ge 1$, there exists $y_n \in K$ such that $\varphi(y_n) \le \varphi_0 + \frac{1}{n}$.

For any $n \ge 1$, set

$$K_n = \left\{ y \in K : \varphi(y) \le \varphi_0 + \frac{1}{n} \right\}.$$

$(K_n)_n$ is a sequence of nonempty ρ-closed convex and ρ-bounded subsets. Indeed, for all $n \ge 1$, K_n is ρ-closed since φ is a ρ-lower semicontinuous function. In fact, let $(y_n)_n$ in K

ρ-converge to $y \in K$. Then

$$\varphi(y) \leq \liminf_{n\to+\infty} \varphi(y_n).$$

Indeed, fix $\varepsilon > 0$ and $R = \operatorname{diam}_\rho(C) > 0$. Using the uniform continuity of ρ, there exists $\eta > 0$ such that

$$\left|\rho(y) - \rho(x+y)\right| \leq \varepsilon \tag{4}$$

whenever $\rho(x) \leq \eta$ and $\rho(y) \leq R$. Since $(y_n)_n$ ρ-converges to y, there exists $n_0 > 0$ such that

$$\rho(y_n - y) \leq \eta$$

for any $n \geq n_0$. Moreover, for $s \geq 0$, let $t \geq s$. As $x_0 \in C$, then $T_t(x_0) \in C$. Thus $\rho(T_t(x_0) - y) \leq R$. Therefore by (4)

$$\left|\rho\big(T_t(x_0) - y\big) - \rho\big(T_t(x_0) - y + y - y_n\big)\right| \leq \varepsilon$$

for any $n \geq n_0$ and $t \geq s$. Hence

$$\left|\rho\big(T_t(x_0) - y\big) - \rho\big(T_t(x_0) - y_n\big)\right| \leq \varepsilon$$

for any $n \geq n_0$ and $t \geq s$. In particular,

$$\rho\big(T_t(x_0) - y\big) \leq \rho\big(T_t(x_0) - y_n\big) + \varepsilon$$

for any $n \geq n_0$ and $t \geq s$. This implies

$$\sup_{t\geq s} \rho\big(T_t(x_0) - y\big) \leq \sup_{t\geq s} \rho\big(T_t(x_0) - y_n\big) + \varepsilon$$

for any $n \geq n_0$. Since $s \geq 0$ is arbitrary, we have

$$\varphi(y) = \inf_{s\geq 0} \sup_{t\geq s} \rho\big(T_t(x_0) - y\big) \leq \inf_{s\geq 0} \sup_{t\geq s} \rho\big(T_t(x_0) - y_n\big) + \varepsilon = \varphi(y_n) + \varepsilon$$

for any $n \geq n_0$. Hence

$$\varphi(y) \leq \liminf_{n\to+\infty} \varphi(y_n) + \varepsilon$$

for any $\varepsilon > 0$. Consequently, $\varphi(y) \leq \liminf_{n\to+\infty} \varphi(y_n)$, that is, φ is ρ-lower semicontinuous. Then K_n is ρ-closed for all $n \geq 1$.

For all $n \geq 1$, K_n is convex since φ is convex. Moreover, K_n is ρ-bounded, and the sequence $(K_n)_n$ is decreasing.

By property (R) the set $K_\infty = \bigcap_{n\geq 1} K_n$ is nonempty ρ-closed convex. Furthermore,

$$K_\infty = \big\{y \in K : \varphi(y) = \varphi_0\big\}.$$

Indeed, if $y \in K_\infty$ then $y \in K_n$ for all $n \geq 1$. Thus $\varphi(y) \leq \varphi_0 + \frac{1}{n}$ for all $n \geq 1$. Hence $\varphi(y) \leq \varphi_0$. Since $\varphi_0 \leq \varphi(y)$, we have $\varphi(y) = \varphi_0$.

Next, we prove that K_∞ is reduced to one point. Let $z_1, z_2 \in K_\infty$ be such that $z_1 \neq z_2$. Set $z = z_1 + z_2$ and let $\varepsilon > 0$. By the definition of φ there exists $s_0 \geq 0$ such that

$$\sup_{t \geq s_0} \rho\bigl(T_t(x_0) - z_i)\bigr) \leq \varphi_0 + \varepsilon, \quad i = 1, 2.$$

Thus

$$\rho\bigl(T_t(x_0) - z_i\bigr) \leq \varphi_0 + \varepsilon, \quad i = 1, 2,$$

for all $t \geq s_0$.

Fixing $t \geq s_0$, we have

$$\rho\bigl(T_t(x_0) - z_1\bigr) \leq \varphi_0 + \varepsilon \quad \text{and} \quad \rho\bigl(T_t(x_0) - z_1 + z_1 - z_2\bigr) = \rho\bigl(T_t(x_0) - z_2\bigr) \leq \varphi_0 + \varepsilon.$$

Since ρ is (UUCED), there exists $\eta(\varphi_0, z) > 0$ such that

$$1 - \frac{1}{\varphi_0 + \varepsilon}\rho\left(T_t(x_0) - z_1 + \frac{z_1 - z_2}{2}\right) \geq \delta(\varphi_0 + \varepsilon, z) \geq \eta(\varphi_0, z).$$

Hence

$$\rho\left(T_t(x_0) - \frac{z_1 + z_2}{2}\right) \leq \bigl(1 - \eta(\varphi_0, z)\bigr)(\varphi_0 + \varepsilon).$$

Since t is arbitrarily fixed such that $t \geq s_0$, we have

$$\sup_{t \geq s_0} \rho\left(T_t(x_0) + \frac{z_1 + z_2}{2}\right) \leq \bigl(1 - \eta(\varphi_0, z)\bigr)(\varphi_0 + \varepsilon).$$

Therefore

$$\varphi\left(\frac{z_1 + z_2}{2}\right) \leq \bigl(1 - \eta(\varphi_0, z)\bigr)(\varphi_0 + \varepsilon).$$

As ε goes to 0^+, we get

$$\varphi\left(\frac{z_1 + z_2}{2}\right) \leq \bigl(1 - \eta(\varphi_0, z)\bigr)\varphi_0.$$

Since K_∞ is convex, $\frac{z_1+z_2}{2} \in K_\infty$. Therefore

$$\varphi_0 = \varphi\left(\frac{z_1 + z_2}{2}\right) \leq \bigl(1 - \eta(\varphi_0, z)\bigr)\varphi_0 < \varphi_0,$$

a contradiction. Then K_∞ is reduced to one point. To finish the proof, we show that K_∞ is reduced to one point if $\varphi = 0$. For x, $y \in K$, we have

$$\rho\left(\frac{x - y}{2}\right) \leq \frac{\varphi(x) + \varphi(y)}{2}.$$

In fact, let $s \geq 0$. Then for every $t \geq s$,

$$\rho\left(\frac{x-y}{2}\right) \leq \rho\left(\frac{x-T_t(x_0)}{2}\right) + \rho\left(\frac{T_t(x_0)-y}{2}\right),$$

and then

$$\rho\left(\frac{x-y}{2}\right) \leq \frac{\varphi(x)+\varphi(y)}{2}$$

for all $x, y \in K$. Especially, for $x, y \in K_\infty$,

$$\rho\left(\frac{x-y}{2}\right) \leq \frac{\varphi(x)+\varphi(y)}{2} = \varphi_0 = 0.$$

Thus $x = y$. In both cases, we have shown that K_∞ is reduced to one point. As a result, φ has a unique minimum point in K. □

The next result is a generalization of Theorem 1.1 in uniformly convex in every direction modular spaces.

Theorem 3.22 *Let ρ be a convex modular uniformly continuous and (UUCED), and let X_ρ be a modular space satisfyiong property (R). Let C be a nonempty ρ-closed convex ρ-bounded subset of X_ρ. Let $\mathcal{S}$ be a monotone ρ-nonexpansive semigroup on C. Assume that there exists $x_0 \in C$ such that $x_0 \leq T_t(x_0)$ (resp., $T_t(x_0) \leq x_0$) for all $t \geq 0$. Then there exists a common fixed point $z \in Fix(\mathcal{S})$ such that $x_0 \leq z$ (resp., $z \leq x_0$).*

Proof Without loss of generality, we assume that $x_0 \leq T_t(x_0)$ for all $t \geq 0$. Let $(s_n)_n$ be a nondecreasing sequence in $\mathbb{R}_+$ such that $s_0 = 0$ and $\lim_n s_n = +\infty$.

For all $n \geq 0$, set

$$K_n = \bigcap_{t \geq s_n} \big[T_t(x_0), \rightarrow\big) \cap C.$$

$(K_n)_n$ is a decreasing sequence of ρ-closed convex and ρ-bounded subsets of C. In fact, for all $h \geq 0$, $x_0 \leq T_h(x_0)$. In particular, for $h = s_n$, we have $x_0 \leq T_{s_n}(x_0)$. Let $t \geq s_n$. Then $T_t(x_0) \leq T_{t+s_n}(x_0) \leq T_{2s_n}(x_0) = x$. Then $x = T_{2s_n}(x_0) \in K_n$. Hence K_n is nonempty for all $n \geq 0$.

For all $n \geq 0$, K_n is ρ-closed. Indeed, let $(y_p)_p$ be a sequence in K_n that ρ-converges to $y \in C$. For all $p \geq 0$, $y_p \in K_n$, that is, $T_t(x_0) \leq y_p$ for all $t \geq s_n$ and $p \geq 0$. Then $y_p - T_t(x_0) \in P$ for all $t \geq s_n$ and $p \geq 0$. Since

$$\lim_{p \to +\infty} \rho\big(y_p - T_t(x_0) - y + T_t(x_0)\big) = \lim_{n \to +\infty} \rho(y_p - y) = 0$$

and P is ρ-closed, we have $y - T_t(x_0) \in P$ for all $t \geq s_n$, that is, $y \in K_n$.

K_n is convex and ρ-bounded, since P is convex and $K_n \subset C$. Moreover, $(K_n)_n$ is decreasing since $(s_n)_n$ is increasing.

By property (R) the set $K = \bigcap_{n \geq 0} K_n$ is nonempty ρ-closed and convex.

K is invariant by $\mathcal{S}$. Indeed, let $x \in K$; then $T_t(x_0) \leq x$ for all $n \geq 0$ and $t \geq s_n$. Letting $\eta \geq 0$, let us prove that $T_t(x_0) \leq T_\eta(x)$ for all $t \geq s_n$.

Let $t \in \mathbb{N}$. If $\eta > t$, then $\varepsilon = \eta - t > 0$, where $t \geq s_n$. Since $x_0 \leq x$, $x_0 \leq T_\varepsilon(x_0) \leq T_\varepsilon(x)$. Hence $T_t(x_0) \leq T_\eta(x)$ for all $t \geq s_n$. Thus $T_\eta(x) \in K_n$.

If $\eta \leq t$, then $t - \eta \geq 0$, where $t \geq s_n$, which implies $T_{t-\eta}(x_0) \leq x$, because $x \in K_0$. Then $T_t(x_0) \leq T_\eta(x)$. Hence $T_\eta(x) \in K_n$. In both cases, we have $\mathcal{S}(K) \subset K$.

Consider the function $\varphi : K \longrightarrow \mathbb{R}_+$ defined by

$$\varphi(y) = \limsup_{t \to +\infty} \rho\bigl(T_t(x_0) - y\bigr).$$

By Lemma 3.21 φ has a unique minimum point $z \in K$.

Fix $s, \eta \geq 0$ and let $t \geq s + \eta$. As $\mathcal{S}$ is a monotone ρ-nonexpansive semigroup, we have

$$\begin{aligned}\rho\bigl(T_t(x_0) - T_s(z)\bigr) &= \rho\bigl(T_s\bigl(T_{t-s}(x_0)\bigr) - T_s(z)\bigr)\\ &\leq \rho\bigl(T_{t-s}(x_0) - z\bigr)\\ &\leq \sup_{\bar{t} \geq \eta} \rho\bigl(T_{\bar{t}}(x_0) - z\bigr),\end{aligned}$$

which implies

$$\varphi\bigl(T_s(y)\bigr) \leq \inf_{\eta \geq 0} \sup_{\bar{t} \geq \eta} \rho\bigl(T_{\bar{t}}(x_0) - y\bigr).$$

Then $\varphi(T_s(z)) \leq \varphi(z)$ for all $s \geq 0$. Thus $T_s(z)$ is also a minimum point of φ for all $s \geq 0$. By the uniqueness of z, $T_s(z) = z$ for all $s \geq 0$. Therefore z is a common fixed point of $\mathcal{S}$. □

3.2 Convergence theorems for common fixed point of a monotone semigroup

First, we introduce the notion of uniformly asymptotic regular semigroups.

Definition 3.23 Let C be a subset of X_ρ. A semigroup $\mathcal{S} = \{T_t : t \geq 0\}$ on C is said to be uniformly asymptotic regular (u.a.r.) if for any $s \geq 0$ and any ρ-bounded subset K of C, we have

$$\lim_{t \to +\infty} \sup_{x \in K} \rho\bigl(T_s\bigl(T_t(x)\bigr) - T_t(x)\bigr) = 0.$$

Example 3.24 Let $X_\rho = \mathbb{R}^2$, and let the modular ρ be defined by $\rho(x) = x_1^2 + x_2^2$ for $x = (x_1, x_2)$ in X_ρ. Let $C = [0, A] \times [0, A]$, where $A > 0$. Consider the one-parameter family $\mathcal{S} = \{T_t : t \geq 0\}$ defined by

$$\begin{aligned}&T : C \longrightarrow C,\\ &x \longmapsto T_t(x) = e^{-t}x\end{aligned}$$

for all $t \geq 0$. It is quite easy to show that $\mathcal{S}$ is a semigroup. Moreover, $\mathcal{S}$ is u.a.r. In fact, let $s \geq 0$, and let K be a ρ-bounded subset of C. Then

$$\begin{aligned}\lim_{t \to +\infty} \sup_{x \in K} \rho\bigl(T_s\bigl(T_t(x)\bigr) - T_t(x)\bigr) &= \lim_{t \to +\infty} \sup_{x \in K} \rho\bigl(e^{-s}\bigl(e^{-t}x\bigr) - e^{-t}x\bigr)\\ &= \lim_{t \to +\infty} \sup_{x \in K} \bigl(e^{-t}\bigr)^2\bigl(e^{-s} - 1\bigr)^2\bigl(x_1^2 + x_2^2\bigr) = 0.\end{aligned}$$

Next, we give some properties of the partial order defined on the modular space X_ρ by a ρ-closed convex cone P.

Definition 3.25 We say that a partial order $\leq$ is ρ-closed if for any two sequences $(x_n)_n$ and $(y_n)_n$ in X_ρ such that $x_n \leq y_n$ for all $n \geq 0$ that ρ-converge to x and y, respectively, then $x \leq y$.

Proposition 3.26 *Let ρ be a convex modular satisfying the Δ_2-type condition. The partial order defined by a ρ-closed convex cone P ($x \leq y \Longleftrightarrow y - x \in P$ for x and y in X_ρ) is ρ-closed.*

Proof Let $(x_n)_n$ and $(y_n)_n$ be two sequences in X_ρ that ρ-converge to x and y, respectively, such that $x_n \leq y_n$ for all $n \geq 0$.

We have $y_n - x_n \in P$ for all $n \geq 0$. Moreover, for all $n \geq 0$,

$$\begin{aligned}\rho\bigl((y_n - x_n) - (y - x)\bigr) &= \rho\bigl((y_n - y) + (x - x_n)\bigr) \\ &\leq \frac{k}{2}\rho(y_n - y) + \frac{k}{2}\rho(x_n - x) \underset{n\to+\infty}{\longrightarrow} 0.\end{aligned}$$

Hence $(y_n - x_n)_n$ ρ-converges to $y - x$. Since the cone P is ρ-closed, we have $y - x \in P$, which equivalent to $x \leq y$. □

Remark 3.27 The partial order "$\leq$" defined by a ρ-closed convex cone P satisfies the following property:

If $(x_n)_n$ is a nondecreasing sequence such that $x_n \xrightarrow{\rho} x$, then $x_n \leq x$ for all n.

Indeed, fix arbitrary $n_0 \in \mathbb{N}$. Since $(x_n)_n$ is a nondecreasing sequence, $x_{n_0} \leq x_n$ for all $n \geq n_0$, which is equivalent to $x_n - x_{n_0} \in P$. Hence

$$x_n \in x_{n_0} + P.$$

Since P is ρ-closed, so is $x_{n_0} + P$. Therefore $x \in x_{n_0} + P$ implies $x - x_{n_0} \in P$. Thus $x_{n_0} \leq x$ for all $n_0 \geq 0$. Hence

$$x_n \leq x \quad \text{for all } n \geq 0.$$

Lemma 3.28 ([4]) *Let ρ be a convex modular $(UUC1)$, and let X_ρ be a modular space. Let $R > 0$ and $(\alpha_n)_n \subset [a, b]$ with $0 < a \leq b < 1$. Let $(u_n)_n$ and $(v_n)_n$ be two sequences in X_ρ. Assume that*

$$\begin{cases} \limsup_{n\to+\infty} \rho(u_n) \leq R, \\ \limsup_{n\to+\infty} \rho(v_n) \leq R, \\ \lim_{n\to+\infty} \rho(\alpha_n u_n + (1 - \alpha_n)v_n) = R. \end{cases}$$

Then

$$\lim_{n\to+\infty} \rho(u_n - v_n) = 0.$$

We further define a new iteration algorithm for monotone ρ-nonexpansive semigroups in modular spaces. Our iteration process is defined as follows: for $x_0 \in C$ such that $x_0 \leq T_s x_0$ for all $s \geq 0$,

$$(Si) \quad \begin{cases} x_{n+1} = (1-\alpha_n)T_{t_n}(x_n) + \alpha_n T_{t_n}(y_n), \\ y_n = (1-\beta_n)x_n + \beta_n T_{t_n}(x_n), \end{cases}$$

where $(\alpha_n)_n$ and $(\beta_n)_n$ are two sequences in $(0,1)$ such that $0 < a \leq \alpha_n \leq b < 1$ and $0 < c \leq \beta_n \leq d < 1$, and $(t_n)_n \subset \mathbb{R}_+$ is a nondecreasing sequence such that $\lim_n t_n = +\infty$ and $T_{t_n}(x) \leq T_{t_{n+1}}(x)$ for all $x \in C$.

The sequence $(x_n)_n$ is nondecreasing, and for all $n \geq 0$,

$$x_n \leq T_{t_n}(x_n) \leq x_{n+1} \leq T_{t_{n+1}}(x_{n+1}). \tag{5}$$

Indeed, for $n = 0$, we have $x_0 \leq T_{t_0}(x_0)$. By the convexity of the order interval $[x_0, T_{t_0}(x_0)]$ we have

$$x_0 \leq y_0 \leq T_{t_0}(x_0). \tag{6}$$

Using the monotonicity of T_{t_0}, we have

$$x_0 \leq y_0 \leq T_{t_0}(x_0) \leq T_{t_0}(y_0). \tag{7}$$

By the convexity of the order interval $[T_{t_0}x_0, T_{t_0}y_0]$ we have

$$x_0 \leq y_0 \leq T_{t_0}(x_0) \leq x_1 \leq T_{t_0}(y_0). \tag{8}$$

Hence by the condition on $(t_n)_n$ and the monotonicity of T_{t_1} we have

$$T_{t_0}(y_0) \leq T_{t_1}(y_0) \leq T_{t_1}(x_1). \tag{9}$$

By (8) and (9) we have

$$x_0 \leq T_{t_0}(x_0) \leq x_1 \leq T_{t_1}(x_1).$$

Assume that inequality (5) is true for $n \geq 0$. Let us prove that

$$x_{n+2} \leq T_{t_{n+2}}(x_{n+2}) \leq x_{n+3} \leq T_{t_{n+3}}(x_{n+3}). \tag{10}$$

We have $x_{n+1} \leq T_{t_{n+1}}(x_{n+1})$, and by the convexity of the order interval $[x_{n+1}, T_{t_{n+1}}(x_{n+1})]$ we get

$$x_{n+1} \leq y_{n+1} \leq T_{t_{n+1}}(x_{n+1}). \tag{11}$$

Since $T_{t_{n+1}}$ is monotone, we have

$$x_{n+1} \leq y_{n+1} \leq T_{t_{n+1}}(x_{n+1}) \leq T_{t_{n+1}}(y_{n+1}).$$

Using the convexity of the order interval $[T_{t_{n+1}}(x_{n+1}), T_{t_{n+1}}(y_{n+1})]$ and the condition on $(t_n)_n$, we get

$$x_{n+1} \preceq T_{t_{n+1}}(x_{n+1}) \preceq x_{n+2} \preceq T_{t_{n+2}}(x_{n+2}). \tag{12}$$

By the same way we prove that

$$x_{n+2} \preceq T_{t_{n+2}}(x_{n+2}) \preceq x_{n+3} \preceq T_{t_{n+3}}(x_{n+3}). \tag{13}$$

Hence, for all $n \geq 0$,

$$x_n \preceq T_{t_n}(x_n) \preceq x_{n+1} \preceq T_{t_{n+1}}(x_{n+1}).$$

Remark 3.29 As an example of a sequence $(t_n)_n$, we can consider the sequence $(2^n s)_n$ where $s > 0$. In fact, for $n = 0$, we show as before that $x_0 \preceq T_s(x_0) \preceq x_1 \preceq T_{2s}(x_1)$. Next, we assume that $x_n \preceq T_{2^n s}(x_n) \preceq x_{n+1} \preceq T_{2^{n+1}s}(x_{n+1})$. As before, we get

$$x_n \preceq y_n \preceq T_{2^n s}(x_n) \preceq x_{n+1} \preceq T_{2^n s}(y_n).$$

Moreover, $T_{2^n s}(y_n) \preceq T_{2^n s}(T_{2^n s}(y_n)) = T_{2^n s+2^n s}(y_n) = T_{2^{n+1}s}(y_n) \preceq T_{2^{n+1}s}(x_{n+1})$. Hence for all $n \geq 0$,

$$x_n \preceq y_n \preceq T_{2^n s}(x_n) \preceq x_{n+1} \preceq T_{2^n s}(y_n) \preceq T_{2^{n+1}s}(x_{n+1}).$$

Then for all $n \geq 0$,

$$x_n \preceq T_{2^n s}(x_n) \preceq x_{n+1} \preceq T_{2^{n+1}s}(x_{n+1}).$$

Lemma 3.30 *Let ρ be a convex modular $(UUC1)$ satisfying the Δ_2-type condition, and let C be a ρ-closed convex ρ-bounded subset of a modular space X_ρ. Let $\mathcal{S}$ be a monotone ρ-nonexpansive semigroup on C, and let $x_0 \in C$ be such that $x_0 \preceq T_s(x_0)$ for all $s \geq 0$. Let $p \in \mathcal{F}ix(\mathcal{S})$ be such that $x_0 \preceq p$. Then*

$$\lim_n \rho\bigl(x_n - T_{t_n}(x_n)\bigr) = 0.$$

Proof It obvious that $x_n \preceq p$ and $y_n \preceq p$ for all $n \geq 0$. Moreover,

$$\begin{aligned} \rho(x_{n+1} - p) &= \rho\bigl((1-\alpha_n)T_{t_n}(x_n) + \alpha_n T_{t_n}(y_n) - p\bigr) \\ &\leq (1-\alpha_n)\rho\bigl(T_{t_n}(x_n) - p\bigr) + \alpha_n \rho\bigl(T_{t_n}(y_n) - p\bigr) \\ &\leq (1-\alpha_n)\rho(x_n - p) + \alpha_n \rho(y_n - p), \end{aligned} \tag{14}$$

or

$$\begin{aligned} \rho(y_n - p) &= \rho\bigl((1-\beta_n)x_n + \beta_n T_{t_n}(x_n) - p\bigr) \\ &\leq (1-\beta_n)\rho(x_n - p) + \beta_n \rho\bigl(T_{t_n}(x_n) - p\bigr) \\ &\leq (1-\beta_n)\rho(x_n - p) + \beta_n \rho(x_n - p) = \rho(x_n - p). \end{aligned} \tag{15}$$

From (14) and (15) we have $\rho(x_{n+1}-p) \leq \rho(x_n-p)$. Hence the sequence $(\rho(x_n-p))_n$ is decreasing in $\mathbb{R}_+$. Then $\lim_n \rho(x_n-p) = R \geq 0$ exists.

If $R = 0$, then there is nothing to prove. Indeed, for all $n \geq 0$,

$$\begin{aligned}\rho(x_n - T_{t_n}x_n) &\leq \frac{k}{2}\rho(x_n-p) + \frac{k}{2}\rho\bigl(T_{t_n}(x_n)-p\bigr)\\ &\leq k\rho(x_n-p) \underset{n\to+\infty}{\longrightarrow} 0.\end{aligned}$$

If $R > 0$, then we put $u_n = x_n - p$ and $v_n = T_{t_n}(x_n) - p$ in Lemma 3.28. Then

$$\limsup_{n\to+\infty} \rho(x_n-p) = R$$

and

$$\limsup_{n\to+\infty} \rho\bigl(T_{t_n}(x_n)-p\bigr) \leq \limsup_{n\to+\infty} \rho(x_n-p) = R.$$

Moreover,

$$\rho(x_{n+1}-p) \leq (1-\alpha_n)\rho(x_n-p) + \alpha_n\rho(y_n-p),$$

which implies

$$\frac{\rho(x_{n+1}-p)-\rho(x_n-p)}{\alpha_n} \leq \rho(y_n-p) - \rho(x_n-p).$$

Since $0 < a \leq \alpha_n \leq b < 1$, we have $\frac{1}{b} \leq \frac{1}{\alpha_n} \leq \frac{1}{a}$. Thus by the previous inequality we have

$$\frac{\rho(x_{n+1}-p)-\rho(x_n-p)}{a} \leq \frac{\rho(x_{n+1}-p)-\rho(x_n-p)}{\alpha_n} \leq \rho(y_n-p) - \rho(x_n-p)$$

because $\rho(x_{n+1}-p) \leq \rho(x_n-p)$. Consequently, as n goes to infinity, we have

$$R \leq \liminf_{n\to+\infty} \rho(y_n-p). \tag{16}$$

Otherwise, $\rho(y_n-p) \leq \rho(x_n-p)$, and then

$$\limsup_{n\to+\infty} \rho(y_n-p) \leq R. \tag{17}$$

By (16) and (17),

$$R \leq \liminf_{n\to+\infty} \rho(y_n-p) \leq \limsup_{n\to+\infty} \rho(y_n-p) \leq R,$$

and thus

$$\lim_{n\to+\infty} \rho(y_n-p) = R,$$

that is,

$$\lim_{n\to+\infty} \rho\big((1-\alpha_n)(x_n-p)+\alpha_n\big(T_{t_n}(x_n)-p\big)\big)=R.$$

By Lemma 3.28,

$$\lim_n \rho\big(x_n - T_{t_n}(x_n)\big)=0. \qquad \square$$

Lemma 3.31 *Let ρ be a convex modular satisfying the Δ_2-type condition. Let C be a nonempty ρ-closed convex subset of a modular space X_ρ, and let $T: C \longrightarrow C$ be a monotone ρ-nonexpansive mapping. Suppose $(x_n)_n$ is a sequence in C such that there exists a subsequence $(x_\varphi(n))_n$ that ρ-converges to $x \in C$, $x_{\varphi(n)} \le T(x_{\varphi(n)}) \le x$ (or $x \le T(x_{\varphi(n)}) \le x_{\varphi(n)}$) for all integer $n \ge 0$, and*

$$\lim_{n\to+\infty} \rho\big(x_{\varphi(n)} - T(x_{\varphi(n)})\big)=0. \tag{18}$$

Then x is a fixed point of T.

Proof Without loss of generality, we assume that $x_{\varphi(n)} \le T(x_{\varphi(n)}) \le x$ for all $n \ge 0$. Since T is monotone ρ-nonexpansive, we have

$$\rho\big(T(x_{\varphi(n)})-T(x)\big) \le \rho(x_{\varphi(n)}-x). \tag{19}$$

Hence

$$\begin{aligned}\rho\left(\frac{x-T(x)}{2}\right) &\le \frac{1}{2}\rho\big(x-T(x_{\varphi(n)})\big)+\frac{1}{2}\rho\big(T(x_{\varphi(n)})-T(x)\big)\\ &\le \frac{1}{2}\rho\big(x-T(x_{\varphi(n)})\big)+\frac{1}{2}\rho(x_{\varphi(n)}-x).\end{aligned} \tag{20}$$

Moreover,

$$\rho\big(x-T(x_{\varphi(n)})\big) \le \frac{k}{2}\rho(x-x_{\varphi(n)})+\frac{k}{2}\rho\big(x_{\varphi(n)}-T(x_{\varphi(n)})\big). \tag{21}$$

Therefore from (18), (20), and (21) we have

$$\rho\left(\frac{x-T(x)}{2}\right) \le \frac{k}{4}\rho(x-x_{\varphi(n)})+\frac{k}{4}\rho\big(x_{\varphi(n)}-T(x_{\varphi(n)})\big)+\frac{1}{2}\rho(x_{\varphi(n)}-x) \underset{n\to+\infty}{\longrightarrow} 0.$$

Hence x is a fixed point of T. $\square$

We further use the fixed point sets with the partial order $\mathcal{F}_x^{\le}(\mathcal{S})$ and $\mathcal{F}_x^{\ge}(\mathcal{S})$ given by

$$\mathcal{F}_x^{\le}(\mathcal{S})=\big\{p\in \mathcal{F}ix(\mathcal{S}) : p \le x\big\} \quad \text{for some } x$$

and

$$\mathcal{F}_x^{\ge}(\mathcal{S})=\big\{p\in \mathcal{F}ix(\mathcal{S}) : p \ge x\big\} \quad \text{for some } x,$$

respectively. Next, we study the convergence of the iteration (Si) for monotone ρ-nonexpansive semigroups u.a.r. $\mathcal{S}$ in uniformly convex modular spaces.

Theorem 3.32 *Let ρ be a convex modular $(UUC1)$ satisfying the Δ_2-type condition. Let C be a nonempty convex ρ-sequentially compact and ρ-bounded subset of a modular space X_ρ. Let $\mathcal{S} = \{T_t : t \geq 0\}$ be a monotone ρ-nonexpansive u.a.r. semigroup on C. Assume that there exists $x_0 \in C$ such that $x_0 \leq T_t(x_0)$ for all $t \geq 0$ and that $\mathcal{F}^{\geq}_{x_0}(\mathcal{S})$ is nonempty. Then the sequence $(x_n)_n$ defined by the iterations (Si) ρ-converge to a common fixed point of the semigroup $\mathcal{S}$.*

Proof Fix $p \in \mathcal{F}^{\geq}_{x_0}(\mathcal{S})$. Without loss of generality, assume that $x_0 \leq T_t(x_0)$ for all $t \geq 0$. We have $x_n \leq p$ for all $n \geq 0$, and by Lemma 3.30

$$\lim_{n\to+\infty} \rho\bigl(x_n - T_{t_n}(x_n)\bigr) = 0. \tag{22}$$

Let us prove that

$$\lim_{n\to+\infty} \rho\bigl(x_n - T_s(x_n)\bigr) = 0 \quad \text{for all } s \geq 0. \tag{23}$$

For all $n \geq 0$,

$$\begin{aligned} \rho\bigl(x_{n+1} - T_s(x_{n+1})\bigr) &= \rho\bigl((1-\alpha_n)T_{t_n}(x_n) + \alpha_n T_{t_n}(y_n) - T_s(x_{n+1})\bigr) \\ &\leq (1-\alpha_n)\rho\bigl(T_{t_n}(x_n) - T_s(x_{n+1})\bigr) + \alpha_n \rho\bigl(T_{t_n}(y_n) - T_s(x_{n+1})\bigr), \end{aligned} \tag{24}$$

$$\begin{aligned} \rho\bigl(T_{t_n}(x_n) - T_s(x_{n+1})\bigr) &= \rho\bigl(T_{t_n}(x_n) - T_s T_{t_n}(x_n) + T_s T_{t_n}(x_n) - T_s(x_{n+1})\bigr) \\ &\leq \frac{k}{2}\rho\bigl(T_{t_n}(x_n) - T_s T_{t_n}(x_n)\bigr) + \frac{k}{2}\rho\bigl(T_s T_{t_n}(x_n) - T_s(x_{n+1})\bigr) \\ &\leq \frac{k}{2}\rho\bigl(T_{t_n}(x_n) - T_s T_{t_n}(x_n)\bigr) + \frac{k}{2}\rho\bigl(T_{t_n}(x_n) - x_{n+1}\bigr). \end{aligned} \tag{25}$$

Since $\mathcal{S}$ is u.a.r., for any ρ-bounded subset K of C, we have

$$\lim_{n\to+\infty} \rho\bigl(T_{t_n}(x_n) - T_s T_{t_n}(x_n)\bigr) \leq \lim_{n\to+\infty} \sup_{x\in K} \rho\bigl(T_{t_n}(x) - T_s T_{t_n}(x)\bigr) = 0.$$

Hence

$$\lim_{n\to+\infty} \rho\bigl(T_{t_n}(x_n) - T_s T_{t_n}(x_n)\bigr) = 0. \tag{26}$$

Moreover, for all $n \geq 0$,

$$\begin{aligned} \rho\bigl(T_{t_n}(x_n) - x_{n+1}\bigr) &= \rho\bigl((1-\alpha_n)T_{t_n}(x_n) + \alpha_n T_{t_n}(y_n) - T_{t_n}(x_n)\bigr) \\ &\leq \alpha_n \rho\bigl(T_{t_n}(y_n) - T_{t_n}(x_n)\bigr) \\ &\leq \alpha_n \rho(y_n - x_n) \\ &\leq \alpha_n \rho\bigl((1-\beta_n)x_n + \beta_n T_{t_n}(x_n) - x_n\bigr) \\ &\leq \alpha_n \beta_n \rho\bigl(T_{t_n}(x_n) - x_n\bigr). \end{aligned}$$

From the hypothesis on $(\alpha_n)_n$ and $(\beta_n)_n$ and (22) we get

$$\lim_n \rho\big(x_{n+1} - T_{t_n}(x_n)\big) = 0. \tag{27}$$

Using (25), (26), and (27), we have

$$\lim_n \rho\big(T_{t_n}(x_n) - T_s(x_{n+1})\big) = 0. \tag{28}$$

Otherwise,

$$\begin{aligned}
\rho\big(T_{t_n}(y_n) - T_s(x_{n+1})\big) &= \rho\big(T_{t_n}(y_n) - T_s T_{t_n}(y_n) + T_s T_{t_n}(y_n) - T_s(x_{n+1})\big) \\
&\leq \frac{k}{2}\rho\big(T_{t_n}(y_n) - T_s T_{t_n}(y_n)\big) + \frac{k}{2}\rho\big(T_s T_{t_n}(y_n) - T_s(x_{n+1})\big) \\
&\leq \frac{k}{2}\rho\big(T_{t_n}(y_n) - T_s T_{t_n}(y_n)\big) + \frac{k}{2}\rho\big(T_{t_n}(y_n) - x_{n+1}\big).
\end{aligned} \tag{29}$$

Since $\mathcal{S}$ is u.a.r., we have

$$\lim_n \rho\big(T_{t_n}(y_n) - T_s T_{t_n}(y_n)\big) = 0. \tag{30}$$

Moreover,

$$\begin{aligned}
\rho\big(T_{t_n}(y_n) - x_{n+1}\big) &= \rho\big(T_{t_n}(y_n) - (1-\alpha_n)T_{t_n}(x_n) - \alpha_n T_{t_n}(y_n)\big) \\
&\leq (1-\alpha_n)\rho(x_n - y_n) \\
&\leq (1-\alpha_n)(1-\beta_n)\rho\big(x_n - T_{t_n}(x_n)\big).
\end{aligned}$$

Therefore

$$\lim_n \rho\big(T_{t_n}(y_n) - x_{n+1}\big) = 0. \tag{31}$$

By (29), (30), and (30)

$$\lim_n \rho\big(T_{t_n}(y_n) - T_s(x_{n+1})\big) = 0. \tag{32}$$

From (24), (28), and (32) we have

$$\lim_n \rho\big(x_{n+1} - T_s(x_{n+1})\big) = 0. \tag{33}$$

Since C is ρ-sequentially compact, $(x_n)_n$ has a subsequence $(x_{\varphi(n)})_n$ ρ-converging to a point $x \in C$ such that $x_{\varphi(n)} \leq x$. Moreover, by (33)

$$\lim_n \rho\big(x_{\varphi(n)} - T_s(x_{\varphi(n)})\big) = 0.$$

Hence by Lemma 3.31 x is a fixed point of T_s for all $s \geq 0$. Then x is a common fixed point of the semigroup $\mathcal{S}$.

To complete the proof, we prove that $(x_n)_n$ ρ-converges to x.

Let $(x_{\psi(n)})_n$ be another subsequence of $(x_n)_n$ that ρ-converges to y. For each $\varphi(n)$, there exists a large enough $\psi(n)$ such that $x_{\varphi(n)} \leq x_{\psi(n)}$. Then by Proposition 3.26 we have $x \leq y$. In the same way, we get $y \leq x$. Therefore $x = y$.

Hence the sequence $(x_n)_n$ has a unique cluster point x, and since C is ρ-sequentially compact, $(x_n)_n$ ρ-converges to x. □

Example 3.33 Let the space $\mathbb{R}$ be equipped with the convex modular $\rho(x) = |x|^2$ for $x \in \mathbb{R}$. It is quite easy to see that ρ is (UUC1) and satisfies the Fatou property and Δ_2-type condition. Consider the usual partial ordering defined on $\mathbb{R}$, that is, $x \preceq y$ if and only if $y - x \in [0, \infty[$. Let $C = [0, 1]$ be a ρ-sequentially compact, ρ-bounded, and convex subset of $\mathbb{R}$.

Let the family $\mathcal{S} = \{T_t : t \geq 0\}$ be such that

$$\begin{aligned} &T_t : C \longrightarrow C, \\ &x \longmapsto T_t(x) = f\bigl(5^{-t} f^{-1}(x)\bigr), \end{aligned}$$

where $f(w) = 1 - w$ for all $w \in C$. It easy to verify that $\mathcal{S}$ is a monotone ρ-nonexpansive semigroup and uniformly asymptotic regular (u.a.r.).

Let $x_0 = 0 \in C$. We have $x_0 \preceq T_t(x_0) = 1 - 5^{-t}$ for all $t \geq 0$. Moreover, $\mathcal{F}^{\geq}_{x_0}(\mathcal{S}) = \{1\}$ is nonempty. Let $\alpha_n = \alpha \in (0, 1)$, $\beta_n = \beta \in (0, 1)$, and $t_n = 2^n$ for all $n \geq 0$. By induction on n we construct the sequence $(x_n)_{n\geq 0}$ given as follows:

$$x_{n+1} = \left(1 - \frac{1}{5^{2^n}}\right)\left(1 + \frac{\alpha\beta}{5^{2^n}}\right) + \frac{x_n}{5^{2^n}}\left(1 - \alpha\beta\left(1 - \frac{1}{5^{2^n}}\right)\right) \tag{34}$$

for $n \geq 0$. In fact, for $n = 0$, we have $x_0 = 0$ and $T_{t_0}(x_0) = 1 - \frac{1}{5}$. Then, using the iteration (Si), we get $y_0 = (1 - \frac{1}{5})\beta$. Thus $x_1 = (1 - \frac{1}{5})(1 + \frac{\alpha\beta}{5})$. Assume that (34) is true until the order n. Let us prove that

$$x_{n+2} = \left(1 - \frac{1}{5^{2^{n+1}}}\right)\left(1 + \frac{\alpha\beta}{5^{2^{n+1}}}\right) + \frac{x_{n+1}}{5^{2^{n+1}}}\left(1 - \alpha\beta\left(1 - \frac{1}{5^{2^{n+1}}}\right)\right).$$

Using the iteration (Si), we obtain $y_{n+1} = \beta(1 - \frac{1}{5^{2^{n+1}}}) + (1 - \beta(1 - \frac{1}{5^{2^{n+1}}}))x_{n+1}$. Thus

$$\begin{aligned} x_{n+2} &= \alpha T_{t_{n+1}}(y_{n+1}) + (1 - \alpha)T_{t_{n+1}}(x_{n+1}) \\ &= \alpha\left(1 - \frac{1}{5^{2^{n+1}}}(1 - y_{n+1})\right) + (1 - \alpha)\left(1 - \frac{1}{5^{2^{n+1}}}(1 - x_{n+1})\right) \\ &= \left(1 - \frac{1}{5^{2^{n+1}}}\right) + \frac{x_{n+1}}{5^{2^{n+1}}} + \frac{\alpha}{5^{2^{n+1}}}(y_{n+1} - x_{n+1}) \\ &= \left(1 - \frac{1}{5^{2^{n+1}}}\right) + \frac{x_{n+1}}{5^{2^{n+1}}} + \frac{\alpha\beta}{5^{2^{n+1}}}(T_{t_{n+1}}x_{n+1} - x_{n+1}) \\ &= \left(1 - \frac{1}{5^{2^{n+1}}}\right)\left(1 + \frac{\alpha\beta}{5^{2^{n+1}}}\right) + \frac{x_{n+1}}{5^{2^{n+1}}}\left(1 - \alpha\beta\left(1 - \frac{1}{5^{2^{n+1}}}\right)\right). \end{aligned}$$

Therefore by Theorem 3.32 the sequence $(x_n)_n$ ρ-converges to 1.

4 Conclusion

We have established some existence results for monotone ρ-nonexpansive semigroups in modular spaces. Then we proposed an iteration scheme with some convergence results for the class of uniformly asymptotic regular monotone ρ-nonexpansive semigroups. Our results of existence are generalizations of several results mentioned in the introduction and the reference sections of this paper.

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Abbreviations
UCi, uniform convexity; UUCi, unique uniform convexity; UCED, uniform convexity in every direction; UUCED, unique uniform convexity in every direction; SC, strict convexity; u.a.r., uniformly asymptotic regular.

Availability of data and materials
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