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Set-valued Leader type contractions, periodic point and endpoint theorems, quasi-triangular spaces, Bellman and Volterra equations

Kazimierz Włodarczyk[1*]

*Correspondence:
kazimierz.wlodarczyk@wmii.uni.lodz.pl
[1]Department of Nonlinear Analysis, Faculty of Mathematics and Computer Science, University of Łódź, Łódź, Poland

Abstract

Set-valued contractions of Leader type in quasi-triangular spaces are constructed, conditions guaranteeing the existence of nonempty sets of periodic points, fixed points and endpoints of such contractions are established, convergence of dynamic processes of these contractions are studied, uniqueness properties are derived, and single-valued cases are considered. Investigated dynamic systems are not necessarily continuous and spaces are not necessarily sequentially complete or Hausdorff. Obtained results suggest, in particular, strategies to new studies of functional Bellman equations and variable discounted Bellman equations in metric spaces and integral Volterra equations in locally convex spaces. Results in this direction are also presented in this paper. More precisely, without continuity of Bellman and Volterra appropriate operators, the sets of solutions of these equations (which are periodic points of these operators) are studied and new and general convergence, existence and uniqueness theorems concerning such equations are proved.

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1 Introduction

The fixed point theory has been advanced by a number of authors. The most popular research in complete metric spaces has risen from the ideas of Banach [5] and Caccioppoli [13] for single-valued maps and Nadler [40, 41] for set-valued maps.

We mention that the results of Leader [32], concerning necessary and sufficient conditions for the existence of contractive fixed points of set-valued and single-valued maps with complete graphs in metric spaces, generalize the results of Banach [5], R. Caccioppoli [13], Burton [12], Rakotch [42], Geraghty [21, 22], Matkowski [34–36], Walter [51], Dugundji [18], Tasković [46], Dugundji and Granas [19], Browder [11], Krasnosel'skiĭ et al. [30], Boyd and Wong [10], Mukherjea [39], Meir and Keeler [38], and many others.

Moreover, Leader's method as presented in [32] does not require the complete assumption of metric spaces. Presentation concerning generality and some structural properties of Leader contractions in metric spaces was fully exploited in Jachymski [23, 24] and Jachymski and Jóźwik [25].

Concerning the Bellman functional equations (see [6–8]), variable δ-discounted Bellman equations (see, e.g., [14, 26, 34]) and Volterra integral equations (see [50]), most of the results contained in several works and books require such assumptions which (by using various techniques or by utilizing various known fixed point theorems) imply that the appropriate Bellman and Volterra operators are continuous (on suitable Banach spaces or complete metric spaces or sequentially complete locally convex vector spaces).

Let X be a (nonempty) set. A *distance* on $X \times X$ is a map $X \times X \to [0;+\infty)$. A *distance space* is a set X together with family of distances $X \times X \to [0;+\infty)$. We will furnish some simple examples of various distance spaces and set-valued and single-valued contractions on these spaces and such that the more than traditional conclusions of known fixed point theorems are valid even if some hypotheses on these spaces or contractions are ignored or weakened or replaced by different and less restrictive ones. Also we will furnish simple examples of various types of Bellman functional equations or variable δ-discounted Bellman equations or Volterra integral equations such that the appropriate Bellman or Volterra operators are discontinuous and without fixed points, whereas solutions of these equations exist and are not necessarily the singletons.

This raises the following natural questions: (i) Are there not necessarily Hausdorff or sequentially complete distance spaces together with families of distances which are not necessarily continuous or vanish on the diagonal or are symmetric or triangular ones? Then we ask whether in such distance spaces there exist not necessarily continuous set-valued and single-valued contractions which have nonempty sets of periodic points or fixed points or endpoints. (ii) How large is the set of these distance spaces? (iii) What structures controlling the limiting behavior of the sequences of dynamic processes and Picard iterations of these set-valued and single-valued contractions, respectively, does this set possess? (iv) What are connections between periodic points or fixed points or endpoints and the limits of dynamic processes or Picard iterations of these set-valued or single-valued contractions? (v) How large and general is the class of such set-valued and single-valued contractions? (vi) Which suitable techniques, methods and ideas are useful for studying these problems as major complications arise in such general spaces? (vii) Are the developments of this sort new in known distance spaces and new even in metric spaces or normed spaces or locally convex spaces or gauge spaces? (viii) Have these investigations the new and different applications, e.g., to studying of Bellman functional equations or variable δ-discounted Bellman equations or Volterra integral equations? (ix) Are possible unifications and generalizations of known spaces and results? (x) Are theorems of this type optimal?

The objects we are interested in are quasi-triangular spaces. The purpose of this paper, by providing efficient tools and techniques for investigating the set-valued and single-valued dynamic systems and Bellman and Volterra type operators in these spaces and also for better understanding generality and specific properties of these spaces, is to show how to answer these questions positively.

Basic definitions, notations and remarks are given in Sects. 2–8.

Motivation for general convergence, periodic point, fixed point and endpoint results presented in Theorems 9.1–9.4 of this paper and concerning the new constructed here set-valued and single-valued contractions in quasi-triangular spaces comes from the fixed point theorems in metric spaces established by Leader in his significant paper [32].

Section 10 contains the proofs of Theorems 9.1–9.4. In Sect. 11, we present a number of examples.

As applications of our convergence, periodic point and fixed point Theorems 9.3 and 9.4 for single-valued contractions in metric and gauge spaces, we provide and prove, without restrictive assumptions, the new and general convergence, existence and uniqueness theorems concerning solutions of Bellman functional equations and variable δ-discounted Bellman equations in metric spaces and Volterra integral equations in locally convex spaces. More precisely, we concentrate on convergence, existence and uniqueness problems concerning periodic and fixed points of appropriate Bellman and Volterra operators. Thus we studied the structure of sets of solutions of these equations in more general setting. Results in this direction are presented in Theorems 12.1–12.4, 13.1–13.3 and 14.1–14.4 of this paper.

2 Quasi-triangular spaces $(X, \mathcal{P}_{C;\mathcal{A}})$. $\mathcal{P}_{C;\mathcal{A}}$-separability. Set-valued and single-valued dynamic systems (X, T) in $(X, \mathcal{P}_{C;\mathcal{A}})$. Fixed points, periodic points and edpoints

Quasi-triangular spaces which unify and generalize an existing body of distance spaces (such as metric, ultra metric, quasi-metric, ultra quasi-metric, b-metric, partial metric, partial b-metric, pseudometric, quasi-pseudometric, ultra quasi-pseudometric, partial quasi-pseudometric, topological, uniform, quasi-uniform, gauge, ultra gauge, partial gauge, quasi-gauge, ultra quasi-gauge, partial quasi-gauge, normed, locally convex spaces, ultra quasi-triangular and partial quasi-triangular (see, e.g., [4, 15, 16, 28, 31, 37, 43, 44, 49, 52–54])) are defined as follows.

Definition 2.1 ([53, 54]) Let X be a (nonempty) set, let $\mathcal{A}$ be an index set, and let $C = \{C_\alpha\}_{\alpha\in\mathcal{A}} \in [1;\infty)^{\mathcal{A}}$.

(A) We say that a family $\mathcal{P}_{C;\mathcal{A}} = \{P_\alpha : \alpha \in \mathcal{A}\}$ of distances $P_\alpha : X^2 \to [0,\infty)$, $\alpha \in \mathcal{A}$, is a *quasi-triangular family on* X if $\forall_{\alpha\in\mathcal{A}}\forall_{u,v,w\in X}\{P_\alpha(u,w) \le C_\alpha[P_\alpha(u,v) + P_\alpha(v,w)]\}$. A *quasi-triangular space* $(X, \mathcal{P}_{C;\mathcal{A}})$ is a set X together with the quasi-triangular family $\mathcal{P}_{C;\mathcal{A}}$ on X.

(B) We say that a family $\mathcal{P}_{\mathcal{A}} = \{P_\alpha : \alpha \in \mathcal{A}\}$ of distances $P_\alpha : X^2 \to [0,\infty)$, $\alpha \in \mathcal{A}$, is a *triangular family on* X if $\forall_{\alpha\in\mathcal{A}}\forall_{u,v,w\in X}\{P_\alpha(u,w) \le P_\alpha(u,v) + P_\alpha(v,w)\}$. A *triangular space* $(X, \mathcal{P}_{\mathcal{A}})$ is a set X together with the triangular family $\mathcal{P}_{\mathcal{A}}$ on X.

(C) If $C_\alpha = 1$ for each $\alpha \in \mathcal{A}$, then $\mathcal{P}_{C;\mathcal{A}}$ is denoted by $\mathcal{P}_{\mathcal{A}}$. If the set $\mathcal{A}$ has only one element, then $\mathcal{P}_{C;\mathcal{A}}$ is denoted by P.

(D) Let $(X, \mathcal{P}_{C;\mathcal{A}})$ be a quasi-triangular space. We say that $\mathcal{P}_{C;\mathcal{A}}$ is *separating on* X if $\forall_{u,w\in X}\{u \neq w \Rightarrow \exists_{\alpha\in\mathcal{A}}\{P_\alpha(u,w) > 0 \vee P_\alpha(w,u) > 0\}\}$.

Remark 2.1 Note that "$C = \{C_\alpha\}_{\alpha\in\mathcal{A}} \in [1;\infty)^{\mathcal{A}}$" means exactly that $[1;\infty)^{\mathcal{A}} = \prod_{\alpha\in\mathcal{A}}[1;\infty) = \{x = \{x_\alpha\}_{\alpha\in\mathcal{A}} : \forall_{\alpha\in\mathcal{A}}\{x_\alpha \in [1;\infty)\}\}$.

Definition 2.2 Let X be a vector space over $\mathbb{K}$ (where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$), $\mathcal{A}$ be an index set, and $C = \{C_\alpha\}_{\alpha\in\mathcal{A}} \in [1;\infty)^{\mathcal{A}}$.

(A) We say that a family $\mathcal{N}_{C;\mathcal{A}} = \{N_\alpha : \alpha \in \mathcal{A}\}$ of maps $N_\alpha : X \to [0,\infty)$, $\alpha \in \mathcal{A}$, is a *locally convex quasi-triangular family* on X if $\forall_{\alpha\in\mathcal{A}}\forall_{u,v\in X}\{N_\alpha(u+v) \le C_\alpha[N_\alpha(u)+N_\alpha(v)]\}$ and $\forall_{\alpha\in\mathcal{A}}\forall_{u\in X}\forall_{\lambda\in\mathbb{K}}\{N_\alpha(\lambda u) = |\lambda|N_\alpha(u)\}$.

(B) A *locally convex quasi-triangular space* $(X,\mathcal{N}_{C;\mathcal{A}})$ is a set X together with the locally convex quasi-triangular family $\mathcal{N}_{C;\mathcal{A}} = \{N_\alpha : \alpha \in \mathcal{A}\}$ on X.

(C) Let $(X,\mathcal{N}_{C;\mathcal{A}})$ be a locally convex quasi-triangular space. We say that $\mathcal{N}_{C;\mathcal{A}}$ *is separating on* X if $\forall_{u\in X}\{u \ne 0 \Rightarrow \exists_{\alpha_0\in\mathcal{A}}\{N_{\alpha_0}(u) > 0\}\}$.

Remark 2.2 We see that each locally convex quasi-triangular space is a symmetric quasi-triangular space. Indeed, if X is a vector space over $\mathbb{K}$ and $\mathcal{N}_{C;\mathcal{A}} = \{N_\alpha : \alpha \in \mathcal{A}\}$, $N_\alpha : X \to [0,\infty)$, $\alpha \in \mathcal{A}$, is a locally convex quasi-triangular family, then $\mathcal{P}_{C;\mathcal{A}} = \{P_\alpha : \alpha \in \mathcal{A}\}$ where $P_\alpha(u,v) = N_\alpha(u-v)$, $\alpha \in \mathcal{A}$, $(u,v) \in X \times X$, is a symmetric and quasi-triangular family and $(X,\mathcal{P}_{C;\mathcal{A}})$ is a symmetric quasi-triangular space.

Remark 2.3 Let $(X,\mathcal{P}_{C;\mathcal{A}})$ be a quasi-triangular space. In general, the distances P_α, $\alpha \in \mathcal{A}$, do not vanish on the diagonal, are asymmetric, and do not satisfy the triangle inequality (i.e., conditions $\forall_{\alpha\in\mathcal{A}}\forall_{u\in X}\{P_\alpha(u,u) = 0\}$ or $\forall_{\alpha\in\mathcal{A}}\forall_{u,w\in X}\{P_\alpha(u,w) = P_\alpha(w,u)\}$ or $\forall_{\alpha\in\mathcal{A}}\forall_{u,v,w\in X}\{P_\alpha(u,w) \le P_\alpha(u,v) + P_\alpha(v,w)\}$ do not necessarily hold).

We will use $(x_m : m \in \mathbb{N}) \subset X$ as a sequence and as a set as the situation demands.

Asymmetry of P_α, $\alpha \in \mathcal{A}$, justify the use of term "left" and term "right". When the symmetry holds, then term "left" and term "right" are identical.

Definition 2.3 Let $(X,\mathcal{P}_{C;\mathcal{A}})$ be a quasi-triangular space.

(A) We say that $(x_m : m \in \mathbb{N}) \subset X$ is *left* (respectively, *right*) $\mathcal{P}_{C;\mathcal{A}}$*-convergent* in X if $\mathrm{LIM}^{L-\mathcal{P}_{C;\mathcal{A}}}_{(x_m:m\in\mathbb{N})} = \{u \in X : \forall_{\alpha\in\mathcal{A}}\{\lim_{m\to\infty} P_\alpha(u,x_m) = 0\}\} \ne \emptyset$ (respectively, $\mathrm{LIM}^{R-\mathcal{P}_{C;\mathcal{A}}}_{(x_m:m\in\mathbb{N})} = \{v \in X : \forall_{\alpha\in\mathcal{A}}\{\lim_{m\to\infty} P_\alpha(x_m,v) = 0\}\} \ne \emptyset$).

(B) We say that $(x_m : m \in \mathbb{N}) \subset X$ is *left* (respectively, *right*) $\mathcal{P}_{C;\mathcal{A}}$*-convergent to* $u \in X$ (respectively, $v \in X$) if $u \in \mathrm{LIM}^{L-\mathcal{P}_{C;\mathcal{A}}}_{(x_m:m\in\mathbb{N})}$ (respectively, $v \in \mathrm{LIM}^{R-\mathcal{P}_{C;\mathcal{A}}}_{(x_m:m\in\mathbb{N})}$).

Let $(X,\mathcal{P}_{C;\mathcal{A}})$ be a quasi-triangular space. The *set-valued dynamic system* on $(X,\mathcal{P}_{C;\mathcal{A}})$ is defined as a pair (X,T), where $T : X \to 2^X$; here, 2^X denotes the family of all nonempty subsets of X. The *single-valued dynamic system* on $(X,\mathcal{P}_{C;\mathcal{A}})$ is defined as a pair (X,T), where T is a single-valued map $T : X \to X$, i.e., $\forall_{x\in X}\{T(x) \in X\}$.

For $q \in \mathbb{N}$ and for set-valued and single-valued dynamic systems (X,T), we define $T^{[q]} = T \circ T \circ \cdots \circ T$ (q-times).

Let (X,T) be a set-valued dynamic system on $(X,\mathcal{P}_{C;\mathcal{A}})$. For each $w^0 \in X$, we denote by $\mathcal{O}_{X,T}(w^0)$ the set of all *dynamic processes* or *trajectories starting at* w^0 or *motions* of the system (X,T) (see [1–3, 56]), i.e.,

$$\mathcal{O}_{X,T}(w^0) = \{(w^m : m \in \{0\}\cup\mathbb{N}) : \forall_{m\in\{0\}\cup\mathbb{N}}\{w^{m+1} \in T(w^m)\}\}.$$

By $\mathrm{Fix}_X(T)$, $\mathrm{Per}_X(T)$ and $\mathrm{End}_X(T)$ we denote the sets of all *fixed points*, *periodic points* and *endpoints* (*stationary points*) of (X,T), respectively, i.e., $\mathrm{Fix}_X(T) = \{w \in X : w \in T(w)\}$, $\mathrm{Per}_X(T) = \{w \in X : w \in T^{[q]}(w)$ for some $q \in \mathbb{N}\}$ and $\mathrm{End}_X(T) = \{w \in X : \{w\} = T(w)\}$.

Let (X,T) be a single-valued dynamic system on $(X,\mathcal{P}_{C;\mathcal{A}})$. For each $w^0 \in X$, a sequence $(w^m = T^{[m]}(w^0) : m \in \{0\}\cup\mathbb{N})$, $T^{[0]} = I_X$ (an identity map on X), is called a *Picard iteration* *starting at* w^0 of the system (X, T). By $\mathrm{Fix}_X(T)$ and $\mathrm{Per}_X(T)$ we denote the sets of all *fixed points* and *periodic points* of (X, T), respectively, i.e., $\mathrm{Fix}_X(T) = \{w \in X : w = T(w)\}$ and $\mathrm{Per}_X(T) = \{w \in X : w = T^{[q]}(w)$ for some $q \in \mathbb{N}\}$.

3 Left (right) families $\mathcal{J}_{C;\mathcal{A}}$ generated by $\mathcal{P}_{C;\mathcal{A}}$, $\mathcal{J}_{C;\mathcal{A}}$-separability, and relation between $\mathcal{J}_{C;\mathcal{A}}$-separability and $\mathcal{P}_{C;\mathcal{A}}$-separability in $(X, \mathcal{P}_{C;\mathcal{A}})$

For given quasi-triangular spaces $(X, \mathcal{P}_{C;\mathcal{A}})$, it is natural to define the notions of the sets $\mathbb{J}^L_{(X,\mathcal{P}_{C;\mathcal{A}})}$ (respectively, $\mathbb{J}^R_{(X,\mathcal{P}_{C;\mathcal{A}})}$) of left (respectively, right) families $\mathcal{J}_{C;\mathcal{A}}$ generated by $\mathcal{P}_{C;\mathcal{A}}$. These sets $\mathbb{J}^L_{(X,\mathcal{P}_{C;\mathcal{A}})}$ (respectively, $\mathbb{J}^R_{(X,\mathcal{P}_{C;\mathcal{A}})}$) provide on X new structures which are richer than structures provided on X by $\mathcal{P}_{C;\mathcal{A}}$.

Definition 3.1 Let $(X, \mathcal{P}_{C;\mathcal{A}})$ be a quasi-triangular space.

(A) The family $\mathcal{J}_{C;\mathcal{A}} = \{J_\alpha : \alpha \in \mathcal{A}\}$ of distances $J_\alpha : X \times X \to [0;\infty)$, $\alpha \in \mathcal{A}$, is said to be a *left family generated by* $\mathcal{P}_{C;\mathcal{A}}$ if the following two conditions hold:
  (A.1) $\forall_{\alpha\in\mathcal{A}}\forall_{u,v,w\in X}\{J_\alpha(u,w) \le C_\alpha[J_\alpha(u,v) + J_\alpha(v,w)]\}$.
  (A.2) For any sequences $(x_m : m \in \mathbb{N})$ and $(y_m : m \in \mathbb{N})$ in X with the properties $\forall_{\alpha\in\mathcal{A}}\{\lim_{m\to\infty}\sup_{n>m} J_\alpha(x_m, x_n) = 0\}$ and $\forall_{\alpha\in\mathcal{A}}\{\lim_{m\to\infty} J_\alpha(y_m, x_m) = 0\}$ we have $\forall_{\alpha\in\mathcal{A}}\{\lim_{m\to\infty} P_\alpha(y_m, x_m) = 0\}$.

(B) The family $\mathcal{J}_{C;\mathcal{A}} = \{J_\alpha : \alpha \in \mathcal{A}\}$ of distances $J_\alpha : X \times X \to [0;\infty)$, $\alpha \in \mathcal{A}$, is said to be a *right family generated by* $\mathcal{P}_{C;\mathcal{A}}$ if the following two conditions hold:
  (B.1) $\forall_{\alpha\in\mathcal{A}}\forall_{u,v,w\in X}\{J_\alpha(u,w) \le C_\alpha[J_\alpha(u,v) + J_\alpha(v,w)]\}$.
  (B.2) For any sequences $(x_m : m \in \mathbb{N})$ and $(y_m : m \in \mathbb{N})$ in X with the properties $\forall_{\alpha\in\mathcal{A}}\{\lim_{m\to\infty}\sup_{n>m} J_\alpha(x_n, x_m) = 0\}$ and $\forall_{\alpha\in\mathcal{A}}\{\lim_{m\to\infty} J_\alpha(x_m, y_m) = 0\}$ we have $\forall_{\alpha\in\mathcal{A}}\{\lim_{m\to\infty} P_\alpha(x_m, y_m) = 0\}$.

(C) Denote by $\mathbb{J}^L_{(X,\mathcal{P}_{C;\mathcal{A}})}$ (respectively, $\mathbb{J}^R_{(X,\mathcal{P}_{C;\mathcal{A}})}$) the set of all left (respectively, right) families $\mathcal{J}_{C;\mathcal{A}}$ generated by $\mathcal{P}_{C;\mathcal{A}}$.

(D) If $C_\alpha = 1$ for each $\alpha \in \mathcal{A}$, then $\mathcal{J}_{C;\mathcal{A}}$, $\mathcal{P}_{C;\mathcal{A}}$, $\mathbb{J}^L_{(X,\mathcal{P}_{C;\mathcal{A}})}$ and $\mathbb{J}^R_{(X,\mathcal{P}_{C;\mathcal{A}})}$ we denoted by $\mathcal{J}_{\mathcal{A}}$, $\mathcal{P}_{\mathcal{A}}$, $\mathbb{J}^L_{(X,\mathcal{P}_{\mathcal{A}})}$ and $\mathbb{J}^R_{(X,\mathcal{P}_{\mathcal{A}})}$, respectively. If the set $\mathcal{A}$ has only one element, then $\mathcal{J}_{C;\mathcal{A}}$, $\mathcal{P}_{C;\mathcal{A}}$, $\mathbb{J}^L_{(X,\mathcal{P}_{C;\mathcal{A}})}$ and $\mathbb{J}^R_{(X,\mathcal{P}_{C;\mathcal{A}})}$ is denoted by J, P, $\mathbb{J}^L_{(X,P)}$ and $\mathbb{J}^R_{(X,P)}$, respectively.

Remark 3.1 The following hold:

(a) $\mathcal{P}_{C;\mathcal{A}} \in \mathbb{J}^L_{(X,\mathcal{P}_{C;\mathcal{A}})} \cap \mathbb{J}^R_{(X,\mathcal{P}_{C;\mathcal{A}})}$.
(b) The structures on X determined by left (respectively, right) families $\mathcal{J}_{C;\mathcal{A}}$ generated by $\mathcal{P}_{C;\mathcal{A}}$ are more general than the structure on X determined by $\mathcal{P}_{C;\mathcal{A}}$.
(c) Let $\mathcal{J}_{C;\mathcal{A}} \in \mathbb{J}^L_{(X,\mathcal{P}_{C;\mathcal{A}})} \cup \mathbb{J}^R_{(X,\mathcal{P}_{C;\mathcal{A}})}$; thus, in particular, let $\mathcal{J}_{C;\mathcal{A}} = \mathcal{P}_{C;\mathcal{A}}$. In general, the distances J_α, $\alpha \in \mathcal{A}$, do not vanish on the diagonal, are asymmetric, and do not satisfy the triangle inequality (i.e., $\forall_{\alpha\in\mathcal{A}}\forall_{u\in X}\{J_\alpha(u,u) = 0\}$ or $\forall_{\alpha\in\mathcal{A}}\forall_{u,w\in X}\{J_\alpha(u,w) = J_\alpha(w,u)\}$ or $\forall_{\alpha\in\mathcal{A}}\forall_{u,v,w\in X}\{J_\alpha(u,w) \le J_\alpha(u,v) + J_\alpha(v,w)\}$ do not necessarily hold).
(d) Asymmetry of J_α, $\alpha \in \mathcal{A}$, justify the use of term "left" and term "right". When the symmetry holds, then term "left" and term "right" are identical.

Definition 3.2 Let $(X, \mathcal{P}_{C;\mathcal{A}})$ be a quasi-triangular space and let $\mathcal{J}_{C;\mathcal{A}} = \{J_\alpha : \alpha \in \mathcal{A}\} \in \mathbb{J}^L_{(X,\mathcal{P}_{C;\mathcal{A}})} \cup \mathbb{J}^R_{(X,\mathcal{P}_{C;\mathcal{A}})}$. We say that $\mathcal{J}_{C;\mathcal{A}}$ is *separating on* X if $\forall_{u,w\in X}\{u \ne w \Rightarrow \exists_{\alpha_0\in\mathcal{A}}\{J_{\alpha_0}(u,w) > 0 \vee J_{\alpha_0}(w,u) > 0\}\}$.

The technique for establishing uniqueness of fixed points and endpoints of dynamic systems is *separation* (Definition 2.1(D)), *contraction property* (conditions (d) of Theorems 9.1 and 9.3) and Proposition 3.1 below.

Proposition 3.1 ([53]) *Let $(X,\mathcal{P}_{C;\mathcal{A}})$ be a quasi-triangular space. If $\mathcal{P}_{C;\mathcal{A}}$ is separating on X, then each $\mathcal{J}_{C;\mathcal{A}} \in \mathbb{J}^L_{(X,\mathcal{P}_{C;\mathcal{A}})} \cup \mathbb{J}^R_{(X,\mathcal{P}_{C;\mathcal{A}})}$ is separating on X.*

Remark 3.2 The distances in uniform spaces (these distances are determined by uniformity and generalize uniformity) first appeared in Vályi [48]. We also mention at this stage that various concepts of distances in metric spaces (X,d) which generalize d, of this sort, are given by Kada *et al.* [27] (w-distances), Lin and Du [33] (τ-functions), Suzuki [45] (τ-distances) and Ume [47] (u-distances). General distances in cone uniform spaces, which generalize these distances and simplify the arguments substantially, are treated in [55]. In the appearing literature, these distances and their generalizations in other spaces provide efficient tools to study various problems of fixed point theory. In this paper, using sets $\mathbb{J}^L_{(X,\mathcal{P}_{C;\mathcal{A}})}$ and $\mathbb{J}^R_{(X,\mathcal{P}_{C;\mathcal{A}})}$ defined above, we also generalize these ideas.

4 Left (respectively, right) $\mathcal{J}_{C;\mathcal{A}}$-convergences of sequences in $(X,\mathcal{P}_{C;\mathcal{A}})$. Left (respectively, right) $\mathcal{J}_{C;\mathcal{A}}$-Hausdorff spaces $(X,\mathcal{P}_{C;\mathcal{A}})$

The above considerations suggest the appropriate definition regarding left (respectively, right) $\mathcal{J}_{C;\mathcal{A}}$-convergence of sequences in $(X,\mathcal{P}_{C;\mathcal{A}})$.

Definition 4.1 Let $(X,\mathcal{P}_{C;\mathcal{A}})$ be a quasi-triangular space. Let $\mathcal{J}_{C;\mathcal{A}} \in \mathbb{J}^L_{(X,\mathcal{P}_{C;\mathcal{A}})} \cup \mathbb{J}^R_{(X,\mathcal{P}_{C;\mathcal{A}})}$; thus, in particular, let $\mathcal{J}_{C;\mathcal{A}} = \mathcal{P}_{C;\mathcal{A}}$.

(A) We say that $(x_m : m \in \mathbb{N}) \subset X$ is *left* (respectively, *right*) *$\mathcal{J}_{C;\mathcal{A}}$-convergent* in X if $\mathcal{J}_{C;\mathcal{A}} \in \mathbb{J}^L_{(X,\mathcal{P}_{C;\mathcal{A}})}$ and $\mathrm{LIM}^{L-\mathcal{J}_{C;\mathcal{A}}}_{(x_m:m\in\mathbb{N})} = \{u \in X : \forall_{\alpha\in\mathcal{A}}\{\lim_{m\to\infty} J_\alpha(u,x_m) = 0\}\} \neq \varnothing$ (respectively, $\mathcal{J}_{C;\mathcal{A}} \in \mathbb{J}^R_{(X,\mathcal{P}_{C;\mathcal{A}})}$ and $\mathrm{LIM}^{R-\mathcal{J}_{C;\mathcal{A}}}_{(x_m:m\in\mathbb{N})} = \{v \in X : \forall_{\alpha\in\mathcal{A}}\{\lim_{m\to\infty} J_\alpha(x_m,v) = 0\}\} \neq \varnothing$).

(B) We say that $(x_m : m \in \mathbb{N}) \subset X$ is *left* (respectively, *right*) *$\mathcal{J}_{C;\mathcal{A}}$-convergent to $u \in X$* (respectively, $v \in X$) if $\mathcal{J}_{C;\mathcal{A}} \in \mathbb{J}^L_{(X,\mathcal{P}_{C;\mathcal{A}})}$ and $u \in \mathrm{LIM}^{L-\mathcal{J}_{C;\mathcal{A}}}_{(x_m:m\in\mathbb{N})}$ (respectively, $\mathcal{J}_{C;\mathcal{A}} \in \mathbb{J}^R_{(X,\mathcal{P}_{C;\mathcal{A}})}$ and $v \in \mathrm{LIM}^{R-\mathcal{J}_{C;\mathcal{A}}}_{(x_m:m\in\mathbb{N})}$).

Remark 4.1 Let $(X,\mathcal{P}_{C;\mathcal{A}})$ be a quasi-triangular space. Assume that $\mathcal{J}_{C;\mathcal{A}} \in \mathbb{J}^L_{(X,\mathcal{P}_{C;\mathcal{A}})}$ (respectively, $\mathcal{J}_{C;\mathcal{A}} \in \mathbb{J}^R_{(X,\mathcal{P}_{C;\mathcal{A}})}$); thus, in particular, $\mathcal{J}_{C;\mathcal{A}} = \mathcal{P}_{C;\mathcal{A}}$. It is clear that if sequence $(x_m : m \in \mathbb{N}) \subset X$ is left (respectively, right) $\mathcal{J}_{C;\mathcal{A}}$-convergent in X, then $\mathrm{LIM}^{L-\mathcal{J}_{C;\mathcal{A}}}_{(x_m:m\in\mathbb{N})} \subset \mathrm{LIM}^{L-\mathcal{J}_{C;\mathcal{A}}}_{(y_m:m\in\mathbb{N})}$ (respectively, $\mathrm{LIM}^{R-\mathcal{J}_{C;\mathcal{A}}}_{(x_m:m\in\mathbb{N})} \subset \mathrm{LIM}^{R-\mathcal{J}_{C;\mathcal{A}}}_{(y_m:m\in\mathbb{N})}$) for each subsequence $(y_m : m \in \mathbb{N})$ of sequence $(x_m : m \in \mathbb{N})$.

Definition 4.2 Let $(X,\mathcal{P}_{C;\mathcal{A}})$ be a quasi-triangular space. Let $\mathcal{J}_{C;\mathcal{A}} \in \mathbb{J}^L_{(X,\mathcal{P}_{C;\mathcal{A}})} \cup \mathbb{J}^R_{(X,\mathcal{P}_{C;\mathcal{A}})}$; thus, in particular, let $\mathcal{J}_{C;\mathcal{A}} = \mathcal{P}_{C;\mathcal{A}}$. We say that $(X,\mathcal{P}_{C;\mathcal{A}})$ is *left* (respectively, *right*) *$\mathcal{J}_{C;\mathcal{A}}$-Hausdorff* if for each left (respectively, right) $\mathcal{J}_{C;\mathcal{A}}$-convergent sequence $(x_m : m \in \mathbb{N})$ in X we have: $\mathcal{J}_{C;\mathcal{A}} \in \mathbb{J}^L_{(X,\mathcal{P}_{C;\mathcal{A}})}$ and $\mathrm{LIM}^{L-\mathcal{J}_{C;\mathcal{A}}}_{(x_m:m\in\mathbb{N})}$ is a singleton (respectively, $\mathcal{J}_{C;\mathcal{A}} \in \mathbb{J}^R_{(X,\mathcal{P}_{C;\mathcal{A}})}$ and $\mathrm{LIM}^{R-\mathcal{J}_{C;\mathcal{A}}}_{(x_m:m\in\mathbb{N})}$ is a singleton).

5 Left (respectively, right) $J_{C;\mathcal{A}}$-admissible on $M \in 2^X$ of set-valued and single-valued dynamic systems (X, T) in $(X, \mathcal{P}_{C;\mathcal{A}})$

Left (respectively, right) $J_{C;\mathcal{A}}$-admissible of set-valued and single-valued dynamic systems (X, T) on $M \in 2^X$ in $(X, \mathcal{P}_{C;\mathcal{A}})$ are defined as follows.

Definition 5.1 Let $(X, \mathcal{P}_{C;\mathcal{A}})$ be a quasi-triangular space, and let (X, T) be a set-valued dynamic system.

(A) Let $\mathcal{J}_{C;\mathcal{A}} \in \mathbb{J}^L_{(X,\mathcal{P}_{C;\mathcal{A}})}$; thus, in particular, let $\mathcal{J}_{C;\mathcal{A}} = \mathcal{P}_{C;\mathcal{A}}$. (X, T) is said to be a *left $\mathcal{J}_{C;\mathcal{A}}$-admissible in a point* $w^0 \in X$ if each dynamic process $(w^m : m \in \{0\} \cup \mathbb{N}) \in \mathcal{O}_{X,T}(w^0)$ starting at w^0, which is *left $\mathcal{J}_{C;\mathcal{A}}$-sequence* (i.e. satisfying the condition $\forall_{\alpha\in\mathcal{A}}\{\lim_{m\to\infty}\sup_{n>m} J_\alpha(w^m, w^n) = 0\}$), is left $\mathcal{J}_{C;\mathcal{A}}$-converging in X (i.e. has property $\mathrm{LIM}^{L-\mathcal{J}_{C;\mathcal{A}}}_{(w^m:m\in\{0\}\cup\mathbb{N})} \neq \emptyset$).

(B) Let $\mathcal{J}_{C;\mathcal{A}} \in \mathbb{J}^R_{(X,\mathcal{P}_{C;\mathcal{A}})}$; thus, in particular, let $\mathcal{J}_{C;\mathcal{A}} = \mathcal{P}_{C;\mathcal{A}}$. (X, T) is said to be a *right $\mathcal{J}_{C;\mathcal{A}}$-admissible in a point* $w^0 \in X$ if each dynamic process $(w^m : m \in \{0\} \cup \mathbb{N}) \in \mathcal{O}_{X,T}(w^0)$ starting at w^0, which is *right $\mathcal{J}_{C;\mathcal{A}}$-sequence* (i.e. satisfying the condition $\forall_{\alpha\in\mathcal{A}}\{\lim_{m\to\infty}\sup_{n>m} J_\alpha(w^n, w^m) = 0\}$), is right $\mathcal{J}_{C;\mathcal{A}}$-converging in X (i.e. has the property $\mathrm{LIM}^{R-\mathcal{J}_{C;\mathcal{A}}}_{(w^m:m\in\{0\}\cup\mathbb{N})} \neq \emptyset$).

(C) Let $M \in 2^X$. (X, T) is said to be a *left* (respectively, *right*) *$\mathcal{J}_{C;\mathcal{A}}$-admissible on M* if $\mathcal{J}_{C;\mathcal{A}} \in \mathbb{J}^L_{(X,\mathcal{P}_{C;\mathcal{A}})}$ (respectively, $\mathcal{J}_{C;\mathcal{A}} \in \mathbb{J}^R_{(X,\mathcal{P}_{C;\mathcal{A}})}$) and (X, T) is a left (respectively, right) $\mathcal{J}_{C;\mathcal{A}}$-admissible in each point $w^0 \in M$.

Definition 5.2 Let $(X, \mathcal{P}_{C;\mathcal{A}})$ be a quasi-triangular space, and let (X, T) be a single-valued dynamic system.

(A) Let $\mathcal{J}_{C;\mathcal{A}} \in \mathbb{J}^L_{(X,\mathcal{P}_{C;\mathcal{A}})}$; thus, in particular, let $\mathcal{J}_{C;\mathcal{A}} = \mathcal{P}_{C;\mathcal{A}}$. (X, T) is said to be a *left $\mathcal{J}_{C;\mathcal{A}}$-admissible in a point* $w^0 \in X$ if a sequence $(T^{[m]}(w^0) : m \in \{0\} \cup \mathbb{N})$, which is *left $\mathcal{J}_{C;\mathcal{A}}$-sequence* (i.e. satisfying the condition

$$\forall_{\alpha\in\mathcal{A}}\Big\{\lim_{m\to\infty}\sup_{n>m} J_\alpha\big(T^{[m]}\big(w^0\big), T^{[n]}\big(w^0\big)\big) = 0\Big\}),$$

is left $\mathcal{J}_{C;\mathcal{A}}$-converging in X (i.e. has the property $\mathrm{LIM}^{L-\mathcal{J}_{C;\mathcal{A}}}_{(T^{[m]}(w^0):m\in\{0\}\cup\mathbb{N})} \neq \emptyset$).

(B) Let $\mathcal{J}_{C;\mathcal{A}} \in \mathbb{J}^R_{(X,\mathcal{P}_{C;\mathcal{A}})}$; thus, in particular, let $\mathcal{J}_{C;\mathcal{A}} = \mathcal{P}_{C;\mathcal{A}}$. (X, T) is said to be a *right $\mathcal{J}_{C;\mathcal{A}}$-admissible in a point* $w^0 \in X$ if a sequence $(T^{[m]}(w^0) : m \in \{0\} \cup \mathbb{N})$, which is *right $\mathcal{J}_{C;\mathcal{A}}$-sequence* (i.e. satisfying the condition

$$\forall_{\alpha\in\mathcal{A}}\Big\{\lim_{m\to\infty}\sup_{n>m} J_\alpha\big(T^{[n]}\big(w^0\big), T^{[m]}\big(w^0\big)\big) = 0\Big\}),$$

is right $\mathcal{J}_{C;\mathcal{A}}$-converging in X (i.e. has the property $\mathrm{LIM}^{R-\mathcal{J}_{C;\mathcal{A}}}_{(T^{[m]}(w^0):m\in\{0\}\cup\mathbb{N})} \neq \emptyset$).

(C) Let $M \in 2^X$. (X, T) is said to be a *left* (respectively, *right*) *$\mathcal{J}_{C;\mathcal{A}}$-admissible on M* iff $\mathcal{J}_{C;\mathcal{A}} \in \mathbb{J}^L_{(X,\mathcal{P}_{C;\mathcal{A}})}$ (respectively, $\mathcal{J}_{C;\mathcal{A}} \in \mathbb{J}^R_{(X,\mathcal{P}_{C;\mathcal{A}})}$) and (X, T) is a left (respectively, right) $\mathcal{J}_{C;\mathcal{A}}$-admissible in each point $w^0 \in M$.

6 Relation between left (respectively, right) $\mathcal{J}_{C;\mathcal{A}}$-admissible property and left (respectively, right) $\mathcal{J}_{C;\mathcal{A}}$-sequential completeness in $(X, \mathcal{P}_{C;\mathcal{A}})$

Here, we record some relation between properties of left (respectively, right) $\mathcal{J}_{C;\mathcal{A}}$-admissible (set-valued or single-valued) dynamic systems (X, T) on $M \in 2^X$ in $(X, \mathcal{P}_{C;\mathcal{A}})$ and left (respectively, right) $\mathcal{J}_{C;\mathcal{A}}$-sequential completeness of $(X, \mathcal{P}_{C;\mathcal{A}})$.

Definition 6.1 Let $(X,\mathcal{P}_{C;\mathcal{A}})$ be a quasi-triangular space. Assume that $\mathcal{J}_{C;\mathcal{A}} \in \mathbb{J}^L_{(X,\mathcal{P}_{C;\mathcal{A}})} \cup \mathbb{J}^R_{(X,\mathcal{P}_{C;\mathcal{A}})}$; thus, in particular, $\mathcal{J}_{C;\mathcal{A}} = \mathcal{P}_{C;\mathcal{A}}$.

(A) We say that a sequence $(u_m : m \in \mathbb{N}) \subset X$ is *left* (respectively, *right*) $\mathcal{J}_{C;\mathcal{A}}$*-sequence* if $\mathcal{J}_{C;\mathcal{A}} \in \mathbb{J}^L_{(X,\mathcal{P}_{C;\mathcal{A}})}$ and $\forall_{\alpha\in\mathcal{A}}\{\lim_{m\to\infty}\sup_{n>m} J_\alpha(u_m,u_n) = 0\}$ (respectively, $\mathcal{J}_{C;\mathcal{A}} \in \mathbb{J}^R_{(X,\mathcal{P}_{C;\mathcal{A}})}$ and $\forall_{\alpha\in\mathcal{A}}\{\lim_{m\to\infty}\sup_{n>m} J_\alpha(u_n,u_m) = 0\}$).

(B) If every left (respectively, right) $\mathcal{J}_{C;\mathcal{A}}$-sequence $(u_m : m \in \mathbb{N}) \subset X$ is left (respectively, right) $\mathcal{J}_{C;\mathcal{A}}$-convergent in X (i.e., $\mathcal{J}_{C;\mathcal{A}} \in \mathbb{J}^L_{(X,\mathcal{P}_{C;\mathcal{A}})}$ and $\mathrm{LIM}^{L-\mathcal{J}_{C;\mathcal{A}}}_{(u_m:m\in\mathbb{N})} \neq \emptyset$ (respectively, $\mathcal{J}_{C;\mathcal{A}} \in \mathbb{J}^R_{(X,\mathcal{P}_{C;\mathcal{A}})}$ and $\mathrm{LIM}^{R-\mathcal{J}_{C;\mathcal{A}}}_{(u_m:m\in\mathbb{N})} \neq \emptyset$)), then $(X,\mathcal{P}_{C;\mathcal{A}})$ is called *left* (respectively, *right*) $\mathcal{J}_{C;\mathcal{A}}$*-sequential complete*.

Remark 6.1 Let $(X,\mathcal{P}_{C;\mathcal{A}})$ be a quasi-triangular space. Assume that $\mathcal{J}_{C;\mathcal{A}} \in \mathbb{J}^L_{(X,\mathcal{P}_{C;\mathcal{A}})} \cup \mathbb{J}^R_{(X,\mathcal{P}_{C;\mathcal{A}})}$); thus, in particular, $\mathcal{J}_{C;\mathcal{A}} = \mathcal{P}_{C;\mathcal{A}}$. If $(X,\mathcal{P}_{C;\mathcal{A}})$ is left (respectively, right) $\mathcal{J}_{C;\mathcal{A}}$-sequentially complete, then each (set-valued and single-valued) dynamic system (X,T) is left (respectively, right) $\mathcal{J}_{C;\mathcal{A}}$-admissible on each $M \in 2^X$.

7 Left (respectively, right) $\mathcal{J}_{C;\mathcal{A}}$-closed on $M \in 2^X$ of set-valued and single-valued dynamic systems $(X,T^{[q]})$, $q \in \mathbb{N}$, in $(X,\mathcal{P}_{C;\mathcal{A}})$

The idea of "closed maps" as generalization of continuity first arose in Berge [9] (see also Klein and Thompson [29]). Our formulation of the notion of left (respectively, right) $\mathcal{J}_{C;\mathcal{A}}$-closed on $M \in 2^X$ of set-valued and single-valued dynamic systems $(X,T^{[q]})$, $q \in \mathbb{N}$, in $(X,\mathcal{P}_{C;\mathcal{A}})$ are defined as follows.

Definition 7.1 Let $(X,\mathcal{P}_{C;\mathcal{A}})$ be a quasi-triangular space. Suppose (X,T) is a set-valued dynamic system, $T : X \to 2^X$, and let $q \in \mathbb{N}$.

(A) Let $w^0 \in X$. Let $\mathcal{J}_{C;\mathcal{A}} \in \mathbb{J}^L_{(X,\mathcal{P}_{C;\mathcal{A}})}$; thus, in particular, let $\mathcal{J}_{C;\mathcal{A}} = \mathcal{P}_{C;\mathcal{A}}$. We say that a set-valued dynamic system $(X,T^{[q]})$ is *left* $\mathcal{J}_{C;\mathcal{A}}$*-closed in a point* w^0, if for each dynamic process $(w^m : m \in \{0\}\cup\mathbb{N}) \in \mathcal{O}_{X,T}(w^0)$ starting at w^0 with property $U = \mathrm{LIM}^{L-\mathcal{J}_{C;\mathcal{A}}}_{(w^m:\{0\}\cup\mathbb{N})} \neq \emptyset$ (i.e. left $\mathcal{J}_{C;\mathcal{A}}$-convergent in X) and containing two left $\mathcal{J}_{C;\mathcal{A}}$-converging in X subsequences $(x_m : m \in \mathbb{N})$ and $(y_m : m \in \mathbb{N})$ (thus, in particular, $\mathrm{LIM}^{L-\mathcal{J}_{C;\mathcal{A}}}_{(w^m:\{0\}\cup\mathbb{N})} \subset \mathrm{LIM}^{L-\mathcal{J}_{C;\mathcal{A}}}_{(x_m:m\in\mathbb{N})} \cap \mathrm{LIM}^{L-\mathcal{J}_{C;\mathcal{A}}}_{(y_m:m\in\mathbb{N})}$) satisfying $\forall_{m\in\mathbb{N}}\{y_m \in T^{[q]}(x_m)\}$, we have $\exists_{u\in U}\{u \in T^{[q]}(u)\}$.

(B) Let $w^0 \in X$. Let $\mathcal{J}_{C;\mathcal{A}} \in \mathbb{J}^R_{(X,\mathcal{P}_{C;\mathcal{A}})}$; thus, in particular, let $\mathcal{J}_{C;\mathcal{A}} = \mathcal{P}_{C;\mathcal{A}}$. We say that a set-valued dynamic system $(X,T^{[q]})$ is *right* $\mathcal{J}_{C;\mathcal{A}}$*-closed in a point* w^0, if for each dynamic process $(w^m : m \in \{0\}\cup\mathbb{N}) \in \mathcal{O}_{X,T}(w^0)$ starting at w^0 with property $V = \mathrm{LIM}^{R-\mathcal{J}_{C;\mathcal{A}}}_{(w^m:\{0\}\cup\mathbb{N})} \neq \emptyset$ (i.e. right $\mathcal{J}_{C;\mathcal{A}}$-convergent in X) and containing two right $\mathcal{J}_{C;\mathcal{A}}$-converging in X subsequences $(x_m : m \in \mathbb{N})$ and $(y_m : m \in \mathbb{N})$ (thus, in particular, $\mathrm{LIM}^{R-\mathcal{J}_{C;\mathcal{A}}}_{(w^m:\{0\}\cup\mathbb{N})} \subset \mathrm{LIM}^{R-\mathcal{J}_{C;\mathcal{A}}}_{(x_m:m\in\mathbb{N})} \cap \mathrm{LIM}^{R-\mathcal{J}_{C;\mathcal{A}}}_{(y_m:m\in\mathbb{N})}$) satisfying $\forall_{m\in\mathbb{N}}\{y_m \in T^{[q]}(x_m)\}$, we have $\exists_{v\in V}\{v \in T^{[q]}(v)\}$.

(C) Let $M \in 2^X$. A set-valued dynamic system $(X,T^{[q]})$ is said to be a *left* (respectively, *right*) $\mathcal{J}_{C;\mathcal{A}}$*-closed on* M, if $\mathcal{J}_{C;\mathcal{A}} \in \mathbb{J}^L_{(X,\mathcal{P}_{C;\mathcal{A}})}$ (respectively, $\mathcal{J}_{C;\mathcal{A}} \in \mathbb{J}^R_{(X,\mathcal{P}_{C;\mathcal{A}})}$) and $(X,T^{[q]})$ is a left (respectively, right) $\mathcal{J}_{C;\mathcal{A}}$-closed in each point $w^0 \in M$.

Definition 7.2 Let $(X,\mathcal{P}_{C;\mathcal{A}})$ be a quasi-triangular space. Suppose (X,T) is a single-valued dynamic system, $T : X \to X$, and let $q \in \mathbb{N}$.

(A) Let $w^0 \in X$. Let $\mathcal{J}_{C;\mathcal{A}} \in \mathbb{J}^L_{(X,\mathcal{P}_{C;\mathcal{A}})}$; thus, in particular, let $\mathcal{J}_{C;\mathcal{A}} = \mathcal{P}_{C;\mathcal{A}}$. We say that a single-valued dynamic system $(X,T^{[q]})$ is a *left* $\mathcal{J}_{C;\mathcal{A}}$*-closed in a point* w^0, if in the case when a sequence $(T^{[m]}(w^0) : m \in \{0\} \cup \mathbb{N})$ has property $U = \mathrm{LIM}^{L-\mathcal{J}_{C;\mathcal{A}}}_{(T^{[m]}(w^0):\{0\}\cup\mathbb{N})} \neq \emptyset$ (i.e. is left $\mathcal{J}_{C;\mathcal{A}}$- converging in X) and contains two left $\mathcal{J}_{C;\mathcal{A}}$- converging in X subsequences $(x_m : m \in \mathbb{N})$ and $(y_m : m \in \mathbb{N})$ (thus, in particular, $\mathrm{LIM}^{L-\mathcal{J}_{C;\mathcal{A}}}_{(T^{[m]}(w^0):\{0\}\cup\mathbb{N})} \subset \mathrm{LIM}^{L-\mathcal{J}_{C;\mathcal{A}}}_{(x_m:m\in\mathbb{N})} \cap \mathrm{LIM}^{L-\mathcal{J}_{C;\mathcal{A}}}_{(y_m:m\in\mathbb{N})}$) satisfying $\forall_{m\in\mathbb{N}}\{y_m = T^{[q]}(x_m)\}$, then we have $\exists_{u\in U}\{u = T^{[q]}(u)\}$.

(B) Let $w^0 \in X$. Let $\mathcal{J}_{C;\mathcal{A}} \in \mathbb{J}^R_{(X,\mathcal{P}_{C;\mathcal{A}})}$; thus, in particular, let $\mathcal{J}_{C;\mathcal{A}} = \mathcal{P}_{C;\mathcal{A}}$. We say that a single-valued dynamic system $(X, T^{[q]})$ is a *right $\mathcal{J}_{C;\mathcal{A}}$-closed in a point w^0*, if in the case when a sequence $(T^{[m]}(w^0) : m \in \{0\} \cup \mathbb{N})$ has property $V = \mathrm{LIM}^{R-\mathcal{J}_{C;\mathcal{A}}}_{(T^{[m]}(w^0):\{0\}\cup\mathbb{N})} \neq \emptyset$ (i.e. is right $\mathcal{J}_{C;\mathcal{A}}$-converging in X) and contains two right $\mathcal{J}_{C;\mathcal{A}}$-converging in X subsequences $(x_m : m \in \mathbb{N})$ and $(y_m : m \in \mathbb{N})$ (thus, in particular, $\mathrm{LIM}^{R-\mathcal{J}_{C;\mathcal{A}}}_{(T^{[m]}(w^0):\{0\}\cup\mathbb{N})} \subset \mathrm{LIM}^{R-\mathcal{J}_{C;\mathcal{A}}}_{(x_m:m\in\mathbb{N})} \cap \mathrm{LIM}^{R-\mathcal{J}_{C;\mathcal{A}}}_{(y_m:m\in\mathbb{N})}$) satisfying $\forall_{m\in\mathbb{N}}\{y_m = T^{[q]}(x_m)\}$, then we have $\exists_{v\in V}\{v = T^{[q]}(v)\}$.

(C) Let $M \in 2^X$. A single-valued dynamic system $(X, T^{[q]})$ is said to be a *left* (respectively, *right*) *$\mathcal{J}_{C;\mathcal{A}}$-closed on M*, if $\mathcal{J}_{C;\mathcal{A}} \in \mathbb{J}^L_{(X,\mathcal{P}_{C;\mathcal{A}})}$ (respectively, $\mathcal{J}_{C;\mathcal{A}} \in \mathbb{J}^R_{(X,\mathcal{P}_{C;\mathcal{A}})}$) and $(X, T^{[q]})$ is a left (respectively, right) $\mathcal{J}_{C;\mathcal{A}}$-closed in each point $w^0 \in M$.

8 $\mathcal{S}_{\mathcal{A}}$-Family of accumulation maps

The notion of families $\mathcal{J}_{C;\mathcal{A}}$ generated by $\mathcal{P}_{C;\mathcal{A}}$ (Sect. 3) and the following notion of $\mathcal{S}_{\mathcal{A}}$-*family* are crucial in constructions of contractions (9.1), (9.11), (9.12) and (9.25).

Definition 8.1 Let $\mathcal{A}$ be an index set. The family $\mathcal{S}_{\mathcal{A}} = \{S_\alpha : \alpha \in \mathcal{A}\}$ is said to be *$\mathcal{S}_{\mathcal{A}}$-family of accumulation maps S_α, $\alpha \in \mathcal{A}$*, (simply, *$\mathcal{S}_{\mathcal{A}}$-family*) if:

(A) $\forall_{\alpha\in\mathcal{A}}\{S_\alpha : [0;\infty) \to [1;\infty)\}$.

(B) S_α, $\alpha \in \mathcal{A}$, are strictly increasing on $(0;\infty)$, i.e., $\forall_{\alpha\in\mathcal{A}}\forall_{0<t_1<t_2}\{S_\alpha(t_1) < S_\alpha(t_2)\}$.

(C) $\forall_{\alpha\in\mathcal{A}}\{S_\alpha(0) = 1\}$.

(D) $\forall_{\alpha\in\mathcal{A}}\{t \to 0 \text{ iff } S_\alpha(t) \to 1\}$.

(E) $\forall_{\alpha\in\mathcal{A}}\forall_{t_1,t_2\in[0;\infty)}\{S_\alpha(t_1 + t_2) \leq S_\alpha(t_1) \cdot S_\alpha(t_2)\}$.

(F) $\forall_{\alpha\in\mathcal{A}}\forall_{t\in[0;\infty)}\forall_{\beta\in[1;\infty)}\{S_\alpha(\beta t) \leq [S_\alpha(t)]^\beta\}$.

Remark 8.1 We record some observations concerning $\mathcal{S}_{\mathcal{A}}$-family. (a) The maps S_α, $\alpha \in \mathcal{A}$, are not necessarily continuous. (b) From (A)–(C) it follows that: (b_1) $\forall_{\alpha\in\mathcal{A}}\forall_{t\in[0;\infty)}\{S_\alpha(t) = 1 \text{ implies } t = 0\}$. (b_2) $\forall_{\alpha\in\mathcal{A}}\forall_{t\in[0;\infty)}\{t > 0 \text{ implies } S_\alpha(t) > 1\}$.

Example 8.1 Let $\mathcal{A}$ be an index set. If family $\mathcal{S}_{\mathcal{A}} = \{S_\alpha : \alpha \in \mathcal{A}\}$, $S_\alpha : [0;\infty) \to [1;\infty)$, $\alpha \in \mathcal{A}$, is such that, for arbitrary and fixed $\alpha \in \mathcal{A}$, the map $S_\alpha : [0;\infty) \to [1;\infty)$ is one from the following:

$$S_\alpha(t) = c_\alpha^t, \quad t \in [0;\infty), \tag{8.1}$$

$$S_\alpha(t) = 1 + \frac{t}{a_\alpha + b_\alpha t}, \quad t \in [0;\infty), \tag{8.2}$$

$$S_\alpha(t) = \begin{cases} 1 + \frac{t}{a_\alpha + b_\alpha t} & \text{if } t > 1, \\ 1 + \frac{t/(n+1)}{a_\alpha + b_\alpha t/(n+1)} & \text{if } \frac{1}{n+1} < t \leq \frac{1}{n}, n \in \mathbb{N}, \\ 1 & \text{if } t = 0, \end{cases} \tag{8.3}$$

$$S_\alpha(t) = 1 + \frac{\ln(1 + t)}{a_\alpha + b_\alpha \ln(1 + t)}, \quad t \in [0;\infty), \tag{8.4}$$

$$S_\alpha(t) = 1 + \frac{t + b_\alpha \ln(1+t)}{a_\alpha + t + b_\alpha \ln(1+t)}, \quad t \in [0;\infty), \tag{8.5}$$

where $c_\alpha \in (1;\infty)$, $a_\alpha \in (0;\infty)$ and $b_\alpha \in [0;\infty)$, then $\mathcal{S}_\mathcal{A}$ is an $\mathcal{S}_\mathcal{A}$-family.

We prove conditions (A)–(F) for map S_α given by (8.2). Clearly (A), (C) and (D) hold. Moreover, the maps S_α, $\alpha \in \mathcal{A}$, are strictly increasing on $(0;\infty)$ since

$$\forall_{\alpha\in\mathcal{A}}\forall_{\tau\in(0;\infty)}\bigl\{S'_\alpha(\tau) = a_\alpha/(a_\alpha + b_\alpha\tau)^2 > 0\bigr\}.$$

Next we see that

$$\begin{aligned}
\forall_{\alpha\in\mathcal{A}}\forall_{\tau_1,\tau_2\in[0;\infty)}\biggl\{&S_\alpha(\tau_1+\tau_2)\\
&= 1 + \frac{\tau_1+\tau_2}{a_\alpha + b_\alpha(\tau_1+\tau_2)} < 1 + \frac{\tau_1}{a_\alpha + b_\alpha\tau_1} + \frac{\tau_2}{a_\alpha + b_\alpha\tau_2}\\
&< \left(1 + \frac{\tau_1}{a_\alpha + b_\alpha\tau_1}\right)\cdot\left(1 + \frac{\tau_2}{a_\alpha + b_\alpha\tau_2}\right) = S_\alpha(\tau_1)\cdot S_\alpha(\tau_2)\biggr\}.
\end{aligned}$$

It remains to prove

$$\begin{aligned}
\forall_{\alpha\in\mathcal{A}}\forall_{\tau\in[0;\infty),\,\beta\in[1;\infty)}\biggl\{S_\alpha(\beta\tau) &= 1 + \frac{\beta\tau}{a_\alpha + b_\alpha\beta\tau}\\
&\leq \left(1 + \frac{\tau}{a_\alpha + b_\alpha\tau}\right)^\beta = \bigl[S_\alpha(\tau)\bigr]^\beta\biggr\}.
\end{aligned} \tag{8.6}$$

It is clear that (8.6) holds for $\beta = 1$. For $\beta > 1$ let

$$h_\alpha(\tau) = \left(1 + \frac{\tau}{a_\alpha + b_\alpha\tau}\right)^\beta - 1 - \frac{\beta\tau}{a_\alpha + b_\alpha\beta\tau}.$$

Observing that

$$\begin{aligned}
h'_\alpha(\tau) &= a_\alpha\beta\left[\left(1 + \frac{\tau}{a_\alpha + b_\alpha\tau}\right)^{\beta-1}\frac{1}{(a_\alpha + b_\alpha\tau)^2} - \frac{1}{(a_\alpha + b_\alpha\beta\tau)^2}\right],\\
h''_\alpha(\tau) &= a_\alpha\beta\left[(\beta-1)\left(1 + \frac{\tau}{a_\alpha + b_\alpha\tau}\right)^{\beta-2}\frac{a_\alpha}{(a_\alpha + b_\alpha\tau)^4}\right.\\
&\quad\left. - \left(1 + \frac{\tau}{a_\alpha + b_\alpha\tau}\right)^{\beta-1}\frac{2b_\alpha}{(a_\alpha + b_\alpha\tau)^3} + \frac{2b_\alpha\beta}{(a_\alpha + b_\alpha\beta\tau)^3}\right],\\
h'_\alpha(0) &= 0, \qquad h''_\alpha(0) = \beta(\beta-1)\frac{2b_\alpha + 1}{a_\alpha^2} > 0,
\end{aligned}$$

we conclude the proof (8.6) for $\beta > 1$.

Now, we prove conditions (A)–(F) for map S_α given by (8.5). The maps S_α are strictly increasing on $(0;\infty)$. Indeed, it is clear that

$$\forall_{t\in(0;\infty)}\left\{S'_\alpha(t) = \frac{a_\alpha(1 + t + b_\alpha)}{(1+t)[(a_\alpha + t + b_\alpha\ln(1+t)]^2} > 0\right\}.$$

Next we see that

$$\begin{aligned}
\forall_{t_1,t_2\in[0;\infty)}\Bigg\{ & S_\alpha(t_1+t_2) \\
&= 1+\frac{t_1+t_2+b_\alpha\ln(1+t_1+t_2)}{a_\alpha+t_1+t_2+b_\alpha\ln(1+t_1+t_2)} \\
&< 1+\frac{t_1+t_2+b_\alpha\ln(1+t_1+t_2+t_1\cdot t_2)}{a_\alpha+t_1+t_2+b_\alpha\ln(1+t_1+t_2)} \\
&= 1+\frac{t_1+t_2+b_\alpha\ln(1+t_1)+b_\alpha\ln(1+t_2)}{a_\alpha+t_1+t_2+b_\alpha\ln(1+t_1+t_2)} \\
&< 1+\frac{t_1+b_\alpha\ln(1+t_1)}{a_\alpha+t_1+b_\alpha\ln(1+t_1)}+\frac{t_2+b_\alpha\ln(1+t_2)}{a_\alpha+t_2+b_\alpha\ln(1+t_2)} \\
&< \left(1+\frac{t_1+b_\alpha\ln(1+t_1)}{a_\alpha+t_1+b_\alpha\ln(1+t_1)}\right)\cdot\left(1+\frac{t_2+b_\alpha\ln(1+t_2)}{a_\alpha+t_2+b_\alpha\ln(1+t_2)}\right) \\
&= S_\alpha(t_1)\cdot S_\alpha(t_2)\Bigg\}.
\end{aligned}$$

It remains to prove

$$\begin{aligned}
\forall_{\alpha\in\mathcal{A}}\forall_{\tau\in[0;\infty),\,\beta\in[1;\infty)}\Bigg\{ & S_\alpha(\beta\tau)=1+\frac{\beta\tau+b_\alpha\ln(1+\beta\tau)}{a_\alpha+\beta\tau+b_\alpha\ln(1+\beta\tau)} \\
&\leq\left[1+\frac{\tau+b_\alpha\ln(1+\tau)}{a_\alpha+\tau+b_\alpha\ln(1+\tau)}\right]^\beta=\left[S_\alpha(\tau)\right]^\beta\Bigg\}.
\end{aligned}\tag{8.7}$$

It is clear that (8.7) holds for $\beta=1$. For $\beta>1$ let

$$h_\alpha(t)=\left[1+\frac{t+b_\alpha\ln(1+t)}{a_\alpha+t+b_\alpha\ln(1+t)}\right]^\beta-1-\frac{\beta t+b_\alpha\ln(1+\beta t)}{a_\alpha+\beta t+b_\alpha\ln(1+\beta t)}.$$

Observing that

$$\begin{aligned}
& h'_\alpha(t) \\
&= a_\alpha\beta\Bigg\{\left[1+\frac{t+b_\alpha\ln(1+t)}{a_\alpha+t+b_\alpha\ln(1+t)}\right]^{\beta-1}\frac{a_\alpha(1+t+b_\alpha)}{[a_\alpha+t+b_\alpha\ln(1+t)]^2(1+t)} \\
&\quad-\left[1+\frac{t+b_\alpha\ln(1+t)}{a_\alpha+t+b_\alpha\ln(1+t)}\right]^{\beta-1}\frac{2(1+t+b_\alpha)^2+b_\alpha[a_\alpha+t+b_\alpha\ln(1+t)]}{[a_\alpha+t+b_\alpha\ln(1+t)]^3(1+t)^2} \\
&\quad+\beta\frac{2(1+\beta t+b_\alpha)^2+b_\alpha[a_\alpha+\beta t+b_\alpha\ln(1+\beta t)]}{[a_\alpha+b_\alpha\ln(1+\beta t)]^3(1+\beta t)^2}\Bigg\},
\end{aligned}$$

$$\begin{aligned}
& h''_\alpha(t) \\
&= a_\alpha\beta\Bigg\{(\beta-1)\left[1+\frac{t+b_\alpha\ln(1+t)}{a_\alpha+t+b_\alpha\ln(1+t)}\right]^{\beta-2}\frac{(1+t+b_\alpha)^2}{[a_\alpha+b_\alpha\ln(1+t)]^4(1+t)^2} \\
&\quad-\left[1+\frac{t+b_\alpha\ln(1+t)}{a_\alpha+t+b_\alpha\ln(1+t)}\right]^{\beta-1}\frac{2(1+t+b_\alpha)^2+b_\alpha[a_\alpha+t+b_\alpha\ln(1+t)]}{[a_\alpha+t+b_\alpha\ln(1+t)]^3(1+t)^2} \\
&\quad+\beta\frac{2(1+\beta t+b_\alpha)^2+b_\alpha[a_\alpha+\beta t+b_\alpha\ln(1+\beta t)]}{[a_\alpha+b_\alpha\ln(1+\beta t)]^3(1+\beta t)^2}\Bigg\},
\end{aligned}$$

$$h'_\alpha(0) = 0, \qquad h''_\alpha(0) = \beta(\beta - 1)\frac{1 + 2a_\alpha(1 + b_\alpha)^2 + a_\alpha^2 b_\alpha}{a_\alpha^3} > 0,$$

we conclude the proof (8.7) for $\beta > 1$. Therefore, (B), (E) and (F) hold. Clearly (A), (C) and (D) hold.

9 Convergence, periodic point, fixed point and endpoint theorems for set-valued and single-valued contractions of Leader type in $(X, \mathcal{P}_{C;\mathcal{A}})$

As a result, our arguments become versatile and can readily extend the study of contractions to more general settings (see Theorems 9.1–9.4).

Theorem 9.1 *Let $(X, \mathcal{P}_{C;\mathcal{A}})$ be a quasi-triangular space, and let (X, T) be a set-valued dynamic system, $T : X \to 2^X$. Assume that*:

(a) $\mathcal{J}_{C;\mathcal{A}} \in \mathbb{J}^L_{(X,\mathcal{P}_{C;\mathcal{A}})}$ (*respectively*, $\mathcal{J}_{C;\mathcal{A}} \in \mathbb{J}^R_{(X,\mathcal{P}_{C;\mathcal{A}})}$).

(b) (X, T) *is left* (*respectively, right*) $\mathcal{J}_{C;\mathcal{A}}$-*admissible on* X.

(c) $\mathcal{S}_\mathcal{A} = \{S_\alpha : \alpha \in \mathcal{A}\}$ *is an* $\mathcal{S}_\mathcal{A}$-*family.*

(d) *The* $\mathcal{S}_\mathcal{A}$-*family* $\mathcal{S}_\mathcal{A} = \{S_\alpha : \alpha \in \mathcal{A}\}$, (X, T) *and* $\mathcal{J}_{C;\mathcal{A}} = \{J_\alpha : \alpha \in \mathcal{A}\} \in \mathbb{J}^L_{(X,\mathcal{P}_{C;\mathcal{A}})}$ (*respectively*, $\mathcal{J}_{C;\mathcal{A}} = \{J_\alpha : \alpha \in \mathcal{A}\} \in \mathbb{J}^R_{(X,\mathcal{P}_{C;\mathcal{A}})}$) *satisfy*

$$\begin{cases} \forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>1}\exists_{\eta>1}\exists_{r\in\mathbb{N}}\forall_{x^0,y^0\in X}\forall_{(x^m:m\in\{0\}\cup\mathbb{N})\in\mathcal{O}_{X,T}(x^0)}\forall_{(y^m:m\in\{0\}\cup\mathbb{N})\in\mathcal{O}_{X,T}(y^0)} \\ \quad \forall_{s,l\in\mathbb{N}}\{S_\alpha(J_\alpha(x^s, y^l)) < \varepsilon \cdot \eta \Rightarrow [S_\alpha(J_\alpha(x^{s+r}, y^{l+r}))]^{C_\alpha} < \varepsilon\}. \end{cases} \tag{9.1}$$

Then the following hold:

(A) Convergence of all dynamic processes. *For each point $w^0 \in X$ and for each dynamic process $(w^m : m \in \{0\} \cup \mathbb{N}) \in \mathcal{O}_{X,T}(w^0)$ starting at w^0, we have*

$$\varnothing \neq \mathrm{LIM}^{L-\mathcal{J}_{C;\mathcal{A}}}_{(w^m:m\in\{0\}\cup\mathbb{N})} \subset \mathrm{LIM}^{L-\mathcal{P}_{C;\mathcal{A}}}_{(w^m:m\in\{0\}\cup\mathbb{N})} \tag{9.2}$$

where $\mathcal{J}_{C;\mathcal{A}} \in \mathbb{J}^L_{(X,\mathcal{P}_{C;\mathcal{A}})}$ (*respectively*,

$$\varnothing \neq \mathrm{LIM}^{R-\mathcal{J}_{C;\mathcal{A}}}_{(w^m:m\in\{0\}\cup\mathbb{N})} \subset \mathrm{LIM}^{R-\mathcal{P}_{C;\mathcal{A}}}_{(w^m:m\in\{0\}\cup\mathbb{N})} \tag{9.3}$$

where $\mathcal{J}_{C;\mathcal{A}} \in \mathbb{J}^R_{(X,\mathcal{P}_{C;\mathcal{A}})}$).

(B) Existence of periodic points of all dynamic processes. *If there exists $q \in \mathbb{N}$ such that the set-valued dynamic system $(X, T^{[q]})$ is left* (*respectively, right*) *$\mathcal{J}_{C;\mathcal{A}}$-closed on X, then*

$$\mathrm{Fix}_X\big(T^{[q]}\big) \neq \varnothing. \tag{9.4}$$

Moreover, for each point $w^0 \in X$ and for each dynamic process $(w^m : m \in \{0\} \cup \mathbb{N}) \in \mathcal{O}_{X,T}(w^0)$ starting at w^0, we see that there exists a point $u \in \mathrm{Fix}_X(T^{[q]})$ (*respectively*, $v \in \mathrm{Fix}_X(T^{[q]})$) *such that*

$$u \in \mathrm{LIM}^{L-\mathcal{J}_{C;\mathcal{A}}}_{(w^m:m\in\{0\}\cup\mathbb{N})} \subset \mathrm{LIM}^{L-\mathcal{P}_{C;\mathcal{A}}}_{(w^m:m\in\{0\}\cup\mathbb{N})} \tag{9.5}$$

where $\mathcal{J}_{C;\mathcal{A}} \in \mathbb{J}^L_{(X,\mathcal{P}_{C;\mathcal{A}})}$ (*respectively*,

$$v \in \mathrm{LIM}^{R-\mathcal{J}_{C;\mathcal{A}}}_{(w^m:m\in\{0\}\cup\mathbb{N})} \subset \mathrm{LIM}^{R-\mathcal{P}_{C;\mathcal{A}}}_{(w^m:m\in\{0\}\cup\mathbb{N})} \tag{9.6}$$

where $\mathcal{J}_{C;\mathcal{A}} \in \mathbb{J}^{R}_{(X,\mathcal{P}_{C;\mathcal{A}})}$).

(C) Existence of unique endpoint of all dynamic processes. *If the set-valued dynamic system* (X,T) *is left* (*respectively, right*) $\mathcal{J}_{C;\mathcal{A}}$*-closed on* X *and the family* $\mathcal{P}_{C;\mathcal{A}} = \{P_\alpha : \alpha \in \mathcal{A}\}$ *is separating on* X, *then there exists a unique endpoint* w *of* T *in* X, *i.e.*

$$\mathrm{End}_X(T) = \{w \in X : \{w\} = T(w)\} = \{w\}, \tag{9.7}$$

satisfying

$$\forall_{\alpha\in\mathcal{A}}\{J_\alpha(w,w) = 0\}. \tag{9.8}$$

Furthermore, for each $w^0 \in X$ *and for each dynamic process* $(w^m : m \in \{0\} \cup \mathbb{N}) \in \mathcal{O}_{X,T}(w^0)$ *starting at* w^0, *we have*

$$w \in \mathrm{LIM}^{L-\mathcal{J}_{C;\mathcal{A}}}_{(w^m:m\in\{0\}\cup\mathbb{N})} \subset \mathrm{LIM}^{L-\mathcal{P}_{C;\mathcal{A}}}_{(w^m:m\in\{0\}\cup\mathbb{N})} \tag{9.9}$$

where $\mathcal{J}_{C;\mathcal{A}} \in \mathbb{J}^{L}_{(X,\mathcal{P}_{C;\mathcal{A}})}$ (*respectively,*

$$w \in \mathrm{LIM}^{R-\mathcal{J}_{C;\mathcal{A}}}_{(w^m:m\in\{0\}\cup\mathbb{N})} \subset \mathrm{LIM}^{R-\mathcal{P}_{C;\mathcal{A}}}_{(w^m:m\in\{0\}\cup\mathbb{N})} \tag{9.10}$$

where $\mathcal{J}_{C;\mathcal{A}} \in \mathbb{J}^{R}_{(X,\mathcal{P}_{C;\mathcal{A}})}$).

Theorem 9.2 *Let* $(X,\mathcal{P}_{C;\mathcal{A}})$ *be a quasi-triangular space, and let* (X,T) *be a set-valued dynamic system,* $T : X \to 2^X$. *Assume that*:

(a) $\mathcal{J}_{C;\mathcal{A}} \in \mathbb{J}^{L}_{(X,\mathcal{P}_{C;\mathcal{A}})}$ (*respectively,* $\mathcal{J}_{C;\mathcal{A}} \in \mathbb{J}^{R}_{(X,\mathcal{P}_{C;\mathcal{A}})}$).
(b) *There exists* $M \in 2^X$ *such that* (X,T) *is left* (*respectively, right*) $\mathcal{J}_{C;\mathcal{A}}$*-admissible on* M.
(c) $\mathcal{S}_{\mathcal{A}} = \{S_\alpha : \alpha \in \mathcal{A}\}$ *is an* $\mathcal{S}_{\mathcal{A}}$*-family.*
(d) *The* $\mathcal{S}_{\mathcal{A}}$*-family* $\mathcal{S}_{\mathcal{A}} = \{S_\alpha : \alpha \in \mathcal{A}\}$, (X,T), M *and* $\mathcal{J}_{C;\mathcal{A}} = \{J_\alpha : \alpha \in \mathcal{A}\} \in \mathbb{J}^{L}_{(X,\mathcal{P}_{C;\mathcal{A}})}$ (*respectively,* $\mathcal{J}_{C;\mathcal{A}} = \{J_\alpha : \alpha \in \mathcal{A}\} \in \mathbb{J}^{R}_{(X,\mathcal{P}_{C;\mathcal{A}})}$) *satisfy*

$$\begin{cases} \forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>1}\exists_{\eta>1}\exists_{r\in\mathbb{N}}\forall_{w^0\in M}\forall_{(w^m:m\in\{0\}\cup\mathbb{N})\in\mathcal{O}_{X,T}(w^0)}\forall_{s,l\in\mathbb{N}} \\ \quad \{S_\alpha(J_\alpha(w^s,w^l)) < \varepsilon\cdot\eta \Rightarrow [S_\alpha(J_\alpha(w^{s+r},w^{l+r}))]^{C_\alpha} < \varepsilon\}. \end{cases} \tag{9.11}$$

Then the following hold:

(A) Convergence of dynamic processes. *For each point* $w^0 \in M$ *and for each dynamic process* $(w^m : m \in \{0\} \cup \mathbb{N}) \in \mathcal{O}_{X,T}(w^0)$ *starting at* w^0, *we have*

$$\emptyset \neq \mathrm{LIM}^{L-\mathcal{J}_{C;\mathcal{A}}}_{(w^m:m\in\{0\}\cup\mathbb{N})} \subset \mathrm{LIM}^{L-\mathcal{P}_{C;\mathcal{A}}}_{(w^m:m\in\{0\}\cup\mathbb{N})} \quad \text{where } \mathcal{J}_{C;\mathcal{A}} \in \mathbb{J}^{L}_{(X,\mathcal{P}_{C;\mathcal{A}})}$$

(*respectively,*

$$\emptyset \neq \mathrm{LIM}^{R-\mathcal{J}_{C;\mathcal{A}}}_{(w^m:m\in\{0\}\cup\mathbb{N})} \subset \mathrm{LIM}^{R-\mathcal{P}_{C;\mathcal{A}}}_{(w^m:m\in\{0\}\cup\mathbb{N})} \quad \text{where } \mathcal{J}_{C;\mathcal{A}} \in \mathbb{J}^{R}_{(X,\mathcal{P}_{C;\mathcal{A}})}).$$

(B) Existence of periodic points of dynamic processes. *If there exist $q \in \mathbb{N}$ and $w^0 \in M$ such that the set-valued dynamic system $(X, T^{[q]})$ is left (respectively, right) $\mathcal{J}_{C;\mathcal{A}}$-closed in a point w^0, then*

$$\operatorname{Fix}_X\bigl(T^{[q]}\bigr) \neq \emptyset.$$

Moreover, for each dynamic process $(w^m : m \in \{0\} \cup \mathbb{N}) \in \mathcal{O}_{X,T}(w^0)$ starting at w^0, there exists a point $u \in \operatorname{Fix}_X(T^{[q]})$ (respectively, $v \in \operatorname{Fix}_X(T^{[q]})$) such that

$$u \in \operatorname{LIM}^{L-\mathcal{J}_{C;\mathcal{A}}}_{(w^m:m\in\{0\}\cup\mathbb{N})} \subset \operatorname{LIM}^{L-\mathcal{P}_{C;\mathcal{A}}}_{(w^m:m\in\{0\}\cup\mathbb{N})} \quad \text{where } \mathcal{J}_{C;\mathcal{A}} \in \mathbb{J}^L_{(X,\mathcal{P}_{C;\mathcal{A}})}$$

(respectively,

$$v \in \operatorname{LIM}^{R-\mathcal{J}_{C;\mathcal{A}}}_{(w^m:m\in\{0\}\cup\mathbb{N})} \subset \operatorname{LIM}^{R-\mathcal{P}_{C;\mathcal{A}}}_{(w^m:m\in\{0\}\cup\mathbb{N})} \quad \text{where } \mathcal{J}_{C;\mathcal{A}} \in \mathbb{J}^R_{(X,\mathcal{P}_{C;\mathcal{A}})}).$$

Theorem 9.3 *Let $(X, \mathcal{P}_{C;\mathcal{A}})$ be a quasi-triangular space, and let (X, T) be a single-valued dynamic system, $T : X \to X$. Assume that:*

(a) $\mathcal{J}_{C;\mathcal{A}} \in \mathbb{J}^L_{(X,\mathcal{P}_{C;\mathcal{A}})}$ *(respectively,* $\mathcal{J}_{C;\mathcal{A}} \in \mathbb{J}^R_{(X,\mathcal{P}_{C;\mathcal{A}})}$*).*
(b) *(X, T) is left (respectively, right) $\mathcal{J}_{C;\mathcal{A}}$-admissible on X.*
(c) *$\mathcal{S}_{\mathcal{A}} = \{S_\alpha : \alpha \in \mathcal{A}\}$ is an $\mathcal{S}_{\mathcal{A}}$-family.*
(d) *The $\mathcal{S}_{\mathcal{A}}$-family $\mathcal{S}_{\mathcal{A}} = \{S_\alpha : \alpha \in \mathcal{A}\}$, (X, T) and $\mathcal{J}_{C;\mathcal{A}} = \{J_\alpha : \alpha \in \mathcal{A}\} \in \mathbb{J}^L_{(X,\mathcal{P}_{C;\mathcal{A}})}$ (respectively, $\mathcal{J}_{C;\mathcal{A}} = \{J_\alpha : \alpha \in \mathcal{A}\} \in \mathbb{J}^R_{(X,\mathcal{P}_{C;\mathcal{A}})}$) satisfy*

$$\begin{cases} \forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>1}\exists_{\eta>1}\exists_{r\in\mathbb{N}}\forall_{x,y\in X}\forall_{s,l\in\mathbb{N}}\{S_\alpha(J_\alpha(T^{[s]}(x), T^{[l]}(y))) \\ \quad < \varepsilon \cdot \eta \Rightarrow [S_\alpha(J_\alpha(T^{[s+r]}(x), T^{[l+r]}(y)))]^{C_\alpha} < \varepsilon\}. \end{cases} \tag{9.12}$$

Then the following hold:

(A) Convergence of all Picard iterations. *For each point $w^0 \in X$, we have*

$$\emptyset \neq \operatorname{LIM}^{L-\mathcal{J}_{C;\mathcal{A}}}_{(T^{[m]}(w^0):m\in\{0\}\cup\mathbb{N})} \subset \operatorname{LIM}^{L-\mathcal{P}_{C;\mathcal{A}}}_{(T^{[m]}(w^0):m\in\{0\}\cup\mathbb{N})} \tag{9.13}$$

where $\mathcal{J}_{C;\mathcal{A}} \in \mathbb{J}^L_{(X,\mathcal{P}_{C;\mathcal{A}})}$ (respectively,

$$\emptyset \neq \operatorname{LIM}^{R-\mathcal{J}_{C;\mathcal{A}}}_{(T^{[m]}(w^0):m\in\{0\}\cup\mathbb{N})} \subset \operatorname{LIM}^{R-\mathcal{P}_{C;\mathcal{A}}}_{(T^{[m]}(w^0):m\in\{0\}\cup\mathbb{N})} \tag{9.14}$$

where $\mathcal{J}_{C;\mathcal{A}} \in \mathbb{J}^R_{(X,\mathcal{P}_{C;\mathcal{A}})}$).

(B) Existence of periodic points of all Picard iterations. *If there exists $q \in \mathbb{N}$ such that the single-valued dynamic system $(X, T^{[q]})$ is left (respectively, right) $\mathcal{J}_{C;\mathcal{A}}$-closed on X, then*

$$\operatorname{Fix}_X\bigl(T^{[q]}\bigr) \neq \emptyset. \tag{9.15}$$

Moreover, for each point $w^0 \in X$, there exists a point $u \in \operatorname{Fix}_X(T^{[q]})$ (respectively, $v \in \operatorname{Fix}_X(T^{[q]})$) such that

$$u \in \operatorname{LIM}^{L-\mathcal{J}_{C;\mathcal{A}}}_{(T^{[m]}(w^0):m\in\{0\}\cup\mathbb{N})} \subset \operatorname{LIM}^{L-\mathcal{P}_{C;\mathcal{A}}}_{(T^{[m]}(w^0):m\in\{0\}\cup\mathbb{N})} \tag{9.16}$$

and

$$\forall_{\alpha\in\mathcal{A}}\forall_{n\in\{1,2,\dots,q\}}\left\{J_\alpha\left(u,T^{[n]}(u)\right)=J_\alpha\left(T^{[n]}(u),u\right)=0\right\} \tag{9.17}$$

where $\mathcal{J}_{C;\mathcal{A}}\in\mathbb{J}^L_{(X,\mathcal{P}_{C;\mathcal{A}})}$ (*respectively,*

$$v\in \mathrm{LIM}^{R-\mathcal{J}_{C;\mathcal{A}}}_{(T^{[m]}(w^0):m\in\{0\}\cup\mathbb{N})}\subset \mathrm{LIM}^{R-\mathcal{P}_{C;\mathcal{A}}}_{(T^{[m]}(w^0):m\in\{0\}\cup\mathbb{N})} \tag{9.18}$$

and

$$\forall_{\alpha\in\mathcal{A}}\forall_{n\in\{1,2,\dots,q\}}\left\{J_\alpha\left(v,T^{[n]}(v)\right)=J_\alpha\left(T^{[n]}(v),v\right)=0\right\} \tag{9.19}$$

where $\mathcal{J}_{C;\mathcal{A}}\in\mathbb{J}^R_{(X,\mathcal{P}_{C;\mathcal{A}})}$).

(C) Existence of unique fixed point of all Picard iterations. *If* (X,T) *is left* (*respectively, right*) $\mathcal{J}_{C;\mathcal{A}}$*-closed on* X *and the family* $\mathcal{P}_{C;\mathcal{A}}=\{P_\alpha:\alpha\in\mathcal{A}\}$ *is separating on* X, *then*

$$\exists_{w\in X}\left\{\mathrm{Fix}_X(T)=\{w\}\right\}. \tag{9.20}$$

Moreover, for each $w^0\in X$,

$$w\in \mathrm{LIM}^{L-\mathcal{J}_{C;\mathcal{A}}}_{(T^{[m]}(w^0):m\in\{0\}\cup\mathbb{N})}\subset \mathrm{LIM}^{L-\mathcal{P}_{C;\mathcal{A}}}_{(T^{[m]}(w^0):m\in\{0\}\cup\mathbb{N})} \tag{9.21}$$

and

$$\forall_{\alpha\in\mathcal{A}}\left\{J_\alpha(w,w)=0\right\} \tag{9.22}$$

where $\mathcal{J}_{C;\mathcal{A}}\in\mathbb{J}^L_{(X,\mathcal{P}_{C;\mathcal{A}})}$ (*respectively,*

$$w\in \mathrm{LIM}^{R-\mathcal{J}_{C;\mathcal{A}}}_{(T^{[m]}(w^0):m\in\{0\}\cup\mathbb{N})}\subset \mathrm{LIM}^{R-\mathcal{P}_{C;\mathcal{A}}}_{(T^{[m]}(w^0):m\in\{0\}\cup\mathbb{N})} \tag{9.23}$$

and

$$\forall_{\alpha\in\mathcal{A}}\left\{J_\alpha(w,w)=0\right\} \tag{9.24}$$

where $\mathcal{J}_{C;\mathcal{A}}\in\mathbb{J}^R_{(X,\mathcal{P}_{C;\mathcal{A}})}$).

Theorem 9.4 *Let* $(X,\mathcal{P}_{C;\mathcal{A}})$ *be a quasi-triangular space, and let* (X,T) *be a single-valued dynamic system,* $T:X\to X$. *Assume that*:

(a) $\mathcal{J}_{C;\mathcal{A}}\in\mathbb{J}^L_{(X,\mathcal{P}_{C;\mathcal{A}})}$ (*respectively,* $\mathcal{J}_{C;\mathcal{A}}\in\mathbb{J}^R_{(X,\mathcal{P}_{C;\mathcal{A}})}$).
(b) *There exists* $M\in 2^X$ *such that* (X,T) *is left* (*respectively, right*) $\mathcal{J}_{C;\mathcal{A}}$*-admissible on* M.
(c) $\mathcal{S}_\mathcal{A}=\{S_\alpha:\alpha\in\mathcal{A}\}$ *is an* $\mathcal{S}_\mathcal{A}$*-family.*
(d) *The* $\mathcal{S}_\mathcal{A}$*-family* $\mathcal{S}_\mathcal{A}=\{S_\alpha:\alpha\in\mathcal{A}\}$, (X,T), M *and* $\mathcal{J}_{C;\mathcal{A}}=\{J_\alpha:\alpha\in\mathcal{A}\}\in\mathbb{J}^L_{(X,\mathcal{P}_{C;\mathcal{A}})}$ (*respectively,* $\mathcal{J}_{C;\mathcal{A}}=\{J_\alpha:\alpha\in\mathcal{A}\}\in\mathbb{J}^R_{(X,\mathcal{P}_{C;\mathcal{A}})}$) *satisfy*

$$\begin{cases}\forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>1}\exists_{\eta>1}\exists_{r\in\mathbb{N}}\forall_{w^0\in M}\forall_{s,l\in\mathbb{N}}\{S_\alpha(J_\alpha(T^{[s]}(w^0),T^{[l]}(w^0)))\\ \quad<\varepsilon\cdot\eta\Rightarrow[S_\alpha(J_\alpha(T^{[s+r]}(w^0),T^{[l+r]}(w^0)))]^{C_\alpha}<\varepsilon\}.\end{cases} \tag{9.25}$$

Then the following hold:

(A) Convergence of Picard iterations. *For each point* $w^0 \in M$, *we have*

$$\emptyset \neq \mathrm{LIM}^{L-\mathcal{J}_{C;\mathcal{A}}}_{(T^{[m]}(w^0):m\in\{0\}\cup\mathbb{N})} \subset \mathrm{LIM}^{L-\mathcal{P}_{C;\mathcal{A}}}_{(T^{[m]}(w^0):m\in\{0\}\cup\mathbb{N})} \quad \textit{where } \mathcal{J}_{C;\mathcal{A}} \in \mathbb{J}^{L}_{(X,\mathcal{P}_{C;\mathcal{A}})}$$

(*respectively*,

$$\emptyset \neq \mathrm{LIM}^{R-\mathcal{J}_{C;\mathcal{A}}}_{(T^{[m]}(w^0):m\in\{0\}\cup\mathbb{N})} \subset \mathrm{LIM}^{R-\mathcal{P}_{C;\mathcal{A}}}_{(T^{[m]}(w^0):m\in\{0\}\cup\mathbb{N})} \quad \textit{where } \mathcal{J}_{C;\mathcal{A}} \in \mathbb{J}^{R}_{(X,\mathcal{P}_{C;\mathcal{A}})}).$$

(B) Existence of periodic points of Picard iterations. *If there exist* $q \in \mathbb{N}$ *and* $w^0 \in M$ *such that the single-valued dynamic system* $(X, T^{[q]})$ *is left* (*respectively*, *right*) $\mathcal{J}_{C;\mathcal{A}}$-*closed in a point* w^0, *then*

$$\mathrm{Fix}_X\big(T^{[q]}\big) \neq \emptyset.$$

Moreover, *there exists a point* $u \in \mathrm{Fix}_X(T^{[q]})$ (*respectively*, $v \in \mathrm{Fix}_X(T^{[q]})$) *such that*

$$u \in \mathrm{LIM}^{L-\mathcal{J}_{C;\mathcal{A}}}_{(T^{[m]}(w^0):m\in\{0\}\cup\mathbb{N})} \subset \mathrm{LIM}^{L-\mathcal{P}_{C;\mathcal{A}}}_{(T^{[m]}(w^0):m\in\{0\}\cup\mathbb{N})}$$

and

$$\forall_{\alpha\in\mathcal{A}}\forall_{n\in\{1,2,\ldots,q\}}\big\{J_\alpha\big(u, T^{[n]}(u)\big) = J_\alpha\big(T^{[n]}(u), u\big) = 0\big\}$$

where $\mathcal{J}_{C;\mathcal{A}} \in \mathbb{J}^{L}_{(X,\mathcal{P}_{C;\mathcal{A}})}$ (*respectively*,

$$v \in \mathrm{LIM}^{R-\mathcal{J}_{C;\mathcal{A}}}_{(T^{[m]}(w^0):m\in\{0\}\cup\mathbb{N})} \subset \mathrm{LIM}^{R-\mathcal{P}_{C;\mathcal{A}}}_{(T^{[m]}(w^0):m\in\{0\}\cup\mathbb{N})}$$

and

$$\forall_{\alpha\in\mathcal{A}}\forall_{n\in\{1,2,\ldots,q\}}\big\{J_\alpha\big(v, T^{[n]}(v)\big) = J_\alpha\big(T^{[n]}(v), v\big) = 0\big\}$$

where $\mathcal{J}_{C;\mathcal{A}} \in \mathbb{J}^{R}_{(X,\mathcal{P}_{C;\mathcal{A}})}$).

Definition 9.1 Let $(X, \mathcal{P}_{C;\mathcal{A}})$ be a quasi-triangular space. If assumptions (a)–(d) of Theorems 9.1, 9.2, 9.3 or 9.4 hold, then a dynamic system (X, T) we call *admissible* $(\mathcal{S}_{\mathcal{A}}, \mathcal{J}_{C;\mathcal{A}})$-*contraction* in $(X, \mathcal{P}_{C;\mathcal{A}})$.

Remark 9.1 Let $(X, \mathcal{P}_{C;\mathcal{A}})$ be a quasi-triangular space. Assume that $\mathcal{J}_{C;\mathcal{A}} \in \mathbb{J}^{L}_{(X,\mathcal{P}_{C;\mathcal{A}})} \cup \mathbb{J}^{R}_{(X,\mathcal{P}_{C;\mathcal{A}})}$; thus, in particular, $\mathcal{J}_{C;\mathcal{A}} = \mathcal{P}_{C;\mathcal{A}}$.

(A) If $(X, \mathcal{P}_{C;\mathcal{A}})$ is left (respectively, right) $\mathcal{J}_{C;\mathcal{A}}$-sequentially complete, then each (set-valued and single-valued) dynamic system (X, T) is left (respectively, right) $\mathcal{J}_{C;\mathcal{A}}$-admissible on each $M \in 2^X$, i.e. then hypothesis (b) of Theorems 9.1–9.4 automatically holds.

(B) In general, $(X, \mathcal{P}_{C;\mathcal{A}})$ are not necessarily left (respectively, right) $\mathcal{J}_{C;\mathcal{A}}$-Hausdorff or left (respectively, right) $\mathcal{J}_{C;\mathcal{A}}$-sequentially complete. Convergence, periodic point, endpoint and fixed point Theorems 9.1–9.4 presented above are without these properties.

10 Proofs of Theorems 9.1–9.4

In the sequel, for $w^0 \in X$, let $(w^m : m \in \{0\} \cup \mathbb{N})$ has property $(w^m : m \in \{0\} \cup \mathbb{N}) \in \mathcal{O}_{X,T}(w^0)$ when (X, T) is set-valued or $(w^m : m \in \{0\} \cup \mathbb{N})$ is of the form $(w^m = T^{[m]}(w^0) : m \in \{0\} \cup \mathbb{N})$ when (X, T) is single-valued. Moreover, we assert that hypotheses (a)–(d) of Theorems 9.1–9.4 hold. This means that the hypotheses of the auxiliary Propositions 10.1–10.3 hold.

Proposition 10.1 *Let* $\mathcal{J}_{C;\mathcal{A}} \in \mathbb{J}^L_{(X,\mathcal{P}_{C;\mathcal{A}})} \cup \mathbb{J}^R_{(X,\mathcal{P}_{C;\mathcal{A}})}$. *Let* $w^0 \in X$ *be such that* $\forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>1}\exists_{\eta>1}\exists_{r\in\mathbb{N}}\forall_{s,l\in\mathbb{N}}\{S_\alpha(J_\alpha(w^s, w^l)) < \varepsilon \cdot \eta \Rightarrow [S_\alpha(J_\alpha(w^{s+r}, w^{l+r}))]^{C_\alpha} < \varepsilon\}$. *For each* $\alpha \in \mathcal{A}$ *and* $k \in \mathbb{N}$, *define*

$$\delta_{(\mathcal{S}_\mathcal{A},\mathcal{J}_{C;\mathcal{A}});\alpha,k}\big(w^0\big) = \inf\big\{\Delta_{(\mathcal{S}_\mathcal{A},\mathcal{J}_{C;\mathcal{A}});\alpha,k}\big(w^0, n\big) : n \in \mathbb{N}\big\} \tag{10.1}$$

where

$$\Delta_{(\mathcal{S}_\mathcal{A},\mathcal{J}_{C;\mathcal{A}});\alpha,k}\big(w^0, n\big) = \max\big\{S_\alpha\big(J_\alpha\big(w^s, w^l\big)\big) : n \le s, l \le n + k\big\}, \quad n \in \mathbb{N}. \tag{10.2}$$

The following holds:

$$\forall_{\alpha\in\mathcal{A}}\forall_{k\in\mathbb{N}}\big\{\delta_{(\mathcal{S}_\mathcal{A},\mathcal{J}_{C;\mathcal{A}});\alpha,k}\big(w^0\big) = 1\big\}. \tag{10.3}$$

Proof of Proposition 10.1 Suppose on the contrary that (10.3) is false. Then, by (10.1),

$$\exists_{\alpha_0\in\mathcal{A}}\exists_{k_0\in\mathbb{N}}\exists_{\varepsilon_0>1}\big\{\delta_{(\mathcal{S}_\mathcal{A},\mathcal{J}_{C;\mathcal{A}});\alpha_0,k_0}\big(w^0\big) = \varepsilon_0\big\} \tag{10.4}$$

where

$$\delta_{(\mathcal{S}_\mathcal{A},\mathcal{J}_{C;\mathcal{A}});\alpha_0,k_0}\big(w^0\big) = \inf\big\{\Delta_{(\mathcal{S}_\mathcal{A},\mathcal{J}_{C;\mathcal{A}});\alpha_0,k_0}\big(w^0, n\big) : n \in \mathbb{N}\big\}. \tag{10.5}$$

Observe that with this choice of α_0 and ε_0, we can use hypothesis and then there exist $\eta_0 > 1$ and $r_0 \in \mathbb{N}$ such that

$$\forall_{s,l\in\mathbb{N}}\big\{S_{\alpha_0}\big(J_{\alpha_0}\big(w^s, w^l\big)\big) < \varepsilon_0 \cdot \eta_0 \Rightarrow \big[S_{\alpha_0}\big(J_{\alpha_0}\big(w^{s+r_0}, w^{l+r_0}\big)\big)\big]^{C_{\alpha_0}} < \varepsilon_0\big\}. \tag{10.6}$$

Also observe that with this choice of α_0, k_0, ε_0 and η_0 from (10.4) and (10.5) we get

$$\exists_{n_0\in\mathbb{N}}\big\{\delta_{(\mathcal{S}_\mathcal{A},\mathcal{J}_{C;\mathcal{A}});\alpha_0,k_0}\big(w^0\big) = \varepsilon_0 \le \Delta_{(\mathcal{S}_\mathcal{A},\mathcal{J}_{C;\mathcal{A}});\alpha_0,k_0}\big(w^0, n_0\big) < \varepsilon_0 \cdot \eta_0\big\}. \tag{10.7}$$

Naturally (10.7) gives

$$\forall_{n_0\le s,l\le n_0+k_0}\big\{S_{\alpha_0}\big(J_{\alpha_0}\big(w^s, w^l\big)\big) < \varepsilon_0 \cdot \eta_0\big\} \tag{10.8}$$

since, by (10.2), $\Delta_{(\mathcal{S}_\mathcal{A},\mathcal{J}_{C;\mathcal{A}});\alpha_0,k_0}(w^0, n_0) = \max\{S_{\alpha_0}(J_{\alpha_0}(w^s, w^l)) : n_0 \le s, l \le n_0 + k_0\}$. Now, from (10.6) and (10.8) one gets

$$\forall_{n_0+r_0\le s,l\le n_0+r_0+k_0}\big\{\big[S_{\alpha_0}\big(J_{\alpha_0}\big(w^s, w^l\big)\big)\big]^{C_{\alpha_0}} < \varepsilon_0\big\}. \tag{10.9}$$

Next, since $\Delta_{(\mathcal{S}_{\mathcal{A}},\mathcal{J}_{C;\mathcal{A}});\alpha_0,k_0}(w^0, n_0 + r_0) = \max\{S_{\alpha_0}(J_{\alpha_0}(w^s, w^l)) : n_0 + r_0 \le s, l \le n_0 + r_0 + k_0\}$, therefore (10.9) implies

$$\left[\Delta_{(\mathcal{S}_{\mathcal{A}},\mathcal{J}_{C;\mathcal{A}});\alpha_0,k_0}\left(w^0, n_0 + r_0\right)\right)\right]^{C_{\alpha_0}} < \varepsilon_0. \tag{10.10}$$

However, by (10.4), (10.5) and (10.10), we obtain

$$\begin{aligned}\varepsilon_0 &= \delta_{(\mathcal{S}_{\mathcal{A}},\mathcal{J}_{C;\mathcal{A}});\alpha_0,k_0}\left(w^0\right) = \inf\left\{\Delta_{(\mathcal{S}_{\mathcal{A}},\mathcal{J}_{C;\mathcal{A}});\alpha_0,k_0}\left(w^0, n\right) : n \in \mathbb{N}\right\} \\ &\le \Delta_{(\mathcal{S}_{\mathcal{A}},\mathcal{J}_{C;\mathcal{A}});\alpha_0,k_0}\left(w^0, n_0 + r_0\right) \le \left[\Delta_{(\mathcal{S}_{\mathcal{A}},\mathcal{J}_{C;\mathcal{A}});\alpha_0,k_0}\left(w^0, n_0 + r_0\right)\right)\right]^{C_{\alpha_0}} < \varepsilon_0\end{aligned}$$

which is impossible. Therefore, (10.3) holds. □

Proposition 10.2 *Let $\mathcal{J}_{C;\mathcal{A}} \in \mathbb{J}^L_{(X,\mathcal{P}_{C;\mathcal{A}})} \cup \mathbb{J}^R_{(X,\mathcal{P}_{C;\mathcal{A}})}$. Let $\alpha_0 \in \mathcal{A}$, $\varepsilon_0 > 1$ and $w^0 \in X$ be such that*

$$\begin{aligned}&\exists_{\eta_0>1}\exists_{r_0\in\mathbb{N}}\forall_{s,l\in\mathbb{N}}\left\{S_{\alpha_0}\left(J_{\alpha_0}\left(w^s, w^l\right)\right) < \varepsilon_0 \cdot \eta_0\right. \\ &\quad\left.\Rightarrow \left[S_{\alpha_0}\left(J_{\alpha_0}\left(w^{s+r_0}, w^{l+r_0}\right)\right)\right]^{C_{\alpha_0}} < \varepsilon_0\right\}.\end{aligned} \tag{10.11}$$

Then the following hold:

$$\exists_{n_0\in\mathbb{N}}\left\{\left[\Delta_{(\mathcal{S}_{\mathcal{A}},\mathcal{J}_{C;\mathcal{A}});\alpha_0,r_0}\left(w^0, n_0\right)\right]^{C_{\alpha_0}} < \min\{\varepsilon_0, \eta_0\}\right\} \tag{10.12}$$

and

$$\forall_{s,l\ge n_0}\left\{S_{\alpha_0}\left(J_{\alpha_0}\left(w^s, w^l\right)\right) < \varepsilon_0^2\right\}. \tag{10.13}$$

Here

$$\begin{aligned}&\left[\Delta_{(\mathcal{S}_{\mathcal{A}},\mathcal{J}_{C;\mathcal{A}});\alpha_0,r_0}\left(w^0, n_0\right)\right]^{C_{\alpha_0}} \\ &\quad= \max\left\{\left[S_{\alpha_0}\left(J_{\alpha_0}\left(w^s, w^l\right)\right)\right]^{C_{\alpha_0}} : n_0 \le s, l \le n_0 + r_0\right\}.\end{aligned} \tag{10.14}$$

Proof of Proposition 10.2 Indeed, let α_0, ε_0, w^0, η_0 and r_0 be as in (10.11).

We prove that (10.12) holds. From (10.3), for α_0, w^0 and $k_0 = r_0$, we get

$$\delta_{(\mathcal{S}_{\mathcal{A}},\mathcal{J}_{C;\mathcal{A}});\alpha_0,r_0}\left(w^0\right) = \inf\left\{\Delta_{(\mathcal{S}_{\mathcal{A}},\mathcal{J}_{C;\mathcal{A}});\alpha_0,r_0}\left(w^0, n\right) : n \in \mathbb{N}\right\} = 1 \tag{10.15}$$

where

$$\begin{aligned}&\Delta_{(\mathcal{S}_{\mathcal{A}},\mathcal{J}_{C;\mathcal{A}});\alpha_0,r_0}\left(w^0, n\right) \\ &\quad= \max\left\{S_{\alpha_0}\left(J_{\alpha_0}\left(w^s, w^l\right)\right) : n \le s, l \le n + r_0\right\}, \quad n \in \mathbb{N}.\end{aligned} \tag{10.16}$$

Observe that (10.15) and (10.16) imply

$$\inf\left\{\left[\Delta_{(\mathcal{S}_{\mathcal{A}},\mathcal{J}_{C;\mathcal{A}});\alpha_0,r_0}\left(w^0, n\right)\right]^{C_{\alpha_0}} : n \in \mathbb{N}\right\} = 1 \tag{10.17}$$

where

$$\begin{aligned}&\left[\Delta_{(\mathcal{S}_{\mathcal{A}},\mathcal{J}_{C;\mathcal{A}});\alpha_0,r_0}\left(w^0,n\right)\right]^{C_{\alpha_0}}\\&\quad=\max\left\{\left[S_{\alpha_0}\left(J_{\alpha_0}\left(w^s,w^l\right)\right)\right]^{C_{\alpha_0}}:n\le s,l\le n+r_0\right\},\quad n\in\mathbb{N}.\end{aligned}\tag{10.18}$$

Now, from (10.17) and (10.18), using the fact that $\min\{\varepsilon_0,\eta_0\}>1$ one gets (10.12).

Now we prove that (10.13) holds. First, we establish that

$$\forall_{l\ge n_0}\left\{\left[S_{\alpha_0}\left(J_{\alpha_0}\left(w^{n_0+r_0},w^l\right)\right)\right]^{C_{\alpha_0}}<\varepsilon_0\right\}.\tag{10.19}$$

If (10.19) is false, then $\exists_{l\ge n_0}\{[S_{\alpha_0}(J_{\alpha_0}(w^{n_0+r_0},w^l))]^{C_{\alpha_0}}\ge\varepsilon_0\}$; in other words,

$$L=\left\{l\in\mathbb{N}:l\ge n_0\wedge\left[S_{\alpha_0}\left(J_{\alpha_0}\left(w^{n_0+r_0},w^l\right)\right)\right]^{C_{\alpha_0}}\ge\varepsilon_0\right\}\ne\varnothing.\tag{10.20}$$

Thus, denoting

$$l_0=\min L,\tag{10.21}$$

by (10.12) and (10.14), the conclusion is that $l_0>n_0$ and in view of (10.20) and (10.21) this implies

$$\forall_{n_0\le l<l_0}\left\{\left[S_{\alpha_0}\left(J_{\alpha_0}\left(w^{n_0+r_0},w^l\right)\right)\right]^{C_{\alpha_0}}<\varepsilon_0\right\}.\tag{10.22}$$

We claim that

$$l_0>n_0+r_0.\tag{10.23}$$

To see this, suppose the contrary and let $l_0\le n_0+r_0$. We claim that then, by virtue of (10.14), $[S_{\alpha_0}(J_{\alpha_0}(w^{n_0+r_0},w^{l_0}))]^{C_{\alpha_0}}\le\max\{[S_{\alpha_0}(J_{\alpha_0}(w^i,w^j))]^{C_{\alpha_0}}:n_0\le i,j\le n_0+r_0\}=[\Delta_{(\mathcal{S}_{\mathcal{A}},\mathcal{J}_{C;\mathcal{A}});\alpha_0,r_0}(w^0,n_0)]^{C_{\alpha_0}}<\min\{\varepsilon_0,\eta_0\}\le\varepsilon_0$, which, in view of (10.20)–(10.22), is impossible. Thus (10.23) holds.

From (10.23) we deduce that $n_0<l_0-r_0<l_0$ and next from (10.22) we conclude that

$$\left[S_{\alpha_0}\left(J_{\alpha_0}\left(w^{n_0+r_0},w^{l_0-r_0}\right)\right)\right]^{C_{\alpha_0}}<\varepsilon_0.\tag{10.24}$$

Next, in view of (A.1) and (B.1) of Definition 3.1, Definition 8.1, (10.14) and (10.24), we obtain $S_{\alpha_0}(J_{\alpha_0}(w^{n_0},w^{l_0-r_0}))\le S_{\alpha_0}(C_{\alpha_0}[J_{\alpha_0}(w^{n_0},w^{n_0+r_0})+J_{\alpha_0}(w^{n_0+r_0},w^{l_0-r_0})])\le[S_{\alpha_0}(J_{\alpha_0}(w^{n_0},w^{n_0+r_0}))]^{C_{\alpha_0}}\cdot[S_{\alpha_0}(J_{\alpha_0}(w^{n_0+r_0},w^{l_0-r_0}))]^{C_{\alpha_0}}<[\Delta_{(\mathcal{S}_{\mathcal{A}},\mathcal{J}_{C;\mathcal{A}});\alpha_0,r_0}(w^0,n_0)]^{C_{\alpha_0}}\cdot\varepsilon_0<\eta_0\cdot\varepsilon_0$. Hence, using (10.11) we therefore have $[S_{\alpha_0}(J_{\alpha_0}(w^{n_0+r_0},w^{l_0}))]^{C_{\alpha_0}}<\varepsilon_0$. In view of (10.20) and (10.21), this is impossible.

The proof of (10.19) is complete.

To continue, we require the following analogue of (10.19). It takes the following form

$$\forall_{s\ge n_0}\left\{\left[S_{\alpha_0}\left(J_{\alpha_0}\left(w^s,w^{n_0+r_0}\right)\right)\right]^{C_{\alpha_0}}<\varepsilon_0\right\}.\tag{10.25}$$

Arguments are very close to those given in the proof of (10.19).

Finally, to establish (10.13), we see that, by (A.1) and (B.1) of Definition 3.1, by Definition 8.1, by (10.25) and by (10.19), we obtain

$$\begin{aligned}&\forall_{s,l\geq n_0}\{S_{\alpha_0}(J_{\alpha_0}(w^s,w^l))\\&\quad\leq S_{\alpha_0}[C_{\alpha_0}J_{\alpha_0}(w^s,w^{n_0+r_0})+C_{\alpha_0}J_{\alpha_0}(w^{n_0+r_0},w^l)]\\&\quad\leq[S_{\alpha_0}(J_{\alpha_0}(w^s,w^{n_0+r_0}))]^{C_{\alpha_0}}\cdot[S_{\alpha_0}(J_{\alpha_0}(w^{n_0+r_0},w^l))]^{C_{\alpha_0}}<\varepsilon_0\cdot\varepsilon_0\}.\end{aligned}$$

Therefore, (10.13) holds. □

Proposition 10.3 *Let* $\mathcal{J}_{C;\mathcal{A}}\in\mathbb{J}^L_{(X,\mathcal{P}_{C;\mathcal{A}})}\cup\mathbb{J}^R_{(X,\mathcal{P}_{C;\mathcal{A}})}$. *Let* $w^0\in X$ *be such that* $\forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>1}\exists_{\eta>1}\exists_{r\in\mathbb{N}}\forall_{s,l\in\mathbb{N}}\{S_\alpha(J_\alpha(w^s,w^l))<\varepsilon\cdot\eta\Rightarrow[S_\alpha(J_\alpha(w^{s+r},w^{l+r}))]^{C_\alpha}<\varepsilon\}$. *Then*

$$\forall_{\alpha\in\mathcal{A}}\Big\{\lim_{m\to\infty}\sup_{n>m}J_\alpha(w^m,w^n)=0\Big\}\quad\textit{where } \mathcal{J}_{C;\mathcal{A}}\in\mathbb{J}^L_{(X,\mathcal{P}_{C;\mathcal{A}})}\tag{10.26}$$

and

$$\forall_{\alpha\in\mathcal{A}}\Big\{\lim_{m\to\infty}\sup_{n>m}J_\alpha(w^n,w^m)=0\Big\}\quad\textit{where } \mathcal{J}_{C;\mathcal{A}}\in\mathbb{J}^R_{(X,\mathcal{P}_{C;\mathcal{A}})}.\tag{10.27}$$

Proof of Proposition 10.3 Indeed, by Proposition 10.2, we get $\forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>1}\exists_{n_0\in\mathbb{N}}\forall_{s,l\geq n_0}\{S_\alpha(J_\alpha(w^s,w^l))<\varepsilon^2\}$ or, equivalently, $\forall_{\alpha\in\mathcal{A}}\{\lim_{m,n\to\infty}S_\alpha(J_\alpha(w^m,w^n))=1\}$. By Definition 8.1, this gives $\forall_{\alpha\in\mathcal{A}}\{\lim_{m,n\to\infty}J_\alpha(w^m,w^n)=0\}$ or, equivalently, $\forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{n_0\in\mathbb{N}}\forall_{s,l\geq n_0}\{J_\alpha(w^s,w^l)<\varepsilon/2\}$. Hence, we obtain, in particular, that $\forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{m_0\in\mathbb{N}}\forall_{n>m\geq m_0}\{J_\alpha(w^m,w^n)<\varepsilon/2\}$ and $\forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{m_0\in\mathbb{N}}\forall_{n>m\geq m_0}\{J_\alpha(w^n,w^m)<\varepsilon/2\}$. From this it follows that $\forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{m_0\in\mathbb{N}}\forall_{m\geq m_0}\{\sup_{n>m}J_\alpha(w^m,w^n)\leq\varepsilon/2<\varepsilon\}$ and also $\forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{m_0\in\mathbb{N}}\forall_{m\geq m_0}\{\sup_{n>m}J_\alpha(w^n,w^m)\leq\varepsilon/2<\varepsilon\}$ and hence (10.26) and (10.27) hold. □

Proof of Theorem 9.1 The proof will be broken into Steps 1–3.

Step 1. The statement (A) *of Theorem* 9.1 *holds.*

To prove (9.2), let $\mathcal{J}_{C;\mathcal{A}}\in\mathbb{J}^L_{(X,\mathcal{P}_{C;\mathcal{A}})}$, let (X,T) be left $\mathcal{J}_{C;\mathcal{A}}$-admissible on X, and let $w^0\in X$ and $(w^m:m\in\{0\}\cup\mathbb{N})\in\mathcal{O}_{X,T}(w^0)$ be arbitrary and fixed. By (10.26), Definition 5.1(A) and hypothesis (b), we get $\mathrm{LIM}^{L-\mathcal{J}_{C;\mathcal{A}}}_{(w^m:m\in\{0\}\cup\mathbb{N})}=\{w\in X:\forall_{\alpha\in\mathcal{A}}\{\lim_{m\to\infty}J_\alpha(w,w^m)=0\}\}\neq\varnothing$. However, by hypothesis (a), $\mathcal{J}_{C;\mathcal{A}}$ is left family generated by $\mathcal{P}_{C;\mathcal{A}}$. Therefore, fixing $w\in\mathrm{LIM}^{L-\mathcal{J}_{C;\mathcal{A}}}_{(w^m:m\in\{0\}\cup\mathbb{N})}$, defining $(x_m=w^m:m\in\{0\}\cup\mathbb{N})$ and $(y_m=w:m\in\{0\}\cup\mathbb{N})$ we get $\forall_{\alpha\in\mathcal{A}}\{\lim_{m\to\infty}\sup_{n>m}J_\alpha(x_m,x_n)=0\}$ and $\forall_{\alpha\in\mathcal{A}}\{\lim_{m\to\infty}J_\alpha(y_m,x_m)=0\}$. Hence, by Definition 3.1(A), we obtain $\forall_{\alpha\in\mathcal{A}}\{\lim_{m\to\infty}P_\alpha(y_m,x_m)=0\}$. Clearly, this means $\forall_{\alpha\in\mathcal{A}}\{\lim_{m\to\infty}P_\alpha(w,w^m)=0\}$, i.e. $w\in\mathrm{LIM}^{L-\mathcal{P}_{C;\mathcal{A}}}_{(w^m:m\in\{0\}\cup\mathbb{N})}$. Consequently, $\varnothing\neq\mathrm{LIM}^{L-\mathcal{J}_{C;\mathcal{A}}}_{(w^m:m\in\{0\}\cup\mathbb{N})}\subset\mathrm{LIM}^{L-\mathcal{P}_{C;\mathcal{A}}}_{(w^m:m\in\{0\}\cup\mathbb{N})}$. Thus (9.2) holds.

If $\mathcal{J}_{C;\mathcal{A}}\in\mathbb{J}^R_{(X,\mathcal{P}_{C;\mathcal{A}})}$, (X,T) is right $\mathcal{J}_{C;\mathcal{A}}$-admissible on X, and if $w^0\in X$ and $(w^m:m\in\{0\}\cup\mathbb{N})\in\mathcal{O}_{X,T}(w^0)$ are arbitrary and fixed, then, using (10.27), a similar computation as above shows that $\varnothing\neq\mathrm{LIM}^{R-\mathcal{J}_{C;\mathcal{A}}}_{(w^m:m\in\{0\}\cup\mathbb{N})}\subset\mathrm{LIM}^{R-\mathcal{P}_{C;\mathcal{A}}}_{(w^m:m\in\{0\}\cup\mathbb{N})}$. Therefore, (9.3) also holds.

Step 2. The statement (B) *of Theorem* 9.1 *holds.*

First, we show that (9.4) and (9.5) hold in the left case. With this aim, let $w^0\in X$ and $(w^m:m\in\{0\}\cup\mathbb{N})\in\mathcal{O}_{X,T}(w^0)$ be arbitrary and fixed. By statement (A), we have $\varnothing\neq U=\mathrm{LIM}^{L-\mathcal{J}_{C;\mathcal{A}}}_{(w^m:m\in\{0\}\cup\mathbb{N})}\subset\mathrm{LIM}^{L-\mathcal{P}_{C;\mathcal{A}}}_{(w^m:m\in\{0\}\cup\mathbb{N})}$. Moreover, $w^{mq+s}\in T^{[q]}(w^{(m-1)q+s})$ where $s=1,2,\dots,q$ and $m\in\mathbb{N}$. Assuming that $s_0\in\{1,2,\dots,q\}$ is arbitrary and fixed, we see that the sequences

$(y_m = w^{mq+s_0} : m \in \mathbb{N})$ and $(x_m = w^{(m-1)q+s_0} : m \in \mathbb{N})$ satisfy $\forall_{m\in\mathbb{N}}\{y_m \in T^{[q]}(x_m)\}$, and, as subsequences of $(w^m : m \in \{0\} \cup \mathbb{N})$, are left $\mathcal{J}_{C;\mathcal{A}}$-convergent to each point of the set $U = \mathrm{LIM}^{L-\mathcal{J}_{C;\mathcal{A}}}_{(w^m:m\in\{0\}\cup\mathbb{N})}$. Furthermore, $\mathrm{LIM}^{L-\mathcal{J}_{C;\mathcal{A}}}_{(w^m:m\in\{0\}\cup\mathbb{N})} \subset \mathrm{LIM}^{L-\mathcal{J}_{C;\mathcal{A}}}_{(y_m:m\in\mathbb{N})}$ and $\mathrm{LIM}^{L-\mathcal{J}_{C;\mathcal{A}}}_{(w^m:m\in\{0\}\cup\mathbb{N})} \subset \mathrm{LIM}^{L-\mathcal{J}_{C;\mathcal{A}}}_{(x_m:m\in\mathbb{N})}$. Hence, we derive $\varnothing \neq U = \mathrm{LIM}^{L-\mathcal{J}_{C;\mathcal{A}}}_{(w^m:m\in\{0\}\cup\mathbb{N})} \subset \mathrm{LIM}^{L-\mathcal{J}_{C;\mathcal{A}}}_{(y_m:m\in\mathbb{N})} \cap \mathrm{LIM}^{L-\mathcal{J}_{C;\mathcal{A}}}_{(x_m:m\in\mathbb{N})}$. By the above, since $T^{[q]}$ is left $\mathcal{J}_{C;\mathcal{A}}$-closed on X, in virtue of Definition 7.1(A), we get $\exists_{u\in U}\{u \in T^{[q]}(u)\}$.

The above considerations lead to the conclusion that $\mathrm{Fix}_X(T^{[q]}) \neq \varnothing$ and that for each $w^0 \in X$ and for each dynamic process $(w^m : m \in \{0\} \cup \mathbb{N}) \in \mathcal{O}_{X,T}(w^0)$ starting w^0 there exists a point $u \in \mathrm{Fix}_X(T^{[q]})$ such that $(w^m : m \in \{0\} \cup \mathbb{N})$ is left $\mathcal{J}_{C;\mathcal{A}}$-convergent and also left $\mathcal{P}_{C;\mathcal{A}}$-convergent to u, so (9.4) and (9.5) hold in the left case.

In a similar way, we show that (9.4) and (9.6) hold in the right case.

Step 3. The statement (C) *of Theorem* 9.1 *holds.*

Part 1. First, we show that

$$\mathrm{Fix}_X(T) = \{w\} \quad \text{for some } w \in X. \tag{10.28}$$

Otherwise, $x^0 \in T(x^0)$, $y^0 \in T(y^0)$ and $x^0 \neq y^0$ for some $x^0, y^0 \in X$; remember that, by statement (B), $\mathrm{Fix}_X(T) \neq \varnothing$. By Proposition 3.1, since the family $\mathcal{P}_{C;\mathcal{A}} = \{P_\alpha : \alpha \in \mathcal{A}\}$ is separating on X, there exists $\alpha_0 \in \mathcal{A}$ such that $J_{\alpha_0}(x^0, y^0) > 0$ or $J_{\alpha_0}(y^0, x^0) > 0$. Suppose $J_{\alpha_0}(x^0, y^0) > 0$. Then, by Definition 8.1, $S_{\alpha_0}(J_{\alpha_0}(x^0, y^0)) = \varepsilon_0$ for some $\varepsilon_0 > 1$. Therefore, by hypothesis (d), we conclude that for these $\alpha_0 \in \mathcal{A}$ and $\varepsilon_0 > 1$, there exist $\eta_0 > 1$ and $r_0 \in \mathbb{N}$, such that dynamic processes $(x^m = x^0 : m \in \{0\} \cup \mathbb{N}) \in \mathcal{O}_{X,T}(x^0)$ and $(y^m = y^0 : m \in \{0\} \cup \mathbb{N}) \in \mathcal{O}_{X,T}(y^0)$ satisfy

$$\begin{aligned}
&\forall_{s,l\in\mathbb{N}}\bigl\{S_{\alpha_0}\bigl(J_{\alpha_0}\bigl(x^s, y^l\bigr)\bigr) = S_{\alpha_0}\bigl(J_{\alpha_0}\bigl(x^0, y^0\bigr)\bigr) = \varepsilon_0 < \varepsilon_0 \cdot \eta_0 \\
&\quad \Rightarrow S_{\alpha_0}\bigl(J_{\alpha_0}\bigl(x^0, y^0\bigr)\bigr) = S_{\alpha_0}\bigl(J_{\alpha_0}\bigl(x^{s+r_0}, y^{l+r_0}\bigr)\bigr) \\
&\quad \leq \bigl[S_{\alpha_0}\bigl(J_{\alpha_0}\bigl(x^{s+r_0}, y^{l+r_0}\bigr)\bigr)\bigr]^{C_{\alpha_0}} < \varepsilon_0\bigr\}.
\end{aligned}$$

Clearly, this is impossible. We obtain a similar conclusion in the case when $J_{\alpha_0}(y^0, x^0) > 0$. We proved that $J_{\alpha_0}(x^0, y^0) = J_{\alpha_0}(y^0, x^0) = 0$. Consequently, by Definition 3.2, (10.28) holds.

Part 2. Let, by (10.28), $w \in \mathrm{Fix}_X(T) = \{w\}$. We show that

$$\forall_{\alpha\in\mathcal{A}}\bigl\{J_\alpha(w, w) = 0\bigr\}. \tag{10.29}$$

Otherwise, there exists $\alpha_0 \in \mathcal{A}$ such that $S_{\alpha_0}(J_{\alpha_0}(w, w)) > 1$. Clearly, $w \in T^{[m]}(w)$ for $m \in \{0\} \cup \mathbb{N}$ which gives $(x^m = w : m \in \{0\} \cup \mathbb{N}) \in \mathcal{O}_{X,T}(x^0)$ and $(y^m = w : m \in \{0\} \cup \mathbb{N}) \in \mathcal{O}_{X,T}(y^0)$. Thus by (9.1) we get that for $\varepsilon_0 = S_{\alpha_0}(J_{\alpha_0}(w, w)) > 1$ there exist $\eta_0 > 1$ and $r_0 \in \mathbb{N}$ such that dynamic processes $(x^m = w : m \in \{0\} \cup \mathbb{N})$ and $(y^m = w : m \in \{0\} \cup \mathbb{N})$ satisfy

$$\begin{aligned}
&\forall_{s,l\in\mathbb{N}}\bigl\{S_{\alpha_0}\bigl(J_{\alpha_0}\bigl(x^s, y^l\bigr)\bigr) = S_{\alpha_0}\bigl(J_{\alpha_0}(w, w)\bigr) = \varepsilon_0 < \varepsilon_0 \cdot \eta_0 \\
&\quad \Rightarrow S_{\alpha_0}\bigl(J_{\alpha_0}(w, w)\bigr) \leq \bigl[S_{\alpha_0}\bigl(J_{\alpha_0}(w, w)\bigr)\bigr]^{C_{\alpha_0}} \\
&\quad = \bigl[S_{\alpha_0}\bigl(J_{\alpha_0}\bigl(x^{s+r_0}, y^{l+r_0}\bigr)\bigr)\bigr]^{C_{\alpha_0}} < \varepsilon_0\bigr\},
\end{aligned}$$

which is impossible. Therefore, (10.29) holds.

Part 3. For each $x^0 \in X$ and $r \in \mathbb{N}$ we say that $u \in T^{[r]}(x^0)$ if there exists a sequence $(x^m : m \in \{0,\ldots,r\}) = (x^0, x^1, \ldots, x^r)$ satisfying $\forall_{m\in\{0,\ldots,r-1\}}\{x^{m+1} \in T(x^m)\}$ and such that $u = x^r$.

Part 4. By (10.28), we get $\mathrm{Fix}_X(T) = \{w\}$ for some $w \in X$. We prove that

$$\forall_{\alpha\in\mathcal{A}}\Big\{\sup_{u,v\in T(w)} J_\alpha(u,v) = 0\Big\}. \tag{10.30}$$

Let $r \in \mathbb{N}$ be arbitrary and fixed and let $(x^m : m \in \{0\} \cup \mathbb{N}) \in \mathcal{O}_{X,T}(w)$ and $(y^m : m \in \{0\} \cup \mathbb{N}) \in \mathcal{O}_{X,T}(w)$ be dynamic processes such that $x^n = y^n = w$ for $n \in \{0,\ldots,r-1\}$ and let $x^r = u \in T^{[r]}(w)$ and $y^r = v \in T^{[r]}(w)$ be arbitrary and fixed. Then, by (10.29), for each $\alpha \in \mathcal{A}$, $\varepsilon > 1$, $\eta > 1$ and $n \in \{0,\ldots,r-1\}$ we get $S_\alpha(J_\alpha(x^n,y^n)) = S_\alpha(J_\alpha(w,w)) = S_\alpha(0) = 1 < \varepsilon\cdot\eta$. Hence, by using (9.1) and since $\forall_{\alpha\in\mathcal{A}}\{C_\alpha \geq 1\}$, we obtain $\forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>1}\exists_{r\in\mathbb{N}}\forall_{u,v\in T^{[r]}(w)}\{S_\alpha(J_\alpha(u,v)) \leq [S_\alpha(J_\alpha(u,v))]^{C_\alpha} < \varepsilon\}$ and this implies that $\forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{r\in\mathbb{N}}\{\sup_{u,v\in T^{[r]}(w)} J_\alpha(u,v) \leq \varepsilon\}$. Observe further that by using property $w \in T(w) \subset T^{[m]}(w)$ for $m \in \mathbb{N}$, we find $\forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\{\sup_{u,v\in T(w)} J_\alpha(u,v) \leq \varepsilon\}$, that is, $\forall_{\alpha\in\mathcal{A}}\{\sup_{u,v\in T(w)} J_\alpha(u,v) = 0\}$.

Part 5. Note that $T(w) = \{w\}$. Otherwise, $u, v \in T(w)$ and $u \neq v$ and, by Proposition 3.1, since the family $\mathcal{P}_{C;\mathcal{A}} = \{P_\alpha : \alpha \in \mathcal{A}\}$ is separating on X, there exists $\alpha_0 \in \mathcal{A}$ such that $J_{\alpha_0}(u,v) > 0$ or $J_{\alpha_0}(v,u) > 0$. Consequently, $\sup_{u,v\in T(w)} J_{\alpha_0}(u,v) > 0$. Clearly, this is impossible by (10.30). Therefore, (9.7) holds.

Part 6. Property (9.8) follows from Parts 2 and 5. By statement (B), properties (9.9) and (9.10) hold. □

Proof of Theorem 9.2 With the notation of the Theorem 9.2, Steps 1 and 2 of the proof of Theorem 9.1 are adapted to $w^0 \in M$ and $(w^m : m \in \{0\} \cup \mathbb{N}) \in \mathcal{O}_{X,T}(w^0)$ satisfying assumptions of Theorem 9.2. □

Proof of Theorem 9.3 The proof will be broken into Steps 1–3.

Step 1. The statement (A) *of Theorem* 9.3 *holds.*

First, we prove (9.13). Let $\mathcal{J}_{C;\mathcal{A}} \in \mathbb{J}^L_{(X,\mathcal{P}_{C;\mathcal{A}})}$ and let (X,T) be left $\mathcal{J}_{C;\mathcal{A}}$-admissible on X. Let $w^0 \in X$ be arbitrary and fixed. Define the sequence $(w^m = T^{[m]}(w^0) : m \in \{0\} \cup \mathbb{N})$. Clearly hypothesis (d) implies hypothesis of Proposition 10.3 and, by (10.26), Definition 5.2(A) and hypothesis (b), we get that this sequence is left $\mathcal{J}_{C;\mathcal{A}}$-convergent in X, i.e.

$$\mathrm{LIM}^{L-\mathcal{J}_{C;\mathcal{A}}}_{(T^{[m]}(w^0):m\in\{0\}\cup\mathbb{N})} = \Big\{w \in X : \forall_{\alpha\in\mathcal{A}}\Big\{\lim_{m\to\infty} J_\alpha\big(w, T^{[m]}\big(w^0\big)\big) = 0\Big\}\Big\} \neq \varnothing. \tag{10.31}$$

However, by hypothesis (a), $\mathcal{J}_{C;\mathcal{A}}$ is the left family generated by $\mathcal{P}_{C;\mathcal{A}}$. Therefore, fixing $w \in \mathrm{LIM}^{L-\mathcal{J}_{C;\mathcal{A}}}_{(T^{[m]}(w^0):m\in\{0\}\cup\mathbb{N})}$, defining

$$\big(x_m = T^{[m]}\big(w^0\big) : m \in \{0\} \cup \mathbb{N}\big) \text{ and } \big(y_m = w : m \in \{0\} \cup \mathbb{N}\big) \tag{10.32}$$

and using (10.26) and (10.28) we get $\forall_{\alpha\in\mathcal{A}}\{\lim_{m\to\infty}\sup_{n>m} J_\alpha(x_m,x_n) = 0\}$ and $\forall_{\alpha\in\mathcal{A}}\{\lim_{m\to\infty} J_\alpha(y_m,x_m) = 0\}$. Hence, by Definition 3.1(A), we obtain $\forall_{\alpha\in\mathcal{A}}\{\lim_{m\to\infty} P_\alpha(y_m, x_m) = 0\}$. Next, by (10.29), we observe that this is of the form $\forall_{\alpha\in\mathcal{A}}\{\lim_{m\to\infty} P_\alpha(w, T^{[m]}(w^0)) = 0\}$. Therefore, $w \in \mathrm{LIM}^{L-\mathcal{P}_{C;\mathcal{A}}}_{(T^{[m]}(w^0):m\in\{0\}\cup\mathbb{N})}$. By (10.28), this means that $\varnothing \neq \mathrm{LIM}^{L-\mathcal{J}_{C;\mathcal{A}}}_{(T^{[m]}(w^0):m\in\{0\}\cup\mathbb{N})} \subset \mathrm{LIM}^{L-\mathcal{P}_{C;\mathcal{A}}}_{(T^{[m]}(w^0):m\in\{0\}\cup\mathbb{N})}$. We proved (9.13).

Let now $\mathcal{J}_{C;\mathcal{A}} \in \mathbb{J}^{R}_{(X,\mathcal{P}_{C;\mathcal{A}})}$ and let (X,T) be right $\mathcal{J}_{C;\mathcal{A}}$-admissible on X. Then, using (10.27), a similar computation shows that, for each $w^0 \in X$, $\varnothing \neq \mathrm{LIM}^{R-\mathcal{J}_{C;\mathcal{A}}}_{(T^{[m]}(w^0):m\in\{0\}\cup\mathbb{N})} \subset \mathrm{LIM}^{R-\mathcal{P}_{C;\mathcal{A}}}_{(T^{[m]}(w^0):m\in\{0\}\cup\mathbb{N})}$, which means that (9.14) holds.

Step 2. Conclusion (B) *of Theorem* 9.3 *holds.*

First, we show that (9.15)–(9.17) hold in the left case.

Let $w^0 \in X$ be arbitrary and fixed. By statement (A), the sequence $(w^m = T^{[m]}(w^0) : m \in \{0\} \cup \mathbb{N})$ satisfies $\varnothing \neq U = \mathrm{LIM}^{L-\mathcal{J}_{C;\mathcal{A}}}_{(w^m:m\in\{0\}\cup\mathbb{N})} \subset \mathrm{LIM}^{L-\mathcal{P}_{C;\mathcal{A}}}_{(w^m:m\in\{0\}\cup\mathbb{N})}$. Moreover, for this sequence we have $w^{mq+s} = T^{[q]}(w^{(m-1)q+s})$ where $s = 1,2,\dots,q$ and $m \in \mathbb{N}$. Assuming that $s_0 \in \{1,2,\dots,q\}$ is arbitrary and fixed, we see that the sequences $(y_m = w^{mq+s_0} : m \in \mathbb{N})$ and $(x_m = w^{(m-1)q+s_0} : m \in \mathbb{N})$ satisfy $\forall_{m\in\mathbb{N}}\{y_m = T^{[q]}(x_m)\}$, and, as subsequences of $(w^m : m \in \{0\} \cup \mathbb{N})$, are left $\mathcal{J}_{C;\mathcal{A}}$-convergent to each point of the set $\mathrm{LIM}^{L-\mathcal{J}_{C;\mathcal{A}}}_{(T^{[m]}(w^0):m\in\{0\}\cup\mathbb{N})}$. Furthermore, $\mathrm{LIM}^{L-\mathcal{J}_{C;\mathcal{A}}}_{(T^{[m]}(w^0):m\in\{0\}\cup\mathbb{N})} \subset \mathrm{LIM}^{L-\mathcal{J}_{C;\mathcal{A}}}_{(y_m:m\in\mathbb{N})}$ and $\mathrm{LIM}^{L-\mathcal{J}_{C;\mathcal{A}}}_{(T^{[m]}(w^0):m\in\{0\}\cup\mathbb{N})} \subset \mathrm{LIM}^{L-\mathcal{J}_{C;\mathcal{A}}}_{(x_m:m\in\mathbb{N})}$. Hence, we derive $\varnothing \neq U = \mathrm{LIM}^{L-\mathcal{J}_{C;\mathcal{A}}}_{(T^{[m]}(w^0):m\in\{0\}\cup\mathbb{N})} \subset \mathrm{LIM}^{L-\mathcal{J}_{C;\mathcal{A}}}_{(y_m:m\in\mathbb{N})} \cap \mathrm{LIM}^{L-\mathcal{J}_{C;\mathcal{A}}}_{(x_m:m\in\mathbb{N})}$. By the above, since $T^{[q]}$ is left $\mathcal{J}_{C;\mathcal{A}}$-closed on X, in virtue of Definition 7.2(A), we get $\exists_{u\in U}\{u = T^{[q]}(u)\}$.

The above considerations lead to the conclusion that $\mathrm{Fix}_X(T^{[q]}) \neq \varnothing$ and that for each $w^0 \in X$ there exists a point $u \in \mathrm{Fix}_X(T^{[q]})$ such that the sequence $(w^m = T^{[m]}(w^0) : m \in \{0\} \cup \mathbb{N})$ is left $\mathcal{J}_{C;\mathcal{A}}$-convergent and left $\mathcal{P}_{C;\mathcal{A}}$-convergent to u, so (9.15) and (9.16) hold in the left case.

Next, we show that (9.17) holds, i.e. that $\forall_{\alpha\in\mathcal{A}}\forall_{n\in\{1,2,\dots,q\}}\{J_\alpha(u,T^{[n]}(u)) = J_\alpha(T^{[n]}(u),u) = 0\}$ holds. Suppose that $\exists_{\alpha_0\in\mathcal{A}}\exists_{n_0\in\{1,2,\dots,q\}}\{J_{\alpha_0}(u,T^{[n_0]}(u)) > 0 \vee J_{\alpha_0}(T^{[n_0]}(u),u) > 0\}$. If $J_{\alpha_0}(u, T^{[n_0]}(u)) > 0$, then, by Definition 8.1, $S_{\alpha_0}(J_{\alpha_0}(u,T^{[n_0]}(u))) > 1$, and putting

$$\varepsilon_0 = S_{\alpha_0}\bigl(J_{\alpha_0}\bigl(u,T^{[n_0]}(u)\bigr)\bigr) \tag{10.33}$$

for some $\varepsilon_0 > 1$, by hypotheses (c) and (d), we get

$$\begin{aligned}&\exists_{\eta_0>1}\exists_{r_0\in\mathbb{N}}\forall_{s,l\in\mathbb{N}}\bigl\{S_{\alpha_0}\bigl(J_{\alpha_0}\bigl(T^{[s]}(u),T^{[l]}(u)\bigr)\bigr) < \varepsilon_0\cdot\eta_0\\ &\qquad\Rightarrow \bigl[S_{\alpha_0}\bigl(J_{\alpha_0}\bigl(T^{[s+r_0]}(u),T^{[l+r_0]}(u)\bigr)\bigr)\bigr]^{C_{\alpha_0}} < \varepsilon_0\bigr\}.\end{aligned} \tag{10.34}$$

Using (9.15), (9.16) and (10.33) we therefore have

$$\forall_{m\in\mathbb{N}}\bigl\{T^{[mq]}(u) = u\bigr\} \tag{10.35}$$

and $S_{\alpha_0}(J_{\alpha_0}(T^{[q]}(u),T^{[q+n_0]}(u))) = S_{\alpha_0}(J_{\alpha_0}(u,T^{[n_0]}(u))) = \varepsilon_0 < \varepsilon_0\cdot\eta_0$. Hence, using (10.34) for $s = q$ and $l = q+n_0$, we get the inequalities $[S_{\alpha_0}(J_{\alpha_0}(T^{[q+r_0]}(u),T^{[q+n_0+r_0]}(u)))]^{C_{\alpha_0}} < \varepsilon_0 < \varepsilon_0\cdot\eta_0$. It follows from this that

$$S_{\alpha_0}\bigl(J_{\alpha_0}\bigl(T^{[q+r_0]}(u),T^{[q+n_0+r_0]}(u)\bigr)\bigr) < \varepsilon_0 < \varepsilon_0\cdot\eta_0 \tag{10.36}$$

since $C_{\alpha_0} \geq 1$ and $S_{\alpha_0}(J_{\alpha_0}(T^{[q+r_0]}(u),T^{[q+n_0+r_0]}(u))) \geq 1$. Similarly, by (10.36), using (10.34) for $s = q+r_0$ and $l = q+n_0+r_0$ we conclude that $[S_{\alpha_0}(J_{\alpha_0}(T^{[q+2r_0]}(u),T^{[q+n_0+2r_0]}(u)))]^{C_{\alpha_0}} < \varepsilon_0 < \varepsilon_0\cdot\eta_0$ and this gives

$$S_{\alpha_0}\bigl(J_{\alpha_0}\bigl(T^{[q+2r_0]}(u),T^{[q+n_0+2r_0]}(u)\bigr)\bigr) < \varepsilon_0 < \varepsilon_0\cdot\eta_0 \tag{10.37}$$

since $C_{\alpha_0} \geq 1$ and $S_{\alpha_0}(J_{\alpha_0}(T^{[q+2r_0]}(u), T^{[q+n_0+2r_0]}(u))) \geq 1$. In view of (10.35) and (10.37), and by induction, using (10.34), we find

$$\forall_{m\in\mathbb{N}}\bigl\{S_{\alpha_0}\bigl(J_{\alpha_0}\bigl(T^{[q+mr_0]}(u), T^{[q+n_0+mr_0]}(u)\bigr)\bigr) \\ \leq \bigl[S_{\alpha_0}\bigl(J_{\alpha_0}\bigl(T^{[q+mr_0]}(u), T^{[q+n_0+mr_0]}(u)\bigr)\bigr)\bigr]^{C_{\alpha_0}} < \varepsilon_0 < \varepsilon_0 \cdot \eta_0\bigr\}. \tag{10.38}$$

If $m = q$ in (10.38), we also have

$$S_{\alpha_0}\bigl(J_{\alpha_0}\bigl(T^{[q+qr_0]}(u), T^{[q+n_0+qr_0]}(u)\bigr)\bigr) < \varepsilon_0 < \varepsilon_0 \cdot \eta_0. \tag{10.39}$$

By using (10.33), (10.35) and (10.39), we obtain $\varepsilon_0 = S_{\alpha_0}(J_{\alpha_0}(u, T^{[n_0]}(u))) = S_{\alpha_0}(J_{\alpha_0}(T^{[q+qr_0]}(u), T^{[q+n_0+qr_0]}(u))) < \varepsilon_0$. It is absurd. Therefore, $J_{\alpha_0}(u, T^{[n_0]}(u)) = 0$. Similarly, we prove that $J_{\alpha_0}(T^{[n_0]}(u), u) = 0$. We proved that (9.17) holds.

In a similar way, we show that (9.15), (9.18) and (9.19) hold in the case of right.

Step 3. The conclusion (C) *of Theorem* 9.3 *holds.*

Since (X, T) is left (respectively, right) $\mathcal{P}_{C;\mathcal{A}}$-closed on X, thus, by conclusion (B), we have $\mathrm{Fix}_X(T) \neq \emptyset$. We show that $\mathrm{Fix}_X(T) = \{w\}$ for some $w \in X$. Otherwise, $x, y \in \mathrm{Fix}_X(T)$ and $x \neq y$ for some $x, y \in X$; remember that, by the above, $\mathrm{Fix}_X(T) \neq \emptyset$. By Proposition 3.1, since the family $\mathcal{P}_{C;\mathcal{A}} = \{P_\alpha : \alpha \in \mathcal{A}\}$ is separating on X, there exists $\alpha_0 \in \mathcal{A}$ such that $J_{\alpha_0}(x, y) > 0$ or $J_{\alpha_0}(y, x) > 0$. Suppose $J_{\alpha_0}(x, y) > 0$. Then, by Definition 8.1, $S_{\alpha_0}(J_{\alpha_0}(x, y)) = \varepsilon_0$ for some $\varepsilon_0 > 1$ and, by hypothesis (d), for these $\alpha_0 \in \mathcal{A}$ and $\varepsilon_0 > 1$, there exist $\eta_0 > 1$ and $r_0 \in \mathbb{N}$, such that

$$\forall_{s,l\in\mathbb{N}}\bigl\{S_{\alpha_0}\bigl(J_{\alpha_0}\bigl(T^{[s]}(x), T^{[l]}(y)\bigr)\bigr) = S_{\alpha_0}\bigl(J_{\alpha_0}(x, y)\bigr) = \varepsilon_0 < \varepsilon_0 \cdot \eta_0 \\ \Rightarrow \bigl[S_{\alpha_0}\bigl(J_{\alpha_0}\bigl(T^{[s+r_0]}(x), T^{[l+r_0]}(y)\bigr)\bigr)\bigr]^{C_{\alpha_0}} < \varepsilon_0\bigr\}. \tag{10.40}$$

However, for each $s, l \in \mathbb{N}$, $S_{\alpha_0}(J_{\alpha_0}(x, y)) = S_{\alpha_0}(J_{\alpha_0}(T^{[s+r_0]}(x), T^{[l+r_0]}(y))) = \varepsilon_0$ and (10.40) imply that $\varepsilon_0 > 1$, $C_{\alpha_0} \geq 1$ and $\varepsilon_0^{C_{\alpha_0}} < \varepsilon_0$. Clearly, this is impossible. We obtain a similar conclusion in the case when $J_{\alpha_0}(y, x) > 0$. We proved that $J_{\alpha_0}(x, y) = J_{\alpha_0}(y, x) = 0$. Consequently, by Definition 3.2, $\mathrm{Fix}_X(T) = \{w\}$ for some $w \in X$, that is (9.20) holds.

Now, (9.20) with (B) means that, for each $w^0 \in X$, the sequence $(w^m = T^{[m]}(w^0) : m \in \{0\} \cup \mathbb{N})$ is left (respectively, right) $\mathcal{P}_{C;\mathcal{A}}$-convergent to w. Thus (9.21) and (9.23) hold.

Finally, we show that (9.22) and (9.24) hold, i.e. that $\forall_{\alpha\in\mathcal{A}}\{S_\alpha(J_\alpha(w, w)) = 1\}$ where $\mathrm{Fix}_X(T) = \{w\}$. Indeed, if we assume that there exists $\alpha_0 \in \mathcal{A}$ such that $S_{\alpha_0}(J_{\alpha_0}(w, w)) > 1$, then, denoting $\varepsilon_0 = S_{\alpha_0}(J_{\alpha_0}(w, w)) > 1$, by (9.12), there exist $\eta_0 > 1$ and $r_0 \in \mathbb{N}$, such that

$$\forall_{s,l\in\mathbb{N}}\bigl\{S_{\alpha_0}\bigl(J_{\alpha_0}\bigl(T^{[s]}(w), T^{[l]}(w)\bigr)\bigr) < \varepsilon_0 \cdot \eta_0 \\ \Rightarrow \bigl[S_{\alpha_0}\bigl(J_{\alpha_0}\bigl(T^{[s+r_0]}(w), T^{[l+r_0]}(w)\bigr)\bigr)\bigr]^{C_{\alpha_0}} < \varepsilon_0\bigr\}. \tag{10.41}$$

However, for each $s, l \in \mathbb{N}$, we have $S_{\alpha_0}(J_{\alpha_0}(T^{[s]}(w), T^{[l]}(w))) = S_{\alpha_0}(J_{\alpha_0}(w, w)) = \varepsilon_0 < \varepsilon_0 \cdot \eta_0$. Thus, using (10.41), we obtain $1 < \varepsilon_0 = S_{\alpha_0}(J_{\alpha_0}(w, w)) = S_{\alpha_0}(J_{\alpha_0}(T^{[s+r_0]}(w), T^{[l+r_0]}(w))) \leq [S_{\alpha_0}(J_{\alpha_0}(T^{[s+r_0]}(w), T^{[l+r_0]}(w)))]^{C_{\alpha_0}} < \varepsilon_0$, i.e., $1 < \varepsilon_0$, $C_{\alpha_0} \geq 1$ and $\varepsilon_0^{C_{\alpha_0}} < \varepsilon_0$, which is impossible. Therefore, (9.22) and (9.24) hold.

The proof of Theorem 9.3 is complete. □

Proof of Theorem 9.4 Thus the condition (9.25) holds. Then, defining $(w^m = T^{[m]}(w^0) : m \in \{0\} \cup \mathbb{N})$ where $w^0 \in M$ and next, for this sequence, using a similar argument as in the proofs of Propositions 10.1–10.3 and Steps 1 and 2 of the proof of Theorem 9.3, we have the assertions. □

11 Examples

Example 11.1 Let $X = (0; 1)$, let $P : X \times X \to [0; +\infty)$ be given by

$$P(u,v) = \begin{cases} 0 & \text{if } u \geq v, \\ (v-u)^4 & \text{if } u < v, \end{cases} \tag{11.1}$$

where $u, v \in X$, and let a set-valued dynamic system (X, T) has the form

$$T(x) = \begin{cases} (x/3 + 1/3; 1/2) \cup (1/2; -x/3 + 2/3) & \text{if } x \in (0; 1/2), \\ \{1/2\} & \text{if } x = 1/2, \\ (-x/3 + 2/3; 1/2) \cup (1/2; x/3 + 1/3) & \text{if } x \in (1/2; 1). \end{cases} \tag{11.2}$$

Let us observe that:

Part 1. P is quasi-triangular distance with $C = 8$ and $(X, \mathcal{P}_{\{8\};\{1\}}) = (X, P)$ is quasi-triangular space. We have $\forall_{u,v,w \in X}\{P(u, w) \leq 8[P(u, v) + P(v, w)]\}$; see Definition 2.1(A) and [53, Example 1, p. 10].

Part 2. For $J = P$, for each $\mathcal{S}_{\{1\}}$-family $\mathcal{S}_{\{1\}} = \{S\}$ (see Definition 8.1) *and for (X, T) the hypotheses (a)–(d) of Theorem* 9.1 *hold.* Indeed, hypothesis (a) holds since, by Remark 3.1(a), $P \in \mathbb{J}^L_{(X,P)} \cap \mathbb{J}^R_{(X,P)}$ (see, Definition 3.1(D)).

Next, we see that (X, T) is left and right P-admissible in each point $w^0 \in X$ (see Definition 5.1). In fact, from (11.2), for $m \in \mathbb{N}$, we get

$$\begin{aligned} T^{[m]}\bigl(X - \{1/2\}\bigr) &= \left(\sum_{i=1}^{m} 1/3^m; 1/2\right) \cup \left(1/2; 1 - \sum_{i=1}^{m} 1/3^m\right) \\ &= \bigl(\bigl(1 - 1/3^m\bigr)/2; 1/2\bigr) \cup \bigl(1/2; 1 - \bigl(1 - 1/3^m\bigr)/2\bigr), \end{aligned} \tag{11.3}$$

$$T^{[m]}\bigl(\{1/2\}\bigr) = \{1/2\}. \tag{11.4}$$

By applying (11.1)–(11.4), by Definition 2.3, and by direct reasoning the calculations show that:

Case A. Let $w^0 \in (0; 1/2) \cup (1/2; 1)$. If $(w^m : m \in \{0\} \cup \mathbb{N}) \in \mathcal{O}_{X,T}(w^0)$ satisfies $\exists_{m_0 \in \mathbb{N}} \forall_{m \geq m_0}\{w^m \in ((1 - 1/3^m)/2; 1/2)\}$, then $\lim_{m\to\infty} \sup_{n>m} P(w^m, w^n) = \lim_{m\to\infty} \sup_{n>m} P(w^n, w^m) = 0$, $[1/2; 1) = \mathrm{LIM}^{L-P}_{(w^m : m \in \{0\} \cup \mathbb{N})}$ and $\{1/2\} = \mathrm{LIM}^{R-P}_{(w^m : m \in \{0\} \cup \mathbb{N})}$.

Case B. Let $w^0 \in (0; 1/2) \cup (1/2; 1)$. If $(w^m : m \in \{0\} \cup \mathbb{N}) \in \mathcal{O}_{X,T}(w^0)$ satisfies $\exists_{m_0 \in \mathbb{N}} \forall_{m \geq m_0}\{w^m \in (1/2; 1 - (1 - 1/3^m)/2)\}$, then $\lim_{m\to\infty} \sup_{n>m} P(w^m, w^n) = \lim_{m\to\infty} \sup_{n>m} P(w^n, w^m) = 0$, $\{1/2\} = \mathrm{LIM}^{L-P}_{(w^m : m \in \{0\} \cup \mathbb{N})}$ and $(0; 1/2] = \mathrm{LIM}^{R-P}_{(w^m : m \in \{0\} \cup \mathbb{N})}$.

Case C. Let $w^0 \in (0; 1/2) \cup (1/2; 1)$. If $(w^m : m \in \{0\} \cup \mathbb{N}) \in \mathcal{O}_{X,T}(w^0)$ satisfies $\forall_{m_0 \in \mathbb{N}} \exists_{m_1 > m_0} \exists_{m_2 > m_0}\{w^{m_1} \in ((1 - 1/3^{m_1})/2; 1/2) \text{ and } w^{m_2} \in (1/2; 1 - (1 - 1/3^{m_2})/2)\}$, then $\lim_{m\to\infty} \sup_{n>m} P(w^m, w^n) = \lim_{m\to\infty} \sup_{n>m} P(w^n, w^m) = 0$, $\{1/2\} = \mathrm{LIM}^{L-P}_{(w^m : m \in \{0\} \cup \mathbb{N})}$ and $\{1/2\} = \mathrm{LIM}^{R-P}_{(w^m : m \in \{0\} \cup \mathbb{N})}$.

Case D. If $w^0 = 1/2$, then $(w^m : m \in \{0\} \cup \mathbb{N}) \in \mathcal{O}_{X,T}(w^0)$ is such that $\forall_{m\in\mathbb{N}}\{w^m = 1/2\}$, $\lim_{m\to\infty} \sup_{n>m} P(w^m, w^n) = \lim_{m\to\infty} \sup_{n>m} P(w^n, w^m) = 0$, $[1/2; 1) = \mathrm{LIM}^{L-P}_{(w^m:m\in\{0\}\cup\mathbb{N})}$ and $(0; 1/2] = \mathrm{LIM}^{R-P}_{(w^m:m\in\{0\}\cup\mathbb{N})}$.

Therefore, by Definition 5.1, hypothesis (b) follows from Cases A–D.

Finally, let us observe that (X, T), $J = P$, and arbitrary $\mathcal{S}_{\{1\}}$-family $\mathcal{S}_{\{1\}} = \{S\}$ satisfy (9.1). Indeed, by (11.1)–(11.4), we note that

$$\begin{cases} \forall_{\varepsilon>0}\exists_{r\in\mathbb{N}}\forall_{x^0,y^0\in X}\forall_{(x^m:m\in\{0\}\cup\mathbb{N})\in\mathcal{O}_{X,T}(x^0)}\forall_{(y^m:m\in\{0\}\cup\mathbb{N})\in\mathcal{O}_{X,T}(y^0)} \\ \quad \forall_{s,l\in\mathbb{N}}\{8 \cdot P(x^{s+r}, y^{l+r}) \leq 8 \cdot (x^{s+r} - y^{l+r})^4 \\ \qquad\qquad\qquad \leq 8 \cdot (1/3^{4r})(1/3^l - 1/3^s)^4/2^4 < \varepsilon\}. \end{cases}$$

Consequently, using Definition 8.1, we obtain

$$\begin{cases} \forall_{\varepsilon>1}\exists_{r\in\mathbb{N}}\forall_{x^0,y^0\in X}\forall_{(x^m:m\in\{0\}\cup\mathbb{N})\in\mathcal{O}_{X,T}(x^0)}\forall_{(y^m:m\in\{0\}\cup\mathbb{N})\in\mathcal{O}_{X,T}(y^0)} \\ \quad \forall_{s,l\in\mathbb{N}}\{[S(P(x^{s+r}, y^{l+r}))]^8 < \varepsilon\}. \end{cases}$$

In view of this, we see that

$$\begin{cases} \forall_{\varepsilon>1}\exists_{\eta>1}\exists_{r\in\mathbb{N}}\forall_{x^0,y^0\in X}\forall_{(x^m:m\in\{0\}\cup\mathbb{N})\in\mathcal{O}_{X,T}(x^0)}\forall_{(y^m:m\in\{0\}\cup\mathbb{N})\in\mathcal{O}_{X,T}(y^0)} \\ \quad \forall_{s,l\in\mathbb{N}}\{S(P(x^s, y^l)) < \varepsilon \cdot \eta \Rightarrow [S(P(x^{s+r}, y^{l+r}))]^8 < \varepsilon\}. \end{cases}$$

Part 3. (X, T) is a left and right P-closed in each point $w^0 \in X$. Indeed, for each $w^0 \in X$ and for each dynamic process $(w^m : m \in \{0\} \cup \mathbb{N}) \in\mathcal{O}_{X,T}(w^0)$, the subsequences $(y_m = w^{m+1} : m \in \mathbb{N})$ and $(x_m = w^m : m \in \mathbb{N})$ of $(w^m : m \in \{0\} \cup \mathbb{N})$ satisfy $\forall_{m\in\mathbb{N}}\{y_m \in T(x_m)\}$. Moreover, in view of Cases A–E, we get $1/2 \in U = \mathrm{LIM}^{L-P}_{(w^m:m\in\{0\}\cup\mathbb{N})} \subset \mathrm{LIM}^{L-P}_{(x_m:m\in\mathbb{N})} \cap \mathrm{LIM}^{L-P}_{(y_m:m\in\mathbb{N})}$ and $1/2 \in V = \mathrm{LIM}^{R-P}_{(w^m:m\in\{0\}\cup\mathbb{N})} \subset \mathrm{LIM}^{R-P}_{(x_m:m\in\mathbb{N})} \cap \mathrm{LIM}^{R-P}_{(y_m:m\in\mathbb{N})}$. Observe also that $1/2 \in T(1/2) = \{1/2\} = \mathrm{End}_X(T)$.

Part 4. P is separating on X. Indeed, for each $x, y \in X$ such that $x \neq y$, we have $P(x, y) > 0$ or $P(y, x) > 0$. This means, by Definition 2.1(D), that P is separating on X.

Part 5. P vanishes on the diagonal. Indeed, for each $x \in X$, $P(x, x) = 0$.

Claim 1. By Parts 1–5, *for $J = P$, $q = 1$, (X, T) and for arbitrary $\mathcal{S}_{\{1\}}$-family $\mathcal{S}_{\{1\}} = \{S\}$, hypotheses* (a)–(d) *and statements* (A), (B) *and* (C) *of Theorem* 9.1 *hold.* We have: (i) $\mathrm{End}_X(T) = \{1/2\}$. (ii) For each $w^0 \in X$, every dynamic process $(w^m : m \in \{0\} \cup \mathbb{N}) \in\mathcal{O}_{X,T}(w^0)$ is left and right P-convergent to $1/2$.

Example 11.2 Let $X = (0; 1)$ and let $P : X \times X \to [0; +\infty)$ be given by (11.1) for $X = (0; 1)$. Suppose also that (X, T) is a set-valued dynamic system defined by

$$T(x) = \begin{cases} (x/3 + 1/3; 1/2) \cup (1/2; -x/3 + 2/3) & \text{for } x \in (0; 1/2), \\ (0; 1) & \text{for } x = 1/2, \\ (-x/3 + 2/3; 1/2) \cup (1/2; x/3 + 1/3) & \text{for } x \in (1/2; 1). \end{cases} \tag{11.5}$$

Part 1. For $J = P$, for $M = X$, for each $\mathcal{S}_{\{1\}}$-family $\mathcal{S}_{\{1\}} = \{S\}$, and for (X, T) the hypotheses (a)–(d) *of Theorem* 9.2 *hold.* Indeed, hypothesis (a) holds since, by Remark 3.1(a), $P \in \mathbb{J}^L_{(X,P)} \cap \mathbb{J}^R_{(X,P)}$.

Next, we see that (X,T) is left and right P-admissible in each point $w^0 \in X$. In fact, from (11.5), for $m \in \mathbb{N}$, we get

$$\begin{aligned} &T^{[m]}\bigl(X-\{1/2\}\bigr) \\ &\quad =\Biggl(\sum_{i=1}^{m} 1/3^m;1/2\Biggr)\cup\Biggl(1/2;1-\sum_{i=1}^{m} 1/3^m\Biggr) \\ &\quad =\bigl(\bigl(1-1/3^m\bigr)/2;1/2\bigr)\cup\bigl(1/2;1-\bigl(1-1/3^m\bigr)/2\bigr), \\ &T^{[m]}\bigl(\{1/2\}\bigr)=(0;1). \end{aligned} \tag{11.6}$$

By applying (11.1) and (11.5)–(11.6), we consider the situations I and II:

I. If $w^0 \in (0;1/2)\cup(1/2;1)$ and $(w^m : m \in \{0\}\cup\mathbb{N}) \in \mathcal{O}_{X,T}(w^0)$, then we see that $\forall_{m\in\mathbb{N}}\{w^m \in ((1-1/3^m)/2;1/2)\cup(1/2;1-(1-1/3^m)/2)\}$ and Cases A–C hold:

Case A. If $w^0 \in (0;1/2)\cup(1/2;1)$ and $\exists_{m_0\in\mathbb{N}}\forall_{m\ge m_0}\{w^m \in ((1-1/3^m)/2;1/2)\}$, then $\lim_{m\to\infty}\sup_{n>m} P(w^m,w^n)=\lim_{m\to\infty}\sup_{n>m} P(w^n,w^m)=0$, $[1/2;1)=\mathrm{LIM}^{L-P}_{(w^m:m\in\{0\}\cup\mathbb{N})}$ and $\{1/2\}=\mathrm{LIM}^{R-P}_{(w^m:m\in\{0\}\cup\mathbb{N})}$.

Case B. If $w^0 \in (0;1/2)\cup(1/2;1)$ and $\exists_{m_0\in\mathbb{N}}\forall_{m\ge m_0}\{w^m \in (1/2;1-(1-1/3^m)/2)\}$, then $\lim_{m\to\infty}\sup_{n>m} P(w^m,w^n)=\lim_{m\to\infty}\sup_{n>m} P(w^n,w^m)=0$, $\{1/2\}=\mathrm{LIM}^{L-P}_{(w^m:m\in\{0\}\cup\mathbb{N})}$ and $(0;1/2]=\mathrm{LIM}^{R-P}_{(w^m:m\in\{0\}\cup\mathbb{N})}$.

Case C. If $w^0 \in (0;1/2)\cup(1/2;1)$ and $\forall_{m_0\in\mathbb{N}}\exists_{m_1>m_0}\exists_{m_2>m_0}\{w^{m_1} \in ((1-1/3^{m_1})/2;1/2)$ and $w^{m_2} \in (1/2;1-(1-1/3^{m_2})/2)\}$, then $\lim_{m\to\infty}\sup_{n>m} P(w^m,w^n)=\lim_{m\to\infty}\sup_{n>m} P(w^n,w^m)=0$, $\{1/2\}=\mathrm{LIM}^{L-P}_{(w^m:m\in\{0\}\cup\mathbb{N})}$ and $\{1/2\}=\mathrm{LIM}^{R-P}_{(w^m:m\in\{0\}\cup\mathbb{N})}$.

II. If $w^0=1/2$ and $(w^m : m \in \{0\}\cup\mathbb{N}) \in \mathcal{O}_{X,T}(w^0)$, then we see that $\forall_{m\in\mathbb{N}}\{w^m \in (0;1)\}$ and Cases D and E hold:

Case D. If $w^0=1/2$ and $\forall_{m\in\mathbb{N}}\{w^m=1/2\}$, then $\lim_{m\to\infty}\sup_{n>m} P(w^m,w^n)=\lim_{m\to\infty}\sup_{n>m} P(w^n,w^m)=0$, $[1/2;1)=\mathrm{LIM}^{L-P}_{(w^m:m\in\{0\}\cup\mathbb{N})}$ and $(0;1/2]=\mathrm{LIM}^{R-P}_{(w^m:m\in\{0\}\cup\mathbb{N})}$.

Case E. If $w^0=1/2$ and $\exists_{m_0\in\mathbb{N}}\forall_{m\in\{1,\dots,m_0\}}\{w^m=1/2\}$ and $w^{m_0+1}\in(0;1/2)\cup(1/2;1)$, then automatically $\forall_{m>m_0}\{w^m \in ((1-1/3^m)/2;1/2)\cup(1/2;1-(1-1/3^m)/2)\}$ and from this we deduce that then Cases A–C hold.

Therefore, by Definition 5.1, hypothesis (b) follows from Cases A–E.

Finally, let us observe that (X,T), $J=P$, and arbitrary $\mathcal{S}_{\{1\}}$-family $\mathcal{S}_{\{1\}}=\{S\}$ satisfy (9.16). Indeed, by (11.1), (11.5)–(11.6) and Cases A–E, we note that

$$\begin{cases} \forall_{\varepsilon>0}\exists_{r\in\mathbb{N}}\forall_{x^0\in M=X}\forall_{(x^m:m\in\{0\}\cup\mathbb{N})\in\mathcal{O}_{X,T}(x^0)}\forall_{s,l\in\mathbb{N}}\{8\cdot P(x^{s+r},x^{l+r}) \\ \quad\le 8\cdot(x^{s+r}-x^{l+r})^4\le 8\cdot(1/3^{4r})(1/3^l-1/3^s)^4/2^4<\varepsilon\}. \end{cases}$$

Consequently, using Definition 8.1, we obtain

$$\forall_{\varepsilon>1}\exists_{r\in\mathbb{N}}\forall_{x^0\in M=X}\forall_{(x^m:m\in\{0\}\cup\mathbb{N})\in\mathcal{O}_{X,T}(x^0)}\forall_{s,l\in\mathbb{N}}\bigl\{\bigl[S\bigl(P\bigl(x^{s+r},x^{l+r}\bigr)\bigr)\bigr]^8<\varepsilon\bigr\}.$$

In view of this, we see that

$$\begin{cases} \forall_{\varepsilon>1}\exists_{\eta>1}\exists_{r\in\mathbb{N}}\forall_{x^0\in M=X}\forall_{(x^m:m\in\{0\}\cup\mathbb{N})\in\mathcal{O}_{X,T}(x^0)} \\ \quad\forall_{s,l\in\mathbb{N}}\{S(P(x^s,x^l))<\varepsilon\cdot\eta\Rightarrow[S(p(x^{s+r},x^{l+r}))]^8<\varepsilon\}. \end{cases}$$

Part 2. (X,T) is a left and right P-closed in each point $w^0 \in X$. Indeed, for each $w^0 \in X$ and for each dynamic process $(w^m : m \in \{0\} \cup \mathbb{N}) \in \mathcal{O}_{X,T}(w^0)$, the subsequences $(y_m = w^{m+1} : m \in \mathbb{N})$ and $(x_m = w^m : m \in \mathbb{N})$ of $(w^m : m \in \{0\} \cup \mathbb{N})$ satisfy $\forall_{m\in\mathbb{N}}\{y_m \in T(x_m)\}$. Moreover, in view of Cases A–E, we get $1/2 \in U = \mathrm{LIM}^{L-P}_{(w^m:m\in\{0\}\cup\mathbb{N})} \subset \mathrm{LIM}^{L-P}_{(x_m:m\in\mathbb{N})} \cap \mathrm{LIM}^{L-P}_{(y_m:m\in\mathbb{N})}$ and $1/2 \in V = \mathrm{LIM}^{R-P}_{(w^m:m\in\{0\}\cup\mathbb{N})} \subset \mathrm{LIM}^{R-P}_{(x_m:m\in\mathbb{N})} \cap \mathrm{LIM}^{R-P}_{(y_m:m\in\mathbb{N})}$. Observe also that $1/2 \in T(1/2) = (0;1)$.

Claim 1. By Parts 1 *and* 2*, for $J = P$, $q = 1$, $M = X$, and for arbitrary $\mathcal{S}_{\{1\}}$-family $\mathcal{S}_{\{1\}} = \{S\}$, hypotheses* (a)–(d) *and statements* (A) *and* (B) *of Theorem* 9.2 *hold.* We have: (i) $\mathrm{Fix}_X(T) = \{1/2\}$ and $T(1/2) = X$. (ii) For each $w^0 \in M = X$, every dynamic process $(w^m : m \in \{0\} \cup \mathbb{N}) \in \mathcal{O}_{X,T}(w^0)$ is left and right P-convergent to $1/2$.

Remark 11.1 We observe that in the cases of left and right we do not apply Theorems 9.1 to (X,T) from Example 11.2. In fact, assume in (9.1) that $(x^m : m \in \{0\} \cup \mathbb{N}) \in \mathcal{O}_{X,T}(x^0)$ and $(y^m : m \in \{0\} \cup \mathbb{N}) \in \mathcal{O}_{X,T}(y^0)$ are such that $x^m = y^m = 1/2$ for $m \in \{0,1,\dots,r-1\}$ but, e.g., $x^r = 1/4$ and $y^r = 3/4$ or $x^r = 3/4$ and $y^r = 1/4$. Then from (9.1) we see that for each $\varepsilon > 1$ and $\eta > 1$ there exists $r \in \mathbb{N}$ such that $S(P(1/2,1/2)) = S(0) = 1 < \varepsilon \cdot \eta \Rightarrow [S(P(x^r,y^r))]^8 = [S((3/4-1/4)^4)]^8 = [S((1/2)^4)]^8 < \varepsilon$. However, since $[S((1/2)^4)]^8 > 1$, this is impossible for each ε such that $1 < \varepsilon < [S((1/2)^4)]^8$.

Example 11.3 Let $X = [0;1]$. Let $P : X \times X \to [0;+\infty]$ be given by (11.1) for $X = [0;1]$. Suppose also that (X,T) is a set-valued dynamic system defined by

$$T(x) = \begin{cases} \{0\} & \text{for } x = 0, \\ (0;x/2) & \text{for } x \in (0;1/2), \\ (0;1) & \text{for } x = 1/2, \\ (x/2+1/2;1) & \text{for } x \in (1/2;1), \\ \{1\} & \text{for } x = 1. \end{cases} \tag{11.7}$$

Part 1. For $J = P$, $M = X$, $q = 1$ and (X,T) given by (11.7)*, harking back to the discussion of Example* 11.2*, we may easily verify the hypotheses of Theorem* 9.2. Indeed, by (11.1) and Definitions 5.1 and 2.3, hypothesis (b) of Theorem 9.2 holds, since:

Case A. If $w^0 = 0$, then $\forall_{m\in\mathbb{N}}\{w^m = 0\}$ and $\lim_{m\to\infty}\sup_{n>m} P(w^m,w^n) = \lim_{m\to\infty}\sup_{n>m} P(w^n,w^m) = 0$. Moreover, $X = [0;1] = \mathrm{LIM}^{L-P}_{(w^m:m\in\{0\}\cup\mathbb{N})}$ and $\{0\} = \mathrm{LIM}^{R-P}_{(w^m:m\in\{0\}\cup\mathbb{N})}$.

Case B. If $w^0 \in (0;1/2)$, then $\forall_{m\in\mathbb{N}}\{w^m \in (0;w^0/2^m)$ and this implies that $\lim_{m\to\infty}\sup_{n>m} P(w^m,w^n) = \lim_{m\to\infty}\sup_{n>m} P(w^n,w^m) = 0$. Moreover, $(0;1] = \mathrm{LIM}^{L-P}_{(w^m:m\in\{0\}\cup\mathbb{N})}$ and $\{0\} = \mathrm{LIM}^{R-P}_{(w^m:m\in\{0\}\cup\mathbb{N})}$.

Case C. If $w^0 = 1/2$ and $\exists_{m_0\in\mathbb{N}}\forall_{m\in\{1,\dots,m_0\}}\{w^m = 1/2\}$ and $w^{m_0+1} \in (0;1/2)$, then $\forall_{m\in\mathbb{N}}\{w^{m_0+1+m} \in (0;w^{m_0+1}/2^m)$ and $\lim_{m\to\infty}\sup_{n>m} P(w^m,w^n) = \lim_{m\to\infty}\sup_{n>m} P(w^n, w^m) = 0$. Moreover, $(0;1] = \mathrm{LIM}^{L-P}_{(w^m:m\in\{0\}\cup\mathbb{N})}$ and $\{0\} = \mathrm{LIM}^{R-P}_{(w^m:m\in\{0\}\cup\mathbb{N})}$.

Case D. If $w^0 = 1/2$ and $\forall_{m\in\mathbb{N}}\{w^m = 1/2\}$, then $\lim_{m\to\infty}\sup_{n>m} P(w^m,w^n) = \lim_{m\to\infty}\sup_{n>m} P(w^n,w^m) = 0$. Moreover, $[1/2;1] = \mathrm{LIM}^{L-P}_{(w^m:m\in\{0\}\cup\mathbb{N})}$ and $[0;1/2] = \mathrm{LIM}^{R-P}_{(w^m:m\in\{0\}\cup\mathbb{N})}$.

Case E. If $w^0 = 1/2$ and $\exists_{m_0\in\mathbb{N}}\forall_{m\in\{1,\dots,m_0\}}\{w^m = 1/2\}$ and $w^{m_0+1} \in (1/2;1)$, then $\forall_{m\in\mathbb{N}}\{w^{m_0+1+m} \in (1-(1-w^{m_0+1})/2^m;1)$. This implies that $\lim_{m\to\infty}\sup_{n>m} P(w^m,w^n) =$

$\lim_{m\to\infty}\sup_{n>m} P(w^n, w^m) = 0$. Moreover, $\{1\} = \mathrm{LIM}^{L-P}_{(w^m:m\in\{0\}\cup\mathbb{N})}$ and $[0;1) = \mathrm{LIM}^{R-P}_{(w^m:m\in\{0\}\cup\mathbb{N})}$.

Case F. If $w^0 \in (1/2;1)$, then $\forall_{m\in\mathbb{N}}\{w^m \in (1-(1-w^0)/2^m;1)$ and also $\lim_{m\to\infty}\sup_{n>m} P(w^m, w^n) = \lim_{m\to\infty}\sup_{n>m} P(w^n, w^m) = 0$. Moreover, $\{1\} = \mathrm{LIM}^{L-P}_{(w^m:m\in\{0\}\cup\mathbb{N})}$ and $[0;1) = \mathrm{LIM}^{R-P}_{(w^m:m\in\{0\}\cup\mathbb{N})}$.

Case G. If $w^0 = 1$, then $\forall_{m\in\mathbb{N}}\{w^m = 1\}$ and $\lim_{m\to\infty}\sup_{n>m} P(w^m, w^n) = \lim_{m\to\infty}\sup_{n>m} P(w^n, w^m) = 0$. Moreover, $\{1\} = \mathrm{LIM}^{L-P}_{(w^m:m\in\{0\}\cup\mathbb{N})}$ and $[0;1] = \mathrm{LIM}^{R-P}_{(w^m:m\in\{0\}\cup\mathbb{N})}$.

From Cases A–G it follows also that (9.16) holds for $M = X$ and that (X, T) is left and right P-closed in each point $w^0 \in M = X$.

Claim 1. For $J = P$, $M = X$ and $q = 1$ the hypotheses and statements of Theorem 9.2 hold. We have: (i) $\mathrm{Fix}_X(T) = \{0, 1/2, 1\}$ and $\mathrm{End}_X(T) = \{0, 1\}$. (ii) For each $w^0 \in X$, every dynamic process $(w^m : m \in \{0\}\cup\mathbb{N}) \in \mathcal{O}_{X,T}(w^0)$ satisfies $1 \in \mathrm{LIM}^{L-P}_{(w^m:m\in\{0\}\cup\mathbb{N})}$ and $0 \in \mathrm{LIM}^{R-P}_{(w^m:m\in\{0\}\cup\mathbb{N})}$.

Example 11.4 Let $X = (0;4)$ and define (X, T) by

$$T(x) = \begin{cases} 7/2 & \text{if } x \in (0;2], \\ (1/2)(x-3)+3 & \text{if } x \in (2;3], \\ 1/2 & \text{if } x \in (3;4). \end{cases} \tag{11.8}$$

For $A = \{1/2\} \cup (5/2;3] \cup \{7/2\} \subset X$, we set

$$P(u, v) = \begin{cases} 0 & \text{if } u = v \text{ or } A \cap \{u, v\} = \{u, v\}, \\ 1 & \text{if } u \neq v \text{ and } A \cap \{u, v\} \neq \{u, v\}, \end{cases} \tag{11.9}$$

where $u, v \in X$. We see that $C = 1$. Observe that:

Part 1. If $J = P$, then (X, T) is left and right P-admissible on X. Indeed, first we note that, for $m \in \mathbb{N}$,

$$T^{[2m]}(x) = \begin{cases} 1/2 & \text{if } x \in (0;2], \\ (1/2^{2m})(x-3)+3 & \text{if } x \in (2;3], \\ 7/2 & \text{if } x \in (3;4), \end{cases} \tag{11.10}$$

and, for $m \in \{0\} \cup \mathbb{N}$,

$$T^{[2m+1]}(x) = \begin{cases} 7/2 & \text{if } x \in (0;2], \\ (1/2^{2m+1})(x-3)+3 & \text{if } x \in (2;3], \\ 1/2 & \text{if } x \in (3;4). \end{cases} \tag{11.11}$$

Using (11.10) and (11.11) we, therefore, have

$$T^{[m]}(X) \subset A. \tag{11.12}$$

Next, (11.9) and (11.12) imply that, for each $w^0 \in X$, a sequence $(w^m = T^{[m]}(w^0) : m \in \{0\} \cup \mathbb{N})$ satisfies $\lim_{m\to\infty} \sup_{n>m} P(w^m, w^n) = \lim_{m\to\infty} \sup_{n>m} P(w^n, w^m) = 0$ and

$$\forall_{w\in A}\Big\{\lim_{m\to\infty} P\big(w, w^m\big) = \lim_{m\to\infty} P\big(w^m, w\big) = 0\Big\}. \tag{11.13}$$

Part 2. If $J = P$, then hypothesis (d) *of Theorem* 9.3 *holds.* Indeed, by (11.9) and (11.11), for each $\mathcal{S}_{\{1\}}$-family $\mathcal{S}_{\{1\}} = \{S\}$ we have $\forall_{\varepsilon>1}\exists_{\eta>1}\exists_{r\in\mathbb{N}}\forall_{x,y\in X}\forall_{s,l\in\mathbb{N}}\{S[P(T^{[s]}(x), T^{[l]}(y))] < \varepsilon \cdot \eta \Rightarrow S[P(T^{[s+r]}(x), T^{[l+r]}(y))] = 1 < \varepsilon\}$.

Part 3. $(X, T^{[2]})$ is left and right P-closed on X. Indeed, if $w^0 \in X$ is arbitrary and fixed, then $(w^m = T^{[m]}(w^0) : m \in \{0\} \cup \mathbb{N})$ is a left and right P-converging sequence to each point of A (see (11.13)), any two subsequences $(y_m : m \in \mathbb{N})$ and $(x_m : m \in \mathbb{N})$ of $(w^m = T^{[m]}(w^0) : m \in \{0\} \cup \mathbb{N})$ satisfying $\forall_{m\in\mathbb{N}}\{y_m = T^{[2]}(x_m)\}$ are left and right P-converging to each point of A, and $\mathrm{Fix}_X(T^{[2]}) = \{w = T^{[2]}(w) : w \in \{1/2, 3, 7/2\}\} \subset A = \mathrm{LIM}^{L-P}_{(w^m:m\in\{0\}\cup\mathbb{N})} = \mathrm{LIM}^{R-P}_{(w^m:m\in\{0\}\cup\mathbb{N})}$. In virtue of Definition 7.2, $(X, T^{[2]})$ is left and right P-closed on X.

Part 4. (X, T) is a left and right $J = P$-closed on X. Indeed, let $w^0 \in X$ be arbitrary and fixed and let $q = 1$. Observe that sequence $(w^m = T^{[m]}(w^0) : m \in \{0\} \cup \mathbb{N})$ satisfies $\{3\} = \mathrm{Fix}_X(T) \subset A = \mathrm{LIM}^{L-P}_{(w^m:m\in\{0\}\cup\mathbb{N})}\} = \mathrm{LIM}^{R-P}_{(w^m:m\in\{0\}\cup\mathbb{N})}$, subsequences $(y_m = w^{m+1} : m \in \mathbb{N})$ and $(x_m = w^m : m \in \mathbb{N})$ of $(w^m = T^{[m]}(w^0) : m \in \{0\} \cup \mathbb{N})$ are left and right P-converging to each point of A, and $\forall_{m\in\mathbb{N}}\{y_m = T(x_m)\}$.

Part 5. P is not separating on X. Indeed, by (11.11), for each $x, y \in X$ such that $x \neq y$ and $A \cap \{x, y\} = \{x, y\}$, we have $\forall_{t\in(0;\infty)}\{P(x, y) = P(y, x) = 0\}$.

Claim 1. By Parts 1–3 *and* 5, *for $J = P$ and $q = 2$, the statements* (A) *and* (B) *of Theorem* 9.3 *hold.* We have: (i) $\mathrm{Fix}_X(T^{[2]}) = \{1/2, 3, 7/2\} \neq \varnothing$. (ii) For each $w^0 \in X$, a sequence $(w^m = T^{[m]}(w^0) : m \in \{0\} \cup \{0\} \cup \mathbb{N})$ is left and right P-convergent to each $w \in \mathrm{Fix}_X(T^{[2]})$. (iii) $\forall_{w\in\mathrm{Fix}_X(T^{[2]})}\{P(w, w) = P(w, T(w)) = P(T(w), w) = 0\}$.

Claim 2. By Parts 1, 2, 4 *and* 5, *for $J = P$ and $q = 1$, the statements* (A) *and* (B) *of Theorem* 9.3 *hold.* We have: (i) $\mathrm{Fix}_X(T) = \{3\} \neq \varnothing$. (ii) For each $w^0 \in X$, a sequence $(w^m = T^{[m]}(w^0) : m \in \{0\} \cup \mathbb{N})$ is left and right P-convergent to 3. (iii) $P(3, 3) = 0$.

Example 11.5 Let $X = (2; 4)$ and $P : X \times X \to [0; +\infty]$ be given by (11.1); thus $C = 8$. Suppose also that (X, T) is a single-valued dynamic system defined by

$$T(x) = \begin{cases} \pi_m & \text{for } x = \pi_{m+1}, m \in \{0\} \cup \mathbb{N}\}, \\ \omega_m & \text{for } x = \omega_{m+1}, m \in \{0\} \cup \mathbb{N}\}, \\ 3 & \text{for } x = \pi_0 = \omega_0 = 3, \\ 3 & \text{for } x \in \Theta, \end{cases} \tag{11.14}$$

where $\Pi = \{\pi_m = 4 - 1/2^m : m \in \{0\} \cup \mathbb{N}\}$, $\Omega = \{\omega_m = 2 + 1/2^m : m \in \{0\} \cup \mathbb{N}\}$, $\Theta = (2; 4)\backslash[\Pi \cup \Omega]$.

Let us observe that:

Part 1. For $J = P$, for each $\mathcal{S}_{\{1\}}$-family $\mathcal{S}_{\{1\}} = \{S\}$, and for dynamic system (X, T) the hypotheses (a)–(c) of Theorem 9.3 *hold.* Indeed, hypothesis (a) holds since, by Remark 3.1(a), $P \in \mathbb{J}^L_{(X,P)} \cap \mathbb{J}^R_{(X,P)}$. Next, we see that dynamic system (X, T) is admissible on X. In fact, using (11.14) we find

$$\forall_{m\in\mathbb{N}}\forall_{x\in\Theta}\big\{T^{[m]}(\pi_m) = T^{[m]}(\pi_0) = T^{[m]}(\omega_m) = T^{[m]}(\omega_0) = T^{[m]}(x) = 3\big\}. \tag{11.15}$$

Next, by (11.14) and (11.15), for each $w^0 \in X$, sequence $(w^m = T^{[m]}(w^0) : m \in \{0\} \cup \mathbb{N})$ satisfies $\lim_{m\to\infty} \sup_{n>m} P(w^m, w^n) = \lim_{m\to\infty} \sup_{n>m} P(w^n, w^m) = 0$ and $\mathrm{LIM}^{L-P}_{(w^m:m\in\{0\}\cup\mathbb{N})} = [3;4)$ and $\mathrm{LIM}^{R-P}_{(w^m:m\in\{0\}\cup\mathbb{N})} = (2;3]$, which means, by Definition 5.2, that hypothesis (b) holds.

Finally, let us observe that (X, T), $J = P$, and arbitrary $\mathcal{S}_{\{1\}}$-family $\mathcal{S}_{\{1\}} = \{S\}$ satisfy hypothesis (d). Indeed, by (11.14) and (11.15), we note that $\exists_{r\in\mathbb{N}} \forall_{x,y\in X} \forall_{s,l\in\mathbb{N}} \{P(T^{[s+r]}(x), T^{[l+r]}(y) = P(3,3) = 0\}$. Consequently, we obtain $\forall_{\varepsilon>1} \exists_{\eta>1} \exists_{r\in\mathbb{N}} \forall_{x,y\in X} \forall_{s,l\in\mathbb{N}} \{S(P(T^{[s]}(x), T^{[l]}(y))) < \varepsilon \cdot \eta \Rightarrow [S(P(T^{[s+r]}(x), T^{[l+r]}(y)))]^8 = [S(0)]^8 = 1 < \varepsilon\}$.

Part 2. (X, T) is a left and right P-closed on X. Indeed, for each $w^0 \in X$, the subsequences $(y_m = w^{m+1} : m \in \mathbb{N})$ and $(x_m = w^m : m \in \mathbb{N})$ of $(w^m = T^{[m]}(w^0) : m \in \{0\} \cup \mathbb{N})$ satisfy $\forall_{m\in\mathbb{N}} \{y_m = T(x_m)\}$ and also $\mathrm{Fix}_X(T) = \{3\}$.

Part 3. P is separating on X (see Part 4 of Example 11.1).

Claim 1. By Parts 1–3, *for $J = P$, for arbitrary $\mathcal{S}_{\{1\}}$-family, and for $q = 1$, hypotheses* (a)–(c) *and statements* (A)–(C) *of Theorem* 9.3 *hold.* We have: (i) $\mathrm{Fix}(T) = \{3\}$. (ii) For each $w^0 \in X$, sequence $(w^m = T^{[m]}(w^0) : m \in \{0\} \cup \mathbb{N})$ is left and right P-convergent to 3. (iii) $P(3, 3) = 0$.

Example 11.6 Let $X = [2;4]$ and let $P : X \times X \to [0;+\infty]$ be of the form (11.1); thus $C = 8$. Let $\varPi = \{\pi_m = 3 - 1/2^m : m \in \{0\} \cup \mathbb{N}\}$, $\varOmega = \{\omega_m = 3 + 1/2^m : m \in \{0\} \cup \mathbb{N}\}$ and $\varTheta = [2;4] \backslash [\varPi \cup \varOmega]$ and let (X, T) be defined by

$$T(x) = \begin{cases} \pi_m & \text{for } x = \pi_{m+1}, m \in \{0\} \cup \mathbb{N}\}, \\ 2 & \text{for } x = \pi_0 = 2, \\ \omega_m & \text{for } x = \omega_{m+1}, m \in \{0\} \cup \mathbb{N}\}, \\ 4 & \text{for } x = \omega_0 = 4, \\ 3 & \text{for } x \in \varTheta. \end{cases} \tag{11.16}$$

Let us observe that:

Part 1. For $J = P$, for each $\mathcal{S}_{\{1\}}$-family $\mathcal{S}_{\{1\}} = \{S\}$, for $M = X$ and for dynamic system (X, T) the hypotheses (a)–(d) *of Theorem* 9.4 *hold.*

Indeed, hypothesis (a) holds since, by Remark 3.1(a), $P \in \mathbb{J}^L_{(X,P)} \cap \mathbb{J}^R_{(X,P)}$.

Next, we see that dynamic system (X, T) is admissible on $M = X$. In fact, by (11.16), observe that $\forall_{m\in\mathbb{N}} \{T^{[m]}(\pi_m) = T^{[m]}(\pi_0) = 2\}$, $\forall_{m\in\mathbb{N}} \{T^{[m]}(\omega_m) = T^{[m]}(\omega_0) = 4\}$, and $\forall_{x\in\varTheta} \forall_{m\in\mathbb{N}} \{T^{[m]}(x) = 3\}$. From this, using (11.1), we see that, for each $w^0 \in M = X$, sequence $(w^m = T^{[m]}(w^0) : m \in \{0\} \cup \mathbb{N})$ satisfies $\lim_{m\to\infty} \sup_{n>m} P(w^m, w^n) = \lim_{m\to\infty} \sup_{n>m} P(w^n, w^m) = 0$. Furthermore, from (11.16) it follows that

$$\mathrm{LIM}^{L-P}_{(w^m:m\in\{0\}\cup\mathbb{N})} = \begin{cases} [2;4] & \text{for } w^0 \in \varPi, \\ \{4\} & \text{for } w^0 \in \varOmega, \\ [3;4] & \text{for } x \in w^0 \in \varTheta, \end{cases}$$

and

$$\mathrm{LIM}^{R-P}_{(w^m:m\in\{0\}\cup\mathbb{N})} = \begin{cases} \{2\} & \text{for } w^0 \in \varPi, \\ [2;4] & \text{for } w^0 \in \varOmega, \\ [2;3] & \text{for } w^0 \in \varTheta. \end{cases}$$

Therefore, hypothesis (b) holds.

Finally, let us observe that (X, T), $J = P$, and arbitrary $\mathcal{S}_{\{1\}}$-family $\mathcal{S}_{\{1\}} = \{S\}$ satisfy (d). In fact, since $\forall_{m\in\{0\}\cup\mathbb{N}}\forall_{s,l\geq m}\{P(T^{[s]}(\pi_m), T^{[l]}(\pi_m)) = P(2,2) = 0\}$, $\forall_{m\in\{0\}\cup\mathbb{N}}\forall_{s,l\geq m}\{P(T^{[s]}(\omega_m), T^{[l]}(\omega_m)) = P(4,4) = 0\}$ and also $\forall_{x\in\Omega}\forall_{s,l\geq 0}\{P(T^{[s]}(x), T^{[l]}(x)) = P(3,3) = 0\}$, thus we have $\forall_{\varepsilon>1}\exists_{\eta>1}\exists_{r\in\mathbb{N}}\forall_{w^0\in M=X}\forall_{s,l\in\mathbb{N}}\{S(P(T^{[s]}(w^0), T^{[l]}(w^0))) < \varepsilon\cdot\eta \Rightarrow [S(P(T^{[s+r]}(w^0), T^{[l+r]}(w^0)))]^8 = [S(0)]^8 = 1 < \varepsilon\}$.

Part 2. (X, T) is a left and right P-closed on X. Indeed, for each $w^0 \in X$, the subsequences $(y_m = w^{m+1} : m \in \mathbb{N})$ and $(x_m = w^m : m \in \mathbb{N})$ of $(w^m = T^{[m]}(w^0) : m \in \{0\} \cup \mathbb{N})$ satisfy $\forall_{m\in\mathbb{N}}\{y_m = T(x_m)\}$. Moreover, $\mathrm{Fix}_X(T) = \{2,3,4\}$.

Part 3. P is separating on X. See Example 11.1 (Part 4).

Claim 1. By Parts 1–3, *for $J = P$ and $q = 1$, hypotheses* (a)–(c) *and statements* (A) and (B) *of Theorem* 9.4 *hold.* We have: (i) $\mathrm{Fix}(T) = \{2,3,4\}$. (ii) If $w^0 \in M = X$, then the sequence $(w^m = T^{[m]}(w^0) : m \in \{0\} \cup \mathbb{N})$ is left and right P-convergent to 2 when $w^0 \in \Pi$, to 4 when $w^0 \in \Omega$, and to 3 when $w^0 \in \Theta$. (iii) $P(2,2) = P(3,3) = P(4,4) = 0$.

12 Convergence, existence and uniqueness results for functional equations of Bellman type

In this section, before proceeding further, let us make the following assumptions and notation:

I. X denotes a nonempty set, $B(X)$ denotes the set of all bounded real-valued maps on X, $(B(X), \|\cdot\|)$ is a normed space with norm

$$\|h\| = \sup\{|h(x)| : x \in X\}, \quad h \in B(X)$$

and $(B(X), P)$ is a metric space with metric $P : B(X) \times B(X) \to [0;\infty)$ defined by

$$P(h,k) = \|h - k\|, \quad h,k \in B(X).$$

Definition 12.1 Let $(B(X), P)$ be a metric space defined above.

(A) The distance $J : B(X) \times B(X) \to [0;\infty)$ is said to be a *left distance generated by P* if the following two conditions hold:

(A.1) $\forall_{h,u,k\in B(X)}\{J(h,k) \leq J(h,u) + J(u,k)\}$.

(A.2) For any sequences $(h_m : m \in \mathbb{N})$ and $(k_m : m \in \mathbb{N})$ in $B(X)$ with the properties $\lim_{m\to\infty}\sup_{n>m} J(h_m, h_n) = 0$ and $\lim_{m\to\infty} J(k_m, h_m) = 0$ we have $\lim_{m\to\infty} P(k_m, h_m) = 0$.

(B) The distance $J : B(X) \times B(X) \to [0;\infty)$ is said to be a *right distance generated by P* if the following two conditions hold:

(B.1) $\forall_{h,u,k\in B(X)}\{J(h,k) \leq J(h,u) + J(u,k)\}$.

(B.2) For any sequences $(h_m : m \in \mathbb{N})$ and $(k_m : m \in \mathbb{N})$ in $B(X)$ with the properties $\lim_{m\to\infty}\sup_{n>m} J(h_n, h_m) = 0$ and $\lim_{m\to\infty} J(h_m, k_m) = 0$ we have $\lim_{m\to\infty} P(h_m, k_m) = 0$.

(C) Denote by $\mathbb{J}^L_{(B(X),P)}$ (respectively, $\mathbb{J}^R_{(B(X),P)}$) the family of all left (respectively, right) distances J generated by P.

Remark 12.1 The following holds: $P \in \mathbb{J}^L_{(B(X),P)} \cap \mathbb{J}^R_{(B(X),P)}$. Here J and P are triangular distances. For details, see Definitions 2.1 and 3.1.

II. Y denotes a nonempty set.

III. We are concerned here with the study of the functional equation of Bellman type of the form

$$h(x) = \sup_{y\in Y}\{f(x,y) + G(x,y,h(\xi(x,y)))\}, \quad x \in X, \tag{12.1}$$

where $f : X \times Y \to \mathbb{R}$ and $G : X \times Y \times \mathbb{R} \to \mathbb{R}$ are given bounded maps, $\xi : X \times Y \to X$ is a given map, and $h \in B(X)$ is an unknown map to be determined.

IV. Put

$$\mathcal{H} = \Big\{h \in B(X) : h(x) = \sup_{y\in Y}\{f(x,y) + G(x,y,h(\xi(x,y)))\}, x \in X\Big\};$$

the set of all solutions $h \in B(X)$ of Eq. (12.1).

V. The operator $\mathcal{B}$ of Bellman type is of the form

$$(\mathcal{B}h)(x) = \sup_{y\in Y}\{f(x,y) + G(x,y,h(\xi(x,y)))\}, \quad h \in B(X), x \in X. \tag{12.2}$$

Here, we define the dynamic system $(B(X),\mathcal{B})$ as follows: For $h \in B(X)$ let $\mathcal{B}h = k$, where $k(x) = \sup_{y\in Y}\{f(x,y) + G(x,y,h(\xi(x,y)))\}$, $x \in X$. Clearly k is bounded, since f and G are bounded and so $k \in B(X)$. Therefore, $\mathcal{B} : B(X) \to B(X)$.

VI. The operators $\mathcal{B}^{[m+1]} : B(X) \to B(X)$, $m \in \{0\} \cup \mathbb{N}$, are defined by

$$(\mathcal{B}^{[m+1]}h)(x) = \sup_{y\in D}\{f(x,y) + G(x,y,(\mathcal{B}^{[m]}h)(\xi(x,y)))\} \tag{12.3}$$

for all $h \in B(X)$ and $x \in X$.

VII. Any fixed point of $(B(X),\mathcal{B})$ is a solution of Eq. (12.1). Moreover, any periodic point of dynamic system $(B(X),\mathcal{B})$, i.e., any point of the set

$$\mathrm{Per}_{B(X)}(\mathcal{B}) = \{h \in B(X) : h = \mathcal{B}^{[q]}h \text{ for some } q \in \mathbb{N}\},$$

is a solution of (12.1).

VIII. Recalling that f and G are bounded, we conclude that, for each $h^0 \in B(X)$, the sequence of iterations

$$\big(h^m = \mathcal{B}^{[m]}h^0 : m \in \{0\} \cup \mathbb{N}\big) \subset B(X) \tag{12.4}$$

starting at $h^0 \in B(X)$ is well defined. Here $\mathcal{B}^{[0]} = I_{B(X)}$-identity on $B(X)$.

Moreover, with the above assumptions and notation, we also record the following two definitions needed in the sequel.

Definition 12.2 Let $(B(X),P)$ be a metric space defined above.

(A) Let $J \in \mathbb{J}^L_{(B(X),P)}$; thus, in particular, let $J = P$. Let $h^0 \in B(X)$. $(B(X),\mathcal{B})$ is said to be a *left J-admissible in h^0* if, in the case when the sequence $(\mathcal{B}^{[m]}h^0 : m \in \{0\} \cup \mathbb{N})$ is *left J-sequence in $B(X)$* (i.e. satisfies the condition $\lim_{m\to\infty}\sup_{n>m} J(\mathcal{B}^{[m]}h^0,\mathcal{B}^{[n]}h^0) = 0$), then the sequence $(\mathcal{B}^{[m]}h^0 : m \in \{0\} \cup \mathbb{N})$ is *left J-convergent in $B(X)$* (i.e. has the property $\emptyset \neq \mathrm{LIM}^{L-J}_{(\mathcal{B}^{[m]}h^0:\{0\}\cup\mathbb{N})} = \{k \in B(X) : \lim_{m\to\infty} J(k,\mathcal{B}^{[m]}h^0) = 0\}$).

(B) Let $J \in \mathbb{J}^R_{(B(X),P)}$; thus, in particular, let $J = P$. Let $h^0 \in B(X)$. $(B(X), \mathcal{B})$ is said to be a *right J-admissible in h^0* if, in the case when the sequence $(\mathcal{B}^{[m]}h^0 : m \in \{0\} \cup \mathbb{N})$ is *right J-sequence in $B(X)$* (i.e. satisfies the condition $\lim_{m\to\infty} \sup_{n>m} J(\mathcal{B}^{[n]}h^0, \mathcal{B}^{[m]}h^0) = 0$), then the sequence $(\mathcal{B}^{[m]}h^0 : m \in \{0\} \cup \mathbb{N})$ is *right J-convergent in $B(X)$* (i.e. has the property $\varnothing \neq \mathrm{LIM}^{R-J}_{(\mathcal{B}^{[m]}h^0:\{0\}\cup\mathbb{N})} = \{k \in B(X) : \lim_{m\to\infty} J(\mathcal{B}^{[m]}h^0, k) = 0\}$).

(C) Let $M \in 2^{B(X)}$. $(B(X), \mathcal{B})$ is said to be a *left* (respectively, *right*) *J-admissible on M* iff $J \in \mathbb{J}^L_{(B(X),P)}$ (respectively, $J \in \mathbb{J}^R_{(B(X),P)}$) and $(B(X), \mathcal{B})$ is a left (respectively, right) J-admissible in each $h^0 \in M$.

Definition 12.3 Let $(B(X), P)$ be a metric space defined above and let $q \in \mathbb{N}$.

(A) Let $J \in \mathbb{J}^L_{(B(X),P)}$; thus, in particular, let $J = P$. Let $h^0 \in B(X)$. $(B(X), \mathcal{B}^{[q]})$ is said to be a *left J-closed in h^0* if, in the case when the sequence $(\mathcal{B}^{[m]}h^0 : m \in \{0\} \cup \mathbb{N})$ is left J-convergent in $B(X)$, i.e. $\varnothing \neq U = \mathrm{LIM}^{L-J}_{(\mathcal{B}^{[m]}h^0:\{0\}\cup\mathbb{N})}$, and contains two right J-converging in $B(X)$ subsequences $(k_m : m \in \mathbb{N})$ and $(w_m : m \in \mathbb{N})$ (i.e., in particular, $\mathrm{LIM}^{R-J}_{(\mathcal{B}^{[m]}h^0:\{0\}\cup\mathbb{N})} \subset \mathrm{LIM}^{R-J}_{(k_m:m\in\mathbb{N})} \cap \mathrm{LIM}^{R-J}_{(w_m:m\in\mathbb{N})}$) satisfying $\forall_{m\in\mathbb{N}}\{k_m = \mathcal{B}^{[q]}w_m\}$, then we have $\exists_{u\in U}\{u = \mathcal{B}^{[q]}u\}$.

(B) Let $J \in \mathbb{J}^R_{(B(X),P)}$; thus, in particular, let $J = P$. Let $h^0 \in B(X)$. $(B(X), \mathcal{B}^{[q]})$ is said to be a *right J-closed in h^0* if, in the case when the sequence $(\mathcal{B}^{[m]}h^0 : m \in \{0\} \cup \mathbb{N})$ is right J-convergent in $B(X)$, i.e. $\varnothing \neq V = \mathrm{LIM}^{R-J}_{(\mathcal{B}^{[m]}h^0:\{0\}\cup\mathbb{N})}$, and contains two right J-converging in $B(X)$ subsequences $(k_m : m \in \mathbb{N})$ and $(w_m : m \in \mathbb{N})$ (i.e., in particular, $\mathrm{LIM}^{R-J}_{(\mathcal{B}^{[m]}h^0:\{0\}\cup\mathbb{N})} \subset \mathrm{LIM}^{R-J}_{(k_m:m\in\mathbb{N})} \cap \mathrm{LIM}^{R-J}_{(w_m:m\in\mathbb{N})}$) satisfying $\forall_{m\in\mathbb{N}}\{k_m = \mathcal{B}^{[q]}k_m\}$, then we have $\exists_{v\in V}\{v = \mathcal{B}^{[q]}v\}$.

(C) Let $M \in 2^{B(X)}$. $(B(X), \mathcal{B})$ is said to be a *left* (respectively, *right*) *J-closed on M* iff $J \in \mathbb{J}^L_{(B(X),P)}$ (respectively, $J \in \mathbb{J}^R_{(B(X),P)}$) and $(B(X), \mathcal{B})$ is a left (respectively, right) J-closed in each $h^0 \in M$.

The general theory concerning existence and uniqueness of solutions of functional equations of Bellman type is currently a very active field, rooted in optimization, dynamic programming, computer programming, invariant imbedding, and applications in engineering and physical sciences. Concerning these existence and uniqueness problems, most of the work requires assumptions that X and Y are Banach spaces and that the operator $\mathcal{B}$ is continuous. For these topics see, e.g., [6–8].

Here we will concentrate on convergence, existence and uniqueness problems concerning fixed and periodic points of operator $\mathcal{B}$ defined by (12.2). Thus we will study the structure of set $\mathcal{H}$ of solutions of the functional Eq. (12.1) of Bellman type in more general setting.

We have the following analogues of Theorems 9.3 and 9.4.

Theorem 12.1 *Assume that* I–VIII *are satisfied. Suppose also that*:

(a) $J \in \mathbb{J}^L_{(B(X),P)}$ (*respectively*, $J \in \mathbb{J}^R_{(B(X),P)}$).

(b) $(B(X), \mathcal{B})$ *is left* (*respectively, right*) *J-admissible on $B(X)$*.

(c) $\mathcal{S} = \{S\}$ *is an $\mathcal{S}$-family.*

(d) *$\mathcal{S}$-family* $\{S\}$, $(B(X), \mathcal{B})$ *and* $J \in \mathbb{J}^L_{(B(X),P)}$ (*respectively*, $J \in \mathbb{J}^R_{(B(X),P)}$) *satisfy*

$$\begin{cases} \forall_{\varepsilon>1}\exists_{\eta>1}\exists_{r\in\mathbb{N}}\forall_{h,k\in B(X)}\forall_{s,l\in\mathbb{N}}\{S(J(\mathcal{B}^{[s]}h, \mathcal{B}^{[l]}k)) \\ \quad < \varepsilon \cdot \eta \Rightarrow S(J(\mathcal{B}^{[s+r]}h, \mathcal{B}^{[l+r]}k)) < \varepsilon\}. \end{cases}$$

Then the following hold:

(A) Convergence property. *For each* $h^0 \in B(X)$, *we have*

$$\emptyset \neq \mathrm{LIM}^{L-J}_{(\mathcal{B}^{[m]}h^0 : m \in \{0\} \cup \mathbb{N})} \subset \mathrm{LIM}^{L-P}_{(\mathcal{B}^{[m]}h^0 : m \in \{0\} \cup \mathbb{N})} \quad \textit{where } J \in \mathbb{J}^{L}_{(B(X),P)}$$

(*respectively*,

$$\emptyset \neq \mathrm{LIM}^{R-J}_{(\mathcal{B}^{[m]}h^0 : m \in \{0\} \cup \mathbb{N})} \subset \mathrm{LIM}^{R-P}_{(\mathcal{B}^{[m]}h^0 : m \in \{0\} \cup \mathbb{N})} \quad \textit{where } J \in \mathbb{J}^{R}_{(B(X),P)}).$$

(B) Existence of solutions and convergence property. *If there exists* $q \in \mathbb{N}$ *such that the single-valued dynamic system* $(B(X), \mathcal{B}^{[q]})$ *is left* (*respectively, right*) J*-closed on* $B(X)$, *then*

$$\emptyset \neq \mathrm{Fix}_{B(X)}\big(\mathcal{B}^{[q]}\big) \subset \mathcal{H}.$$

Moreover, for each $h^0 \in B(X)$, *there exists* $u \in \mathrm{Fix}_{B(X)}(\mathcal{B}^{[q]})$ (*respectively*, $v \in \mathrm{Fix}_{B(X)}(\mathcal{B}^{[q]})$) *such that*

$$u \in \mathrm{LIM}^{L-J}_{(\mathcal{B}^{[m]}h^0 : m \in \{0\} \cup \mathbb{N})} \subset \mathrm{LIM}^{L-P}_{(\mathcal{B}^{[m]}h^0 : m \in \{0\} \cup \mathbb{N})}$$

and

$$\forall_{n \in \{1,2,\ldots,q\}} \big\{ J\big(u, \mathcal{B}^{[n]}(u)\big) = J\big(\mathcal{B}^{[n]}(u), u\big) = 0 \big\} \quad \textit{where } J \in \mathbb{J}^{L}_{(B(X),P)}$$

(*respectively*,

$$v \in \mathrm{LIM}^{R-J}_{(\mathcal{B}^{[m]}h^0 : m \in \{0\} \cup \mathbb{N})} \subset \mathrm{LIM}^{R-P}_{(\mathcal{B}^{[m]}h^0 : m \in \{0\} \cup \mathbb{N})}$$

and

$$\forall_{n \in \{1,2,\ldots,q\}} \big\{ J\big(v, \mathcal{B}^{[n]}(v)\big) = J\big(\mathcal{B}^{[n]}(v), v\big) = 0 \big\} \quad \textit{where } J \in \mathbb{J}^{R}_{(B(X),P)}).$$

(C) Existence of unique solutions and convergence property. *If* $(B(X), \mathcal{B})$ *is left* (*respectively, right*) J*-closed on* $B(X)$ *then*

$$\exists_{h \in B(X)} \big\{ \mathcal{H} = \mathrm{Fix}_{B(X)}(\mathcal{B}) = \{h\} \big\}.$$

Moreover, for each $h^0 \in B(X)$,

$$h \in \mathrm{LIM}^{L-J}_{(\mathcal{B}^{[m]}h^0 : m \in \{0\} \cup \mathbb{N})} \subset \mathrm{LIM}^{L-P}_{(\mathcal{B}^{[m]}h^0 : m \in \{0\} \cup \mathbb{N})} \quad \textit{and} \quad J(h,h) = 0$$

where $J \in \mathbb{J}^{L}_{(B(X),P)}$ (*respectively*,

$$h \in \mathrm{LIM}^{R-J}_{(\mathcal{B}^{[m]}h^0 : m \in \{0\} \cup \mathbb{N})} \subset \mathrm{LIM}^{R-P}_{(\mathcal{B}^{[m]}h^0 : m \in \{0\} \cup \mathbb{N})} \quad \textit{and} \quad J(h,h) = 0,$$

where $J \in \mathbb{J}^{R}_{(B(X),P)}$).

Theorem 12.2 *Assume that* I–VIII *are satisfied. Suppose also that*:

(a) $J \in \mathbb{J}^L_{(B(X),P)}$ (*respectively*, $J \in \mathbb{J}^R_{(B(X),P)}$).

(b) *There exists* $M \in 2^{B(X)}$ *such that* $(B(X), \mathcal{B})$ *is left* (*respectively, right*) J-*admissible on* M.

(c) $\mathcal{S} = \{S\}$ *is an* $\mathcal{S}$-*family*.

(d) $\mathcal{S}$-*family* $\mathcal{S} = \{S\}$, $(B(X), \mathcal{B})$, M *and* $J \in \mathbb{J}^L_{(B(X),P)}$ (*respectively*, $J \in \mathbb{J}^R_{(B(X),P)}$) *satisfy*

$$\begin{cases} \forall_{\varepsilon>1} \exists_{\eta>1} \exists_{r\in\mathbb{N}} \forall_{h^0 \in M} \forall_{s,l\in\mathbb{N}} \{ S(J(\mathcal{B}^{[s]}h^0, \mathcal{B}^{[l]}h^0)) \\ \quad < \varepsilon \cdot \eta \Rightarrow S(J(\mathcal{B}^{[s+r]}h^0, \mathcal{B}^{[l+r]}h^0)) < \varepsilon \}. \end{cases}$$

Then the following hold:

(A) Convergence property. *If* $h^0 \in M$, *then*

$$\emptyset \neq \mathrm{LIM}^{L-J}_{(\mathcal{B}^{[m]}h^0 : m\in\{0\}\cup\mathbb{N})} \subset \mathrm{LIM}^{L-P}_{(\mathcal{B}^{[m]}h^0 : m\in\{0\}\cup\mathbb{N})} \quad \textit{where } J \in \mathbb{J}^L_{(B(X),P)}$$

(*respectively*,

$$\emptyset \neq \mathrm{LIM}^{R-J}_{(\mathcal{B}^{[m]}h^0 : m\in\{0\}\cup\mathbb{N})} \subset \mathrm{LIM}^{R-P}_{(\mathcal{B}^{[m]}h^0 : m\in\{0\}\cup\mathbb{N})} \quad \textit{where } J \in \mathbb{J}^R_{(B(X),P)}).$$

(B) Existence of solutions and convergence property. *If there exist* $q \in \mathbb{N}$ *and* $h^0 \in M$ *such that the single-valued dynamic system* $(B(X), \mathcal{B}^{[q]})$ *is left* (*respectively, right*) J-*closed in* h^0, *then*

$$\emptyset \neq \mathrm{Fix}_{B(X)}(\mathcal{B}^{[q]}) \subset \mathcal{H}.$$

Moreover, there exists $u \in \mathrm{Fix}_{B(X)}(\mathcal{B}^{[q]})$ (*respectively*, $v \in \mathrm{Fix}_{B(X)}(\mathcal{B}^{[q]})$) *such that*

$$u \in \mathrm{LIM}^{L-J}_{(\mathcal{B}^{[m]}h^0 : m\in\{0\}\cup\mathbb{N})} \subset \mathrm{LIM}^{L-P}_{(\mathcal{B}^{[m]}h^0 : m\in\{0\}\cup\mathbb{N})}$$

and

$$\forall_{n\in\{1,2,\ldots,q\}} \{ J(u, \mathcal{B}^{[n]}(u)) = J(\mathcal{B}^{[n]}(u), u) = 0 \}$$

where $J \in \mathbb{J}^L_{(B(X),P)}$ (*respectively*,

$$v \in \mathrm{LIM}^{R-J}_{(\mathcal{B}^{[m]}h^0 : m\in\{0\}\cup\mathbb{N})} \subset \mathrm{LIM}^{R-P}_{(\mathcal{B}^{[m]}h^0 : m\in\{0\}\cup\mathbb{N})}$$

and

$$\forall_{n\in\{1,2,\ldots,q\}} \{ J(v, \mathcal{B}^{[n]}(v)) = J(\mathcal{B}^{[n]}(v), v) = 0 \}$$

where $J \in \mathbb{J}^R_{(B(X),P)}$).

The following results are special cases of the above theorems.

Theorem 12.3 *Assume that* I–VIII *are satisfied. Suppose also that*:

(a) $(B(X),\mathcal{B})$ *is* P*-admissible on* $B(X)$.

(b) $\mathcal{S}=\{S\}$ *is an* $\mathcal{S}$*-family.*

(c) $\mathcal{S}$*-family* $\mathcal{S}=\{S\}$ *and* $(B(X),\mathcal{B})$ *satisfy*

$$\begin{cases} \forall_{\varepsilon>1}\exists_{\eta>1}\exists_{r\in\mathbb{N}}\forall_{h,k\in B(X)}\forall_{s,l\in\mathbb{N}} \\ \quad\{S(\sup_{x,t\in X,y\in Y}|G(x,y,(\mathcal{B}^{[s]}h)(t)) \\ \quad - G(x,y,(\mathcal{B}^{[l]}k)(t))|) < \varepsilon\cdot\eta \\ \quad\Rightarrow S(\sup_{x,t\in X,y\in Y}|G(x,y,(\mathcal{B}^{[s+r]}h)(t)) \\ \quad - G(x,y,(\mathcal{B}^{[l+r]}k)(t))|) < \varepsilon\}. \end{cases} \tag{12.5}$$

Then the following hold:

(A) Convergence property. *For each* $h^0\in B(X)$, *there exists* $h\in B(X)$ *such that a sequence* $(h^m=\mathcal{B}^{[m]}h^0:m\in\{0\}\cup\mathbb{N})$ *is* P*-convergent to* h.

(B) Existence of solutions and convergence property. *If there exists* $q\in\mathbb{N}$ *such that the dynamic system* $(B(X),\mathcal{B}^{[q]})$ *is* P*-closed on* $B(X)$, *then*:

(B1) $\emptyset\neq\mathrm{Fix}_{B(X)}(\mathcal{B}^{[q]})\subset\mathcal{H}$.

(B2) *For each* $h^0\in B(X)$, *there exists* $h\in\mathrm{Fix}_{B(X)}(\mathcal{B}^{[q]})$ *such that the sequence* $(h^m=\mathcal{B}^{[m]}h^0:m\in\{0\}\cup\mathbb{N})$ *is* P*-convergent to* h.

(C) Existence of unique solution and convergence property. *If the dynamic system* $(B(X),\mathcal{B})$ *is* P*-closed on* $B(X)$, *then*:

(C1) *There exists* $h\in B(X)$ *such that* $\mathcal{H}=\mathrm{Fix}_{B(X)}(\mathcal{B})=\{h\}$.

(C2) *For each* $h^0\in B(X)$, *the sequence* $(h^m=\mathcal{B}^{[m]}h^0:m\in\{0\}\cup\mathbb{N})$ *is* P*-convergent to* h.

Theorem 12.4 *Assume that* I–VIII *are satisfied. Suppose also that*:

(a) *There exists* $M\in 2^{B(X)}$ *such that* $(B(X),\mathcal{B})$ *is* P*-admissible on* M.

(b) $\mathcal{S}=\{S\}$ *is an* $\mathcal{S}$*-family.*

(c) $\mathcal{S}$*-family* $\mathcal{S}=\{S\}$, M *and* $(B(X),\mathcal{B})$ *satisfy*

$$\begin{cases} \forall_{\varepsilon>1}\exists_{\eta>1}\exists_{r\in\mathbb{N}}\forall_{h^0\in M}\forall_{s,l\in\mathbb{N}} \\ \quad\{S(\sup_{x,t\in X,y\in Y}|G(x,y,(\mathcal{B}^{[s]}h^0)(t)) \\ \quad - G(x,y,(\mathcal{B}^{[l]}h^0)(t))|) < \varepsilon\cdot\eta \\ \quad\Rightarrow S(\sup_{x,t\in X,y\in Y}|G(x,y,(\mathcal{B}^{[s+r]}h^0)(t)) \\ \quad - G(x,y,(\mathcal{B}^{[l+r]}h^0)(t))|) < \varepsilon\}. \end{cases} \tag{12.6}$$

Then the following hold:

(A) Convergence property. *For each* $h^0\in M$, *there exists* $h\in B(X)$ *such that a sequence* $(h^m=\mathcal{B}^{[m]}h^0:m\in\{0\}\cup\mathbb{N})$ *is* P*-convergent to* h.

(B) Existence of solutions and convergence property. *If there exists* $h^0\in M$ *such that the dynamic system* $(B(X),\mathcal{B})$ *is* P*-closed in* h^0, *then*:

(B1) $\emptyset\neq\mathrm{Fix}_{B(X)}(\mathcal{B})\subset\mathcal{H}$.

(B2) *There exists* $h\in\mathrm{Fix}_{B(X)}(\mathcal{B})$ *such that the sequence* $(h^m=\mathcal{B}^{[m]}h^0:m\in\{0\}\cup\mathbb{N})$ *is* P*-convergent to* h.

Remark 12.2 We record some observations concerning the implications of Theorems 12.1–12.4.

(i) If $(B(X),P)$ is P-complete, then $(B(X),\mathcal{B})$ is P-admissible on each $M\in 2^{B(X)}$ (see also Sect. 6).

(ii) If there exist $q\in\mathbb{N}$ and $h^0\in B(X)$ such that the dynamic system $(B(X),\mathcal{B}^{[q]})$ is P-continuous in h^0, then $(B(X),\mathcal{B}^{[q]})$ is P-closed in h^0.

(iii) Here X and Y are nonempty sets. In the literature, X and Y are Banach spaces or complete metric spaces. We see that Theorems 12.1–12.4 are new even when X and Y are these spaces.

Proof of Theorem 12.3 We first note that P is symmetric on $B(X)$ and $(B(X),P)$ is not necessarily P-sequentially complete (see Sect. 6).

We prove that (12.5) implies

$$\begin{aligned}&\forall_{\varepsilon>1}\exists_{\eta>1}\exists_{r\in\mathbb{N}}\forall_{h,k\in B(X)}\forall_{s,l\in\mathbb{N}}\bigl\{S\bigl[P\bigl(\mathcal{B}^{[s+1]}h,\mathcal{B}^{[l+1]}k\bigr)\bigr]<\varepsilon\cdot\eta\\&\quad\Longrightarrow S\bigl[P\bigl(\mathcal{B}^{[s+r+1]}h,\mathcal{B}^{[l+r+1]}k\bigr)\bigr]<\varepsilon\bigr\}.\end{aligned}\tag{12.7}$$

To establish this, let $x\in X$ and $h,k\in B(X)$ be arbitrary and fixed. Then, for arbitrary $\mu>0$, in view of (12.3), there exist $y_1,y_2\in Y$ such that

$$\bigl(\mathcal{B}^{[s+1]}h\bigr)(x)<f(x,y_1)+G\bigl(x,y_1,\bigl(\mathcal{B}^{[s]}h\bigr)\bigl(\xi(x,y_1)\bigr)\bigr)+\mu,\tag{12.8}$$

$$\bigl(\mathcal{B}^{[l+1]}k\bigr)(x)<f(x,y_2)+G\bigl(x,y_2,\bigl(\mathcal{B}^{[l]}k\bigr)\bigl(\xi(x,y_2)\bigr)\bigr)+\mu.\tag{12.9}$$

Observe also that (12.3) implies

$$\bigl(\mathcal{B}^{[l+1]}k\bigr)(x)\ge f(x,y_1)+G\bigl(x,y_1,\bigl(\mathcal{B}^{[l]}k\bigr)\bigl(\xi(x,y_1)\bigr)\bigr),\tag{12.10}$$

$$\bigl(\mathcal{B}^{[s+1]}h\bigr)(x)\ge f(x,y_2)+G\bigl(x,y_2,\bigl(\mathcal{B}^{[s]}h\bigr)\bigl(\xi(x,y_2)\bigr)\bigr).\tag{12.11}$$

Restating (12.8)–(12.11) as

$$\begin{aligned}&\bigl(\mathcal{B}^{[s+1]}h\bigr)(x)-\bigl(\mathcal{B}^{[l+1]}k\bigr)(x)\\&\quad<G\bigl(x,y_1,\bigl(\mathcal{B}^{[s]}h\bigr)\bigl(\xi(x,y_1)\bigr)\bigr)-G\bigl(x,y_1,\bigl(\mathcal{B}^{[l]}k\bigr)\bigl(\xi(x,y_1)\bigr)\bigr)+\mu\\&\quad\le\bigl|G\bigl(x,y_1,\bigl(\mathcal{B}^{[s]}h\bigr)\bigl(\xi(x,y_1)\bigr)\bigr)-G\bigl(x,y_1,\bigl(\mathcal{B}^{[l]}k\bigr)\bigl(\xi(x,y_1)\bigr)\bigr)\bigr|+\mu\end{aligned}$$

and

$$\begin{aligned}&\bigl(\mathcal{B}^{[l+1]}k\bigr)(x)-\bigl(\mathcal{B}^{[s+1]}h\bigr)(x)\\&\quad<G\bigl(x,y_2,\bigl(\mathcal{B}^{[l]}k\bigr)\bigl(\xi(x,y_2)\bigr)\bigr)-G\bigl(x,y_2,\bigl(\mathcal{B}^{[s]}h\bigr)\bigl(\xi(x,y_2)\bigr)\bigr)+\mu\end{aligned}$$

or

$$\begin{aligned}&\bigl(\mathcal{B}^{[s+1]}h\bigr)(x)-\bigl(\mathcal{B}^{[l+1]}k\bigr)(x)\\&\quad>G\bigl(x,y_2,\bigl(\mathcal{B}^{[s]}h\bigr)\bigl(\xi(x,y_2)\bigr)\bigr)-G\bigl(x,y_2,\bigl(\mathcal{B}^{[l]}k\bigr)\bigl(\xi(x,y_2)\bigr)\bigr)-\mu\end{aligned}$$

$$\geq -\big|G(x,y_2,(\mathcal{B}^{[s]}h)(\xi(x,y_2)) - G(x,y_2,(\mathcal{B}^{[l]}k)(\xi(x,y_2))\big| - \mu,$$

we see that they imply

$$\begin{aligned}
&\big|(\mathcal{B}^{[s+1]}h)(x) - (\mathcal{B}^{[l+1]}k)(x)\big| \\
&\quad < \max\big\{\big|G(x,y_1,(\mathcal{B}^{[s]}h)(\xi(x,y_1)) - G(x,y_1,(\mathcal{B}^{[l]}k)(\xi(x,y_1))\big|, \\
&\big|G(x,y_2,(\mathcal{B}^{[s]}h)(\xi(x,y_2)) - G(x,y_2,(\mathcal{B}^{[l]}k)(\xi(x,y_2))\big|\big\} + \mu \\
&\quad \leq \sup_{y\in Y}\big|G(x,y,(\mathcal{B}^{[s]}h)(\xi(x,y)) - G(x,y,(\mathcal{B}^{[l]}k)(\xi(x,y))\big| + \mu.
\end{aligned}$$

Recalling that $\mu > 0$ is arbitrary, we conclude $|(\mathcal{B}^{[s+1]}h)(x) - (\mathcal{B}^{[l+1]}k)(x)| \leq \sup_{y\in Y}|G(x,y,(\mathcal{B}^{[s]}h)(\xi(x,y)) - G(x,y,(\mathcal{B}^{[l]}k)(\xi(x,y))|$ and in view of (12.1) and (12.3) this implies

$$\begin{aligned}
&P(\mathcal{B}^{[s+1]}h,\mathcal{B}^{[l+1]}k) \\
&\quad = \sup_{x\in X}\big|(\mathcal{B}^{[s+1]}h)(x) - (\mathcal{B}^{[l+1]}k)(x)\big| \\
&\quad = \sup_{x\in X}\big|(\mathcal{B}^{[s+1]}h)(x) - (\mathcal{B}^{[l+1]}k)(x)\big| \\
&\quad \leq \sup_{x\in X,y\in Y}\big|G(x,y,(\mathcal{B}^{[s]}h)(\xi(x,y))) - G(x,y,(\mathcal{B}^{[l]}k)(\xi(x,y)))\big| \\
&\quad \leq \sup_{x,t\in X,y\in Y}\big|G(x,y,(\mathcal{B}^{[s]}h)(t)) - G(x,y,(\mathcal{B}^{[l]}k)(t))\big|. \qquad (12.12)
\end{aligned}$$

A similar computation shows that

$$\begin{aligned}
&P(\mathcal{B}^{[s+r+1]}h,\mathcal{B}^{[l+r+1]}k) \\
&\quad = \sup_{x\in X}\big|(\mathcal{B}^{[s+r+1]}h)(x) - (\mathcal{B}^{[l+r+1]}k)(x)\big| \\
&\quad \leq \sup_{x\in X,y\in Y}\big|G(x,y,(\mathcal{B}^{[s+r]}h)(\xi(x,y))) - G(x,y,(\mathcal{B}^{[l+r]}k)(\xi(x,y)))\big| \\
&\quad \leq \sup_{x.t\in X,y\in Y}\big|G(x,y,(\mathcal{B}^{[s+r]}h)(t)) - G(x,y,(\mathcal{B}^{[l+r]}k)(t))\big|. \qquad (12.13)
\end{aligned}$$

Therefore, using (12.12), (12.13) and (12.5), we find (12.7).

Therefore, the contractive condition (12.7) holds and also it remains to see that P is separating on $B(X)$ (see Sects. 2 and 3). From this, using (12.7), the statements of Theorem 12.3 are now immediate consequences of Theorem 9.3 when $\mathcal{J} = \mathcal{P} = P$. □

Proof of Theorem 12.4 We deduce from (12.6) that

$$\begin{aligned}
&\forall_{\varepsilon>1}\exists_{\eta>1}\exists_{r\in\mathbb{N}}\forall_{h^0\in M}\forall_{s,l\in\mathbb{N}}\big\{S\big[P(\mathcal{B}^{[s+1]}h^0,\mathcal{B}^{[l+1]}h^0)\big] \\
&\quad < \varepsilon\cdot\eta \Longrightarrow S\big[P(\mathcal{B}^{[s+r+1]}h^0,\mathcal{B}^{[l+r+1]}h^0)\big] < \varepsilon\big\}. \qquad (12.14)
\end{aligned}$$

Next, using (12.14) and a similar argument as in the proofs of Theorems 9.4 and 12.3, we have the assertions. □

13 Discount maps δ and convergence, existence and uniqueness results for variable δ-discounted equations of Bellman type

Here let us make the following assumptions and notation:

I. Assume that a discount map $\delta : \mathbb{R} \to \mathbb{R}$ is such that $\delta(D)$ is a bounded set for each bounded set $D \subset \mathbb{R}$ and that there exists a continuous map $\gamma : [0;\infty) \to [0;\infty)$ satisfying

$$\forall_{t_1,t_2\in[0;\infty)}\bigl\{t_1 < t_2 \Longrightarrow \gamma(t_1) \le \gamma(t_2)\bigr\}, \tag{13.1}$$

$$\forall_{t\in[0;\infty)}\Bigl\{\lim_{n\to\infty} \gamma^{[n]}(t) = 0\Bigr\} \tag{13.2}$$

and, for each bounded set $D \subset \mathbb{R}$,

$$\forall_{\tau_1,\tau_2\in D}\bigl\{\bigl|\delta(\tau_2) - \delta(\tau_1)\bigr| \le \gamma\bigl(|\tau_2 - \tau_1|\bigr)\bigr\}. \tag{13.3}$$

II. X and A are nonempty sets.

III. $B(X)$ is the set of all bounded real-valued maps on X, $(B(X), \|\cdot\|)$ is a normed space with norm $\|h\| = \sup\{|h(x)| : x \in X\}$, $h \in B(X)$, and $(B(X), P)$ is a metric space with metric $P : B(X) \times B(X) \to [0;\infty)$ defined by

$$P(h,k) = \|h - k\|, \quad h,k \in B(X). \tag{13.4}$$

IV. Assume now that $\Psi : X \to 2^A$, $f : X \times A \to X$ and that the map $u : X \times A \to \mathbb{R}$ is bounded. The variable δ-discounted equation of Bellman type studied in this section is of the form

$$h(x) = \sup_{a\in\Psi(x)} \bigl\{u(x,a) + \delta\bigl(h\bigl(f(x,a)\bigr)\bigr)\bigr\}, \quad x \in X, \tag{13.5}$$

where $h \in B(X)$ is an unknown map to be determined.

V. The operator $\mathcal{B}$ of Bellman type is of the form

$$(\mathcal{B}h)(x) = \sup_{a\in\Psi(x)} \bigl\{u(x,a) + \delta\bigl(h\bigl(f(x,a)\bigr)\bigr)\bigr\}, \quad x \in X, h \in B(X). \tag{13.6}$$

Here the dynamic system $(B(X), \mathcal{B})$ is defined as follows: For $h \in B(X)$ we define $\mathcal{B}h = k$, where $k(x) = \sup_{a\in\Psi(x)}\{u(x,a) + \delta(h(f(x,a)))\}$, $x \in X$. Clearly k is bounded, since u and δ are bounded and so $k \in B(X)$. Therefore, $\mathcal{B} : B(X) \to B(X)$.

VI. The operators $\mathcal{B}^{[m+1]} : B(X) \to B(X)$, $m \in \{0\} \cup \mathbb{N}$, are defined by

$$\bigl(\mathcal{B}^{[m+1]}h\bigr)(x) = \sup_{a\in\Psi(x)} \bigl\{u(x,a) + \delta\bigl[\bigl(\mathcal{B}^{[m]}h\bigr)\bigl(f(x,a)\bigr)\bigr]\bigr\} \tag{13.7}$$

for all $h \in B(X)$ and $x \in X$.

VII. Recalling that u and δ are bounded, we conclude that, for each $h^0 \in B(X)$, the sequence of iterations $(h^m = \mathcal{B}^{[m]}h^0 : m \in \{0\} \cup \mathbb{N}) \subset B(X)$ starting at $h^0 \in B(X)$ is well defined. Here $\mathcal{B}^{[0]} = I_{B(X)}$—the identity on $B(X)$.

VIII. Any fixed point of $(B(X),\mathcal{B})$ is a solution of Eq. (13.5). Moreover, any periodic point of $(B(X),\mathcal{B})$, i.e., any point of the set

$$\mathrm{Per}_{B(X)}(\mathcal{B}) = \bigl\{h \in B(X) : h = \mathcal{B}^{[q]}h \text{ for some } q \in \mathbb{N}\bigr\},$$

is a solution of (13.5).

IX. Put

$$\mathcal{H}_\delta = \Bigl\{h \in B(X) : h(x) = \sup_{a\in\Psi(x)} \bigl\{u(x,a) + \delta\bigl(h\bigl(f(x,a)\bigr)\bigr)\bigr\}, x \in X\Bigr\};$$

the set of all solutions $h \in B(X)$ of Eq. (13.5).

We turn to some examples of maps δ and γ satisfying (13.1)–(13.3).

Example 13.1 We note that if $\delta : \mathbb{R} \to \mathbb{R}$ is defined by

$$\delta(\tau) = |\tau|/\bigl[a + b|\tau|\bigr], \quad \tau \in \mathbb{R},$$

where $a \in [1;\infty)$ and $b \in (0;\infty)$, then $\delta(D)$ is a bounded set for each bounded set $D \subset \mathbb{R}$. Moreover, for each $\tau_1, \tau_2 \in \mathbb{R}$, we have

$$\begin{aligned}\bigl|\delta(\tau_2) - \delta(\tau_1)\bigr| &= \frac{a||\tau_2| - |\tau_1||}{a^2 + ab(|\tau_2| + |\tau_1|) + b^2|\tau_2\tau_1|} \\ &\le \frac{a|\tau_2 - \tau_1|}{a^2 + ab(|\tau_2| + |\tau_1|)} = \frac{|\tau_2 - \tau_1|}{a + b(|\tau_2| + |\tau_1|)} \le \frac{|\tau_2 - \tau_1|}{a + b|\tau_2 - \tau_1|}.\end{aligned}$$

Now we note that if $\gamma(t) = t/[a + bt]$, $t \in [0;\infty)$, then γ is strictly increasing since $\forall_{t\in[0;\infty)}\{\gamma'(t) = a/(a + bt)^2 > 0\}$, $\forall_{\tau_1,\tau_2\in\mathbb{R}}\{|\delta(\tau_2) - \delta(\tau_1)| \le \gamma(|\tau_2 - \tau_1|)\}$ and $\forall_{t\in[0;\infty)}\{\lim_{n\to\infty}\gamma^{[n]}(t) = \lim_{n\to\infty} t/[a^n + b(a^{n-1} + a^{n-2} + \cdots + a + 1)t] = 0\}$.

Example 13.2 Let $\delta : \mathbb{R} \to \mathbb{R}$ be defined by

$$\delta(\tau) = \frac{|\tau| + b\ln(1 + |\tau|)}{a + |\tau| + b\ln(1 + |\tau|)}, \quad \tau \in \mathbb{R},$$

where $b \in (0;\infty)$ and $1 + b < a$. We see that $\delta(D)$ is a bounded set for each bounded set $D \subset \mathbb{R}$ and, for each $\tau_1, \tau_2 \in \mathbb{R}$, we have

$$\begin{aligned}\bigl|\delta(\tau_2) - \delta(\tau_1)\bigr| &= \frac{|a(|\tau_2| - |\tau_1|) + ab\ln\frac{1+|\tau_2|}{1+|\tau_1|}|}{[a + |\tau_2| + b\ln(1 + |\tau_2|)][a + |\tau_1| + b\ln(1 + |\tau_1|)]} \\ &\le \frac{a||\tau_2| - |\tau_1|| + ab\ln(1 + \frac{|\tau_2|-|\tau_1|}{1+|\tau_1|})}{a^2 + a(|\tau_2| + |\tau_1|) + ab\ln[(1 + |\tau_2|)(1 + |\tau_1|)]} \\ &\le \frac{||\tau_2| - |\tau_1|| + b\ln(1 + ||\tau_2| - |\tau_1||)}{a + (|\tau_2| + |\tau_1|) + b\ln(1 + |\tau_2| + |\tau_1|)} \\ &\le \frac{|\tau_2 - \tau_1| + b\ln(1 + |\tau_2 - \tau_1|)}{a + |\tau_2 - \tau_1| + b\ln(1 + |\tau_2 - \tau_1|)}.\end{aligned}$$

We claim that if γ is defined by

$$\gamma(t) = \frac{t + b\ln(1 + t)}{a + t + b\ln(1 + t)}, \quad t \in [0;\infty),$$

then $\forall_{\tau_1,\tau_2\in\mathbb{R}}\{|\delta(\tau_2)-\delta(\tau_1)|\le\gamma(|\tau_2-\tau_1|)\}$ and γ is strictly increasing since

$$\forall_{t\in[0;\infty)}\left\{\gamma'(t)=\left(1+\frac{b}{1+t}\right)\frac{a}{[a+t+b\ln(1+t)]^2}>0\right\}.$$

Furthermore, $\forall_{t\in[0;\infty)}\{\lim_{n\to\infty}\gamma^{[n]}(t)=0\}$ is evident. Here we rely on the fact that $\gamma(t)\le(1+b)t/a$ for $t\in[0;\infty)$, and, consequently, $\forall_{n\in\mathbb{N}}\forall_{t\in[0;\infty)}\{0\le\gamma^{[n]}(t)\le[(1+b)/a]^n t\}$.

We begin with the following auxiliary result concerning crucial properties of operators (13.7).

Theorem 13.1 *Assume that* I–IX *are satisfied. Then* $(B(X),\mathcal{B})$ *and each* $\mathcal{S}$*-family* $\mathcal{S}=\{S\}$ *satisfy*

$$\begin{cases}\forall_{\varepsilon>1}\exists_{\eta>1}\exists_{r\in\mathbb{N}}\forall_{h,k\in B(X)}\forall_{s,l\in\mathbb{N}}\{S[P(\mathcal{B}^{[s+1]}h,\mathcal{B}^{[l+1]}k)]<\varepsilon\cdot\eta\\ \quad\Longrightarrow S[P(\mathcal{B}^{[s+r+1]}h,\mathcal{B}^{[l+r+1]}k)]<\varepsilon\}.\end{cases}\tag{13.8}$$

In particular,

$$\begin{cases}\forall_{\varepsilon>1}\exists_{\eta>1}\exists_{r\in\mathbb{N}}\forall_{M\in 2^{B(X)}}\forall_{h^0\in M}\forall_{s,l\in\mathbb{N}}\{S[P(\mathcal{B}^{[s+1]}h^0,\mathcal{B}^{[l+1]}h^0)]\\ \quad<\varepsilon\cdot\eta\Longrightarrow S[P(\mathcal{B}^{[s+r+1]}h^0,\mathcal{B}^{[l+r+1]}h^0)]<\varepsilon\}.\end{cases}\tag{13.9}$$

Proof First we prove that

$$\begin{aligned}&\forall_{s,l,r\in\mathbb{N}}\forall_{h,k\in B(X)}\big\{P\big(\mathcal{B}^{[s+r+1]}h,\mathcal{B}^{[l+r+1]}k\big)\\ &\quad\le\gamma^{[r]}\big[P\big(\mathcal{B}^{[s]}h,\mathcal{B}^{[l]}k\big)\big]\big\}.\end{aligned}\tag{13.10}$$

To establish this, let $x\in X$, $h,k\in B(X)$ and $s,l,r\in\mathbb{N}$ be arbitrary and fixed. Then, for arbitrary $\mu>0$, in view of (13.7), there exist $a_1,a_2\in\Psi(x)$ such that

$$\big(\mathcal{B}^{[s+1]}h\big)(x)<u(x,a_1)+\delta\big[\big(\mathcal{B}^{[s]}h\big)\big(\big(f(x,a_1)\big)\big]+\mu,\tag{13.11}$$

$$\big(\mathcal{B}^{[l+1]}k\big)(x)<u(x,a_2)+\delta\big[\big(\mathcal{B}^{[l]}k\big)\big(\big(f(x,a_2)\big)\big]+\mu.\tag{13.12}$$

Observe also that (13.5) implies

$$\big(\mathcal{B}^{[l+1]}k\big)(x)\ge u(x,a_1)+\delta\big[\big(\mathcal{B}^{[l]}k\big)\big(\big(f(x,a_1)\big)\big],\tag{13.13}$$

$$\big(\mathcal{B}^{[s+1]}h\big)(x)\ge u(x,a_2)+\delta\big[\big(\mathcal{B}^{[s]}h\big)\big(\big(f(x,a_2)\big)\big].\tag{13.14}$$

Restating (13.11) and (13.13) as

$$\begin{aligned}&\big(\mathcal{B}^{[s+1]}h\big)(x)-\big(\mathcal{B}^{[l+1]}k\big)(x)\\ &\quad<\delta\big[\big(\mathcal{B}^{[s]}h\big)\big(\big(f(x,a_1)\big)\big]-\delta\big[\big(\mathcal{B}^{[l]}k\big)\big(\big(f(x,a_1)\big)\big]+\mu\\ &\quad\le\big|\delta\big[\big(\mathcal{B}^{[s]}h\big)\big(\big(f(x,a_1)\big)\big]-\delta\big[\big(\mathcal{B}^{[l]}k\big)\big(\big(f(x,a_1)\big)\big]\big|+\mu\end{aligned}$$

and (13.12) and (13.14) as

$$\begin{aligned}&\big(\mathcal{B}^{[l+1]}k\big)(x)-\big(\mathcal{B}^{[s+1]}h\big)(x)\\&\quad<\delta\big[\big(\mathcal{B}^{[l]}k\big)\big(\big(f(x,a_2)\big)\big]-\delta\big[\big(\mathcal{B}^{[s]}h\big)\big(\big(f(x,a_2)\big)\big]+\mu\end{aligned}$$

or

$$\begin{aligned}&\big(\mathcal{B}^{[s+1]}h\big)(x)-\big(\mathcal{B}^{[l+1]}k\big)(x)\\&\quad>\delta\big[\big(\mathcal{B}^{[s]}h\big)\big(\big(f(x,a_2)\big)\big]-\delta\big[\big(\mathcal{B}^{[l]}k\big)\big(\big(f(x,a_2)\big)\big]-\mu\\&\quad\geq-\big|\delta\big[\big(\mathcal{B}^{[s]}h\big)\big(\big(f(x,a_2)\big)\big]-\delta\big[\big(\mathcal{B}^{[l]}k\big)\big(\big(f(x,a_2)\big)\big]\big|-\mu,\end{aligned}$$

we see that they imply

$$\begin{aligned}&\big|\big(\mathcal{B}^{[s+1]}h\big)(x)-\big(\mathcal{B}^{[l+1]}k\big)(x)\big|\\&\quad<\max\big\{\big|\delta\big[\big(\mathcal{B}^{[s]}h\big)\big(\big(f(x,a_1)\big)\big]-\delta\big[\big(\mathcal{B}^{[l]}k\big)\big(\big(f(x,a_1)\big)\big]\big|,\\&\big|\delta\big[\big(\mathcal{B}^{[s]}h\big)\big(\big(f(x,a_2)\big)\big]-\delta\big[\big(\mathcal{B}^{[l]}k\big)\big(\big(f(x,a_2)\big)\big]\big|\big\}+\mu\\&\quad\leq\sup_{a\in\Psi(x)}\big|\delta\big[\big(\mathcal{B}^{[s]}h\big)\big(\big(f(x,a)\big)\big]-\delta\big[\big(\mathcal{B}^{[l]}k\big)\big(\big(f(x,a)\big)\big]\big|+\mu.\end{aligned}$$

Recalling that $\mu>0$ is arbitrary and $\Psi:X\to 2^A$ and $g:X\times A\to X$, we conclude $|(\mathcal{B}^{[s+1]}h)(x)-(\mathcal{B}^{[l+1]}k)(x)|\leq\sup_{t\in X}|\delta[(\mathcal{B}^{[s]}h)(t)]-\delta[(\mathcal{B}^{[l]}k)(t)]|\leq\sup_{t\in X}\gamma(|(\mathcal{B}^{[s]}h)(t)-(\mathcal{B}^{[l]}k)(t)|)\leq\sup_{t\in X}\gamma(|(\mathcal{B}^{[s]}h)(t)-(\mathcal{B}^{[l]}k)(t)|)\leq\gamma(\sup_{t\in X}|(\mathcal{B}^{[s]}h)(t)-(\mathcal{B}^{[l]}k)(t)|)=\gamma[p(\mathcal{B}^{[s]}h,\mathcal{B}^{[l]}k)]$ and this implies $P(\mathcal{B}^{[s+1]}h,\mathcal{B}^{[l+1]}k)=\sup_{x\in X}|(\mathcal{B}^{[s+1]}h)(x)-(\mathcal{B}^{[l+1]}k)(x)|\leq\gamma[P(\mathcal{B}^{[s]}h,\mathcal{B}^{[l]}k)]$. A similar computation shows that $P(\mathcal{B}^{[s+r+1]}h,\mathcal{B}^{[l+r+1]}k)=\sup_{x\in X}|(\mathcal{B}^{[s+r+1]}h)(x)-(\mathcal{B}^{[l+r+1]}k)(x)|\leq\gamma[P(\mathcal{B}^{[s+r]}h,\mathcal{B}^{[l+r]}k)]\leq\gamma^{[r]}[P(\mathcal{B}^{[s]}h,\mathcal{B}^{[l]}k)]$. Therefore, (13.10) holds.

Now we prove that if $\mathcal{S}$-family $\mathcal{S}=\{S\}$ is arbitrary and fixed, then $\mathcal{S}=\{S\}$ and operators $\mathcal{B}^{[m+1]}:B(X)\to B(X)$, $m\in\{0\}\cup\mathbb{N}$, defined by (13.7) satisfy

$$\begin{aligned}&\forall_{\varepsilon>1}\exists_{\eta>1}\exists_{r\in\mathbb{N}}\forall_{h,k\in B(X)}\forall_{s,l\in\mathbb{N}}\big\{S\big[P\big(\mathcal{B}^{[s+1]}h,\mathcal{B}^{[l+1]}k\big)\big]\\&\quad<\varepsilon\cdot\eta\Longrightarrow S\big[P\big(\mathcal{B}^{[s+r+1]}h,\mathcal{B}^{[l+r+1]}k\big)\big]<\varepsilon\big\}.\end{aligned}\tag{13.15}$$

Indeed, observe that $P(\mathcal{B}^{[s]}h,\mathcal{B}^{[l]}k)$ is bounded for each $s,l\in\mathbb{N}$ and $h,k\in B(X)$. Thus, it is clear that

$$\lim_{r\to\infty}\gamma^{[r]}\big(P\big[\mathcal{B}^{[s]}h,\mathcal{B}^{[l]}k\big]\big)=0\tag{13.16}$$

and consequence of (13.10) and (13.16) is

$$\lim_{r\to\infty}P\big[\mathcal{B}^{[s+r+1]}h,\mathcal{B}^{[l+r+1]}k\big]=0.\tag{13.17}$$

Next we deduce from (13.17) that

$$\begin{aligned}&\forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{h,k\in B(X)}\forall_{s,l\in\mathbb{N}}\big\{P\big[\mathcal{B}^{[s+1]}h,\mathcal{B}^{[l+1]}k\big]\\&\quad<\varepsilon+\eta\Longrightarrow P\big[\mathcal{B}^{[s+r+1]}h,\mathcal{B}^{[l+r+1]}k\big]<\varepsilon\big\}.\end{aligned}\tag{13.18}$$

Finally, using $\mathcal{S}$-family $\mathcal{S} = \{S\}$ and property (13.17), the contractive condition (13.9) may then be constructed by modification of condition (13.18). □

In the sequel, we need the following definitions.

Definition 13.1 Let $(B(X), P)$ be a metric space with metric P defined by (13.4).

(A) Let $h^0 \in B(X)$. $(B(X), \mathcal{B})$ is said to be a *P-admissible in h^0* if, in the case when the sequence $(\mathcal{B}^{[m]}h^0 : m \in \{0\} \cup \mathbb{N})$ is *P-sequence in $B(X)$* (i.e. satisfies the condition $\lim_{m\to\infty} \sup_{n>m} P(\mathcal{B}^{[m]}h^0, \mathcal{B}^{[n]}h^0) = 0$), then the sequence $(\mathcal{B}^{[m]}h^0 : m \in \{0\} \cup \mathbb{N})$ is *P-convergent in $B(X)$* (i.e. has the property $\exists_{k\in B(X)}\{\lim_{m\to\infty} P(\mathcal{B}^{[m]}h^0, k) = 0\}$).

(B) Let $M \in 2^{B(X)}$. $(B(X), \mathcal{B})$ is said to be a *P-admissible on M* if $(B(X), \mathcal{B})$ is P-admissible in each $h^0 \in M$.

Definition 13.2 Let $(B(X), P)$ be a metric space with metric p defined by (13.4) and let $q \in \mathbb{N}$.

(A) Let $h^0 \in B(X)$. $(B(X), \mathcal{B}^{[q]})$ is said to be a *P-closed in h^0* if, in the case when the sequence $(\mathcal{B}^{[m]}h^0 : m \in \{0\} \cup \mathbb{N})$ is P-converging in $B(X)$ and contains two P-convergent in $B(X)$ subsequences $(u_m : m \in \mathbb{N})$ and $(v_m : m \in \mathbb{N})$ satisfying $\forall_{m\in\mathbb{N}}\{u_m = \mathcal{B}^{[q]}v_m\}$, then there exists $h \in B(X)$ such that $h = \mathcal{B}^{[q]}h$.

(B) Let $M \in 2^{B(X)}$. $(B(X), \mathcal{B}^{[q]})$ is said to be a *P-closed on M* if $(B(X), \mathcal{B}^{[q]})$ is P-closed in each $h^0 \in M$.

Now, we prove the following two results.

Theorem 13.2 *Assume that* I–IX *are satisfied and suppose that the dynamic system* $(B(X), \mathcal{B})$ *is P-admissible on* $B(X)$. *Then the following hold*:

(A) Convergence property. *For each* $h^0 \in B(X)$, *there exists* $h \in B(X)$ *such that a sequence* $(h^m = \mathcal{B}^{[m]}h^0 : m \in \{0\} \cup \mathbb{N})$ *is P-convergent to* h.

(B) Existence of solutions and convergence property. *If there exists* $q \in \mathbb{N}$ *such that the dynamic system* $(B(X), \mathcal{B}^{[q]})$ *is P-closed on* $B(X)$, *then*:

(B1) $\emptyset \neq \mathrm{Fix}_{B(X)}(\mathcal{B}^{[q]}) \subset \mathcal{H}_\delta$.

(B2) *For each* $h^0 \in B(X)$, *there exists* $h \in \mathrm{Fix}_{B(X)}(\mathcal{B}^{[q]})$ *such that the sequence* $(h^m = \mathcal{B}^{[m]}h^0 : m \in \{0\} \cup \mathbb{N})$ *is P-convergent to* h.

(C) Existence of unique solution and convergence property. *If the dynamic system* $(B(X), \mathcal{B})$ *is P-closed on* $B(X)$, *then*:

(C1) *There exists* $h \in B(X)$ *such that* $\mathcal{H}_\delta = \mathrm{Fix}_{B(X)}(\mathcal{B}) = \{h\}$.

(C2) *For each* $h^0 \in B(X)$, *the sequence* $(h^m = \mathcal{B}^{[m]}h^0 : m \in \{0\} \cup \mathbb{N})$ *is P-convergent to* h.

Theorem 13.3 *Assume that* I–IX *are satisfied and suppose that there exists* $M \in 2^{B(X)}$ *such that the dynamic system* $(B(X), \mathcal{B})$ *is P-admissible on* M.

Then the following hold:

(A) Convergence property. *For each* $h^0 \in M$, *there exists* $h \in B(X)$ *such that a sequence* $(h^m = \mathcal{B}^{[m]}h^0 : m \in \{0\} \cup \mathbb{N})$ *is P-convergent to* h.

(B) Existence of solutions and convergence property. *If there exist* $q \in \mathbb{N}$ *and* $h^0 \in M$ *such that the dynamic system* $(B(X), \mathcal{B}^{[q]})$ *is P-closed in* h^0, *then*:

(B1) $\emptyset \neq \mathrm{Fix}_{B(X)}(\mathcal{B}^{[q]}) \subset \mathcal{H}_\delta$.

(B2) *There exists* $h \in \mathrm{Fix}_{B(X)}(\mathcal{B}^{[q]})$ *such that the sequence* $(h^m = \mathcal{B}^{[m]}h^0 : m \in \{0\} \cup \mathbb{N})$ *is P-convergent to h.*

Proof of Theorem 13.2 Therefore, $(B(X), P)$ is a metric space, single-valued dynamic system $(B(X), \mathcal{B})$ is $\mathcal{J} = \mathcal{P} = P$-admissible on $B(X)$, contractive condition (13.8) holds (see Theorem 13.1), and $\mathcal{P} = P$ is separating on $B(X)$ (since P is metric). Consequently, the statements of Theorem 13.2 are now immediate consequences of Theorem 9.3. □

Proof of Theorems 13.3 The statements of Theorem 13.3 are consequences of contractive condition (13.9) of Theorem 13.1 and Theorem 9.4. □

Remark 13.1 Let us observe here that X and A are nonempty sets. In the literature, discount maps δ and variable δ-discounted Bellman equations are studied in the case when X is a complete metric space, A is a metric space, $B(X)$ is the set of all continuous bounded real-valued maps on X, $(B(X), P)$ is a complete metric space, u and f are continuous, $\Psi(x)$ is a compact set for each $x \in X$, and the dynamic system $(B(X), \mathcal{B})$, $\mathcal{B} : B(X) \to B(X)$, is a continuous generalized Matkowski contraction (see, e.g., [14, 26, 34]).

14 Convergence, existence and uniqueness results for integral equations of Volterra type in locally convex spaces

First, we record two definitions needed in the sequel.

Definition 14.1 *Let E* be a vector space over $\mathbb{R}$, and let $\mathcal{A}$ be an index set.

(A) The map $P : E \to [0; \infty)$ is called a *seminorm* on E if:
  (i) $\forall_{u \in E} \forall_{\lambda \in \mathbb{R}} \{P(\lambda u) = |\lambda| \cdot P(u)\}$ (*homogeneity*). So, in particular, $P(0) = 0$.
  (ii) $\forall_{u,v \in E} \{P(u + v) \le P(u) + P(v)\}$ (*triangle inequality*).

(B) A topological vector space $(E, \mathcal{T})$, such that there is a family $\mathcal{P}_{\mathcal{A}} = \{P_\alpha : \alpha \in \mathcal{A}\}$ of continuous seminorms $P_\alpha : E \to [0; \infty)$, $\alpha \in \mathcal{A}$, on E and $\mathcal{T}$ is a locally convex topology on E generated by the family $\mathcal{P}_{\mathcal{A}}$, is called a *locally convex space* and is denoted by $(E, \mathcal{P}_{\mathcal{A}})$.

(C) The family $\mathcal{P}_{\mathcal{A}} = \{P_\alpha : \alpha \in \mathcal{A}\}$ of seminorms $P_\alpha : E \to [0; \infty)$, $\alpha \in \mathcal{A}$, on E is called *separating* if $\forall_{u \in E} \{u \ne 0 \Rightarrow \exists_{\alpha_0 \in \mathcal{A}} \{P_{\alpha_0}(u) > 0\}\}$.

(D) If the family $\mathcal{P}_{\mathcal{A}} = \{P_\alpha : \alpha \in \mathcal{A}\}$ is separating on E, then $\mathcal{T}$ is a Hausdorff locally convex topology on E and $(E, \mathcal{P}_{\mathcal{A}})$ is called a *Hausdorff locally convex space*.

Definition 14.2 (see [17]) *Let X* be a (nonempty) set, and let $\mathcal{A}$ be an index set.

(A) The distance $D : X^2 \to [0; \infty)$ is called a *pseudometric* (or the *gauge*) on X if:
  (i) $\forall_{u \in X} \{D(u, u) = 0\}$. It is not required that $D(u, v) = 0$ implies $u = v$.
  (ii) $\forall_{u,v \in X} \{D(u, v) = D(v, u)\}$ (*symmetry*).
  (iii) $\forall_{u,v,w \in X} \{D(u, v) \le D(u, w) + D(w, v)\}$ (*triangle inequality*).

(B) Each family $\mathcal{D}_{\mathcal{A}} = \{D_\alpha : \alpha \in \mathcal{A}\}$ of pseudometrics $D_\alpha : X^2 \to [0; \infty)$, $\alpha \in \mathcal{A}$, on X is called a *gauge* on X.

(C) The gauge $\mathcal{D}_{\mathcal{A}} = \{D_\alpha : \alpha \in \mathcal{A}\}$ on X is called *separating* if $\forall_{u,w \in E} \{u \ne w \Rightarrow \exists_{\alpha_0 \in \mathcal{A}} \{D_{\alpha_0}(u, w) > 0\}\}$.

(D) Let the family $\mathcal{D}_{\mathcal{A}} = \{D_\alpha : \alpha \in \mathcal{A}\}$ be gauge on X. The topology $\mathcal{T}(\mathcal{D}_{\mathcal{A}})$ having as a subbase the family $\mathcal{B}(\mathcal{D}_{\mathcal{A}}) = \{B(u, D_\alpha, \varepsilon_\alpha) : u \in X, \varepsilon_\alpha > 0, \alpha \in \mathcal{A}\}$ of all balls

$B(u, D_\alpha, \varepsilon_\alpha) = \{v \in X : D_\alpha(u, v) < \varepsilon_\alpha\}$ with $u \in X$, $\varepsilon_\alpha > 0$, and $\alpha \in \mathcal{A}$ is called topology *induced* by $\mathcal{D}_\mathcal{A}$ on X.

(E) A topological space $(X, \mathcal{T})$ such that there is a gauge $\mathcal{D}_\mathcal{A}$ on X with $\mathcal{T} = \mathcal{T}(\mathcal{D}_\mathcal{A})$ is called a *gauge space* and is denoted by $(X, \mathcal{D}_\mathcal{A})$.

(F) If the family $\mathcal{D}_\mathcal{A} = \{D_\alpha : \alpha \in \mathcal{A}\}$ is separating on X, then the topology $\mathcal{T}(\mathcal{D}_\mathcal{A})$ is Hausdorff and $(X, \mathcal{D}_\mathcal{A})$ is called a *Hausdorff gauge space*.

Before proceeding, let us make the following assumptions and notations:

I. $(E, \mathcal{P}_\mathcal{A})$ is a Hausdorff sequentially complete locally convex space.

II. The integral equation of Volterra type studied in this section is given in the form

$$y(t) = f(t) + \int_{I(t)} K\bigl(t, \tau, y\bigl(h(\tau)\bigr)\bigr)\, d\tau, \quad t \in I, \tag{14.1}$$

where $I = [0;1]^n$ and $I(t) = \{\tau = (\tau_1, \ldots, \tau_n) \in \mathbb{R}^n : 0 \le \tau_i \le t_i, i = 1, \ldots, n\}$ for $t = (t_1, \ldots, t_n) \in I$. Here: $f : I \to E$ is continuous (that is, $f \in \mathcal{C}(I, E)$), $h : I \to I$ is continuous (that is, $h \in \mathcal{C}(I, I)$) and $K : I \times I \times E \to E$ is continuous and bounded (that is, $K \in \mathcal{CB}(I \times I \times E, E)$); the maps f, h and K are given maps; $y \in \mathcal{C}(I, E)$ is an unknown map to be determined; and, for each $t \in I$, $\int_{I(t)} K(t, \tau, y(h(\tau)))\, d\tau$ denote the Riemann integral on $I(t)$. For a Riemann integral in locally convex space, see e.g. [20, Appendix 1].

III. The operator $\mathcal{V}$ of Volterra type is defined by

$$(\mathcal{V}y)(t) = f(t) + \int_{I(t)} K\bigl(t, \tau, y\bigl(h(\tau)\bigr)\, d\tau, \tag{14.2}$$

$t \in I$, $y \in \mathcal{C}(I, E)$. Here we define the dynamic system $(\mathcal{C}(I, E), \mathcal{V})$ as follows: For $y \in \mathcal{C}(I, E)$ we define $\mathcal{V}y = x$, where $x(t) = f(t) + \int_{I(t)} K(t, \tau, y(h(\tau))\, d\tau$, $t \in I$. Clearly $x \in \mathcal{C}(I, E)$. Therefore, $\mathcal{V} : \mathcal{C}(I, E) \to \mathcal{C}(I, E)$.

IV. The operators $\mathcal{V}^{[m+1]} : \mathcal{C}(I, E) \to \mathcal{C}(I, E)$, $m \in \{0\} \cup \mathbb{N}$, are defined by

$$\bigl(\mathcal{V}^{[m+1]} y\bigr)(t) = f(t) + \int_{I(t)} K\bigl(t, \tau, \bigl(\mathcal{V}^{[m]} y\bigr)\bigl(h(\tau)\bigr)\bigr)\, d\tau, \tag{14.3}$$

$t \in I$, $y \in \mathcal{C}(I, E)$.

V. For each $y^0 \in \mathcal{C}(I, E)$ and for $(\mathcal{V}^{[m+1]} : m \in \{0\} \cup \mathbb{N})$ defined by (14.3), the sequence of iterations

$$\bigl(y^m = \mathcal{V}^{[m]} y^0 : m \in \{0\} \cup \mathbb{N}\bigr) \subset \mathcal{C}(I, E)$$

starting at $y^0 \in \mathcal{C}(I, E)$ is well defined. Here $\mathcal{V}^{[0]} = I_{\mathcal{C}(I,E)}$—the identity on $\mathcal{C}(I, E)$.

VI. Any fixed point of dynamic system $(\mathcal{C}(I, E), \mathcal{V})$ is a solution of Eq. (14.1). Moreover, any periodic point of $(\mathcal{C}(I, E), \mathcal{V})$, i.e., any point of the set

$$\mathrm{Per}_{\mathcal{C}(I,E)}(\mathcal{V}) = \bigl\{h \in \mathcal{C}(I, E) : y = \mathcal{V}^{[q]} y \text{ for some } q \in \mathbb{N}\bigr\},$$

is a solution of (14.1).

VII. Put

$$\mathcal{Y} = \left\{y \in \mathcal{C}(I, E) : y(t) = f(t) + \int_{I(t)} K\bigl(t, \tau, y\bigl(h(\tau)\bigr)\bigr)\, d\tau, t \in I\right\};$$

the set of all solutions $y \in \mathcal{C}(I,E)$ of Eq. (14.1).

VIII. $(\mathcal{C}(I,E),\mathcal{D}_{\mathcal{A}})$ is a gauge space with gauge $\mathcal{D}_{\mathcal{A}} = \{D_\alpha : \alpha \in \mathcal{A}\}$ defined by

$$\forall_{\alpha\in\mathcal{A}}\forall_{x,y\in\mathcal{C}(I,E)}\Big\{D_\alpha(x,y) = \max_{t\in I} P_\alpha\big[x(t) - y(t)\big]\Big\}. \tag{14.4}$$

We need the following two definitions.

Definition 14.3 Let $(E,\mathcal{P}_{\mathcal{A}})$ be a Hausdorff locally convex space and let $(\mathcal{C}(I,E),\mathcal{D}_{\mathcal{A}})$ be a gauge space with gauge $\mathcal{D}_{\mathcal{A}} = \{D_\alpha : \alpha \in \mathcal{A}\}$ defined by (14.4).

(A) Let $\mathcal{J}_{\mathcal{A}} \in \mathbb{J}^L_{(\mathcal{C}(I,E),\mathcal{D}_{\mathcal{A}})}$; thus, in particular, let $\mathcal{J}_{\mathcal{A}} = \mathcal{D}_{\mathcal{A}}$. We say that the single-valued dynamic system $(\mathcal{C}(I,E),\mathcal{V})$ is *left $\mathcal{J}_{\mathcal{A}}$-admissible in y^0* $\in \mathcal{C}(I,E)$ if, in the case when the sequence $(\mathcal{V}^{[m]}y^0 : m \in \{0\}\cup\mathbb{N})$ is *left $\mathcal{J}_{\mathcal{A}}$-sequence in* $\mathcal{C}(I,E)$ (i.e., $\forall_{\alpha\in\mathcal{A}}\{\lim_{m\to\infty}\sup_{n>m} J_\alpha(\mathcal{V}^{[m]}y^0,\mathcal{V}^{[n]}y^0) = 0\}$), then the sequence $(\mathcal{V}^{[m]}y^0 : m \in \{0\}\cup\mathbb{N})$ is *left $\mathcal{J}_{\mathcal{A}}$-convergent in* $\mathcal{C}(I,E)$ (i.e., there exists $u \in \mathcal{C}(I,E)$ such that $\forall_{\alpha\in\mathcal{A}}\{\lim_{m\to\infty} J_\alpha(u,\mathcal{V}^{[m]}y^0) = 0\}$).

(B) Let $\mathcal{J}_{\mathcal{A}} \in \mathbb{J}^R_{(\mathcal{C}(I,E),\mathcal{D}_{\mathcal{A}})}$; thus, in particular, let $\mathcal{J}_{\mathcal{A}} = \mathcal{D}_{\mathcal{A}}$. We say that the single-valued dynamic system $(\mathcal{C}(I,E),\mathcal{V})$ is *right $\mathcal{J}_{\mathcal{A}}$-admissible in y^0* $\in \mathcal{C}(I,E)$ if, in the case when the sequence $(\mathcal{V}^{[m]}y^0 : m \in \{0\}\cup\mathbb{N})$ is *right $\mathcal{J}_{\mathcal{A}}$-sequence in* $\mathcal{C}(I,E)$ (i.e., $\forall_{\alpha\in\mathcal{A}}\{\lim_{m\to\infty}\sup_{n>m} J_\alpha(\mathcal{V}^{[n]}y^0,\mathcal{V}^{[m]}y^0,) = 0\}$), then the sequence $(\mathcal{V}^{[m]}y^0 : m \in \{0\}\cup\mathbb{N})$ is *right $\mathcal{J}_{\mathcal{A}}$-convergent in* $\mathcal{C}(I,E)$ (i.e., there exists $v \in \mathcal{C}(I,E)$ such that $\forall_{\alpha\in\mathcal{A}}\{\lim_{m\to\infty} J_\alpha(\mathcal{V}^{[m]}y^0,v) = 0\}$).

(C) Let $M \in 2^{\mathcal{C}(I,E)}$. $(\mathcal{C}(I,E),\mathcal{V})$ is said to be a *left* (respectively, *right*) *$\mathcal{J}_{\mathcal{A}}$-admissible on M* if $\mathcal{J}_{\mathcal{A}} \in \mathbb{J}^L_{(\mathcal{C}(I,E),\mathcal{D}_{\mathcal{A}})}$ (respectively, $\mathcal{J}_{\mathcal{A}} \in \mathbb{J}^R_{(\mathcal{C}(I,E),\mathcal{D}_{\mathcal{A}})}$) and $(\mathcal{C}(I,E),\mathcal{V})$ is a left (respectively, right) $\mathcal{J}_{\mathcal{A}}$-admissible in each $y^0 \in M$.

Definition 14.4 Let $(E,\mathcal{P}_{\mathcal{A}})$ be a locally convex space and let $(\mathcal{C}(I,E),\mathcal{D}_{\mathcal{A}})$ be a gauge space with gauge $\mathcal{D}_{\mathcal{A}} = \{D_\alpha : \alpha \in \mathcal{A}\}$ defined by (14.4) and let $q \in \mathbb{N}$.

(A) Let $y^0 \in \mathcal{C}(I,E)$. Let $\mathcal{J}_{\mathcal{A}} \in \mathbb{J}^L_{(\mathcal{C}(I,E),\mathcal{D}_{\mathcal{A}})}$; thus, in particular, let $\mathcal{J}_{\mathcal{A}} = \mathcal{D}_{\mathcal{A}}$.We say that the single-valued dynamic system $(\mathcal{C}(I,E),\mathcal{V}^{[q]})$ is *left $\mathcal{J}_{\mathcal{A}}$-closed in y^0* if, in the case when the sequence $(\mathcal{V}^{[m]}y^0 : m \in \{0\}\cup\mathbb{N})$ is left $\mathcal{J}_{\mathcal{A}}$-convergent in $\mathcal{C}(I,E)$, i.e. $U = \mathrm{LIM}^{L-\mathcal{J}_{\mathcal{A}}}_{(\mathcal{V}^{[m]}y^0:\{0\}\cup\mathbb{N})} \neq \emptyset$, and contains two left $\mathcal{J}_{\mathcal{A}}$-converging in $\mathcal{C}(I,E)$ subsequences $(k_m : m \in \mathbb{N})$ and $(w_m : m \in \mathbb{N})$ satisfying $\forall_{m\in\mathbb{N}}\{k_m = \mathcal{V}^{[q]}w_m\}$, then there exists $u \in U$ such that $u = \mathcal{V}^{[q]}u$.

(B) Let $y^0 \in \mathcal{C}(I,E)$. Let $\mathcal{J}_{\mathcal{A}} \in \mathbb{J}^R_{(\mathcal{C}(I,E),\mathcal{D}_{\mathcal{A}})}$; thus, in particular, let $\mathcal{J}_{\mathcal{A}} = \mathcal{D}_{\mathcal{A}}$.We say that the single-valued dynamic system $(\mathcal{C}(I,E),\mathcal{V}^{[q]})$ is *right $\mathcal{J}_{\mathcal{A}}$-closed in y^0* if, in the case when the sequence $(\mathcal{V}^{[m]}y^0 : m \in \{0\}\cup\mathbb{N})$ is right $\mathcal{J}_{\mathcal{A}}$-convergent in $\mathcal{C}(I,E)$, i.e. $V = \mathrm{LIM}^{R-\mathcal{J}_{\mathcal{A}}}_{(\mathcal{V}^{[m]}y^0:\{0\}\cup\mathbb{N})} \neq \emptyset$, and contains two right $\mathcal{J}_{\mathcal{A}}$-converging in $\mathcal{C}(I,E)$ subsequences $(k_m : m \in \mathbb{N})$ and $(w_m : m \in \mathbb{N})$ satisfying $\forall_{m\in\mathbb{N}}\{k_m = \mathcal{V}^{[q]}w_m\}$, then there exists $v \in V$ such that $v = \mathcal{V}^{[q]}v$.

(C) Let $M \in 2^{\mathcal{C}(I,E)}$. We say that $(\mathcal{C}(I,E),\mathcal{V}^{[q]})$ is *left* (respectively, *right*) *$\mathcal{J}_{\mathcal{A}}$-closed on M* if $\mathcal{J}_{\mathcal{A}} \in \mathbb{J}^L_{(\mathcal{C}(I,E),\mathcal{D}_{\mathcal{A}})}$ (respectively, $\mathcal{J}_{\mathcal{A}} \in \mathbb{J}^R_{(\mathcal{C}(I,E),\mathcal{D}_{\mathcal{A}})}$) and $(\mathcal{C}(I,E),\mathcal{V}^{[q]})$ is a left (respectively, right) $\mathcal{J}_{\mathcal{A}}$-closed in each $y^0 \in M$.

Remark 14.1 The following hold: (a) $\mathcal{D}_{\mathcal{A}} \in \mathbb{J}^L_{(\mathcal{C}(I,E),\mathcal{D}_{\mathcal{A}})} \cap \mathbb{J}^R_{(\mathcal{C}(I,E),\mathcal{D}_{\mathcal{A}})}$. (b) J_α, D_α, and P_α, $\alpha \in \mathcal{A}$, are triangular distances. (c) $(E,\mathcal{P}_{\mathcal{A}})$ and $(\mathcal{C}(I,E),\mathcal{D}_{\mathcal{A}})$ are triangular spaces.

The fundamental papers of Volterra concerning integral equations were initiated by Ref. [50]. These papers and papers of many researchers in this field provide new perspectives on the investigations and new ideas and techniques together with the different areas in which the topic of solutions of integral equations has had its influence. In particular, the theory of Volterra integral equations in abstract settings (e.g., in Banach spaces, Fréchet spaces, and locally convex spaces) has received increasing attention.

In this section we will concentrate on convergence, existence and uniqueness problems concerning solutions of Eq. (14.1) of Volterra type. More precisely, we have the analogues of Theorems 9.3 and 9.4. They take the following forms.

Theorem 14.1 *Let $(E,\mathcal{P}_{\mathcal{A}})$ be a Hausdorff sequentially complete locally convex space with the topology defined by the family $\mathcal{P}_{\mathcal{A}}=\{P_\alpha:\alpha\in\mathcal{A}\}$ of continuous seminorms on E. Let $(\mathcal{C}(I,E),\mathcal{D}_{\mathcal{A}})$ be a gauge space with the gauge $\mathcal{D}_{\mathcal{A}}=\{D_\alpha:\alpha\in\mathcal{A}\}$ defined by (14.4). Suppose also that*:

(a) $\mathcal{J}_{\mathcal{A}}\in\mathbb{J}^L_{(\mathcal{C}(I,E),\mathcal{D}_{\mathcal{A}})}$ *(respectively, $\mathcal{J}_{\mathcal{A}}\in\mathbb{J}^R_{(\mathcal{C}(I,E),\mathcal{D}_{\mathcal{A}})}$).*

(b) *$(\mathcal{C}(I,E),\mathcal{V})$ is left (respectively, right) $\mathcal{J}_{\mathcal{A}}$-admissible on $\mathcal{C}(I,E)$.*

(c) *$\mathcal{S}_{\mathcal{A}}=\{S_\alpha:\alpha\in\mathcal{A}\}$ is an $\mathcal{S}_{\mathcal{A}}$-family.*

(d) *The $\mathcal{S}_{\mathcal{A}}$-family $\mathcal{S}_{\mathcal{A}}=\{S_\alpha:\alpha\in\mathcal{A}\}$, $(\mathcal{C}(I,E),\mathcal{V})$ and $\mathcal{J}_{\mathcal{A}}=\{J_\alpha:\alpha\in\mathcal{A}\}\in\mathbb{J}^L_{(\mathcal{C}(I,E),\mathcal{D}_{\mathcal{A}})}$ (respectively, $\mathcal{J}_{\mathcal{A}}=\{J_\alpha:\alpha\in\mathcal{A}\}\in\mathbb{J}^R_{(\mathcal{C}(I,E),\mathcal{D}_{\mathcal{A}})}$) satisfy*

$$\begin{cases}\forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>1}\exists_{\eta>1}\exists_{r\in\mathbb{N}}\forall_{x,y\in\mathcal{C}(I,E)}\forall_{s,l\in\mathbb{N}}\{S_\alpha(J_\alpha(\mathcal{V}^{[s]}x,\mathcal{V}^{[l]}y))\\ \quad<\varepsilon\cdot\eta\Rightarrow S_\alpha(J_\alpha(\mathcal{V}^{[s+r]}x,\mathcal{V}^{[l+r]}y))<\varepsilon\}.\end{cases}$$

Then the following hold:

(A) Convergence property. *For each $y^0\in\mathcal{C}(I,E)$, we have*

$$\emptyset\neq\mathrm{LIM}^{L-\mathcal{J}_{\mathcal{A}}}_{(\mathcal{V}^{[m]}y^0:m\in\{0\}\cup\mathbb{N})}\subset\mathrm{LIM}^{L-\mathcal{D}_{\mathcal{A}}}_{(\mathcal{V}^{[m]}y^0:m\in\{0\}\cup\mathbb{N})}$$

where $\mathcal{J}_{\mathcal{A}}\in\mathbb{J}^L_{(\mathcal{C}(I,E),\mathcal{D}_{\mathcal{A}})}$ (respectively,

$$\emptyset\neq\mathrm{LIM}^{R-\mathcal{J}_{\mathcal{A}}}_{(\mathcal{V}^{[m]}y^0:m\in\{0\}\cup\mathbb{N})}\subset\mathrm{LIM}^{R-\mathcal{D}_{\mathcal{A}}}_{(\mathcal{V}^{[m]}y^0:m\in\{0\}\cup\mathbb{N})}$$

where $\mathcal{J}_{\mathcal{A}}\in\mathbb{J}^R_{(\mathcal{C}(I,E),\mathcal{D}_{\mathcal{A}})}$).

(B) Existence of solutions and convergence property. *If there exists $q\in\mathbb{N}$ such that $(\mathcal{C}(I,E),\mathcal{V}^{[q]})$ is left (respectively, right) $\mathcal{J}_{\mathcal{A}}$-closed on $\mathcal{C}(I,E)$, then*

$$\emptyset\neq\mathrm{Fix}_{\mathcal{C}(I,E)}\big(\mathcal{V}^{[q]}\big)\subset\mathcal{Y}.$$

Moreover, for each $y^0\in\mathcal{C}(I,E)$, there exists $u\in\mathrm{Fix}_{\mathcal{C}(I,E)}(\mathcal{V}^{[q]})$ (respectively, $v\in\mathrm{Fix}_{\mathcal{C}(I,E)}(\mathcal{V}^{[q]})$) such that

$$u\in\mathrm{LIM}^{L-\mathcal{J}_{\mathcal{A}}}_{(\mathcal{V}^{[m]}y^0:m\in\{0\}\cup\mathbb{N})}\subset\mathrm{LIM}^{L-\mathcal{D}_{\mathcal{A}}}_{(\mathcal{V}^{[m]}y^0:m\in\{0\}\cup\mathbb{N})}$$

and

$$\forall_{\alpha\in\mathcal{A}}\forall_{n\in\{1,2,\dots,q\}}\big\{J_\alpha\big(u,\mathcal{V}^{[n]}u\big)=J_\alpha\big(\mathcal{V}^{[n]}u,u\big)=0\big\}$$

where $\mathcal{J}_{\mathcal{A}} \in \mathbb{J}^{L}_{(\mathcal{C}(I,E),\mathcal{D}_{\mathcal{A}})}$ (*respectively,*

$$v \in \mathrm{LIM}^{R-\mathcal{J}_{\mathcal{A}}}_{(\mathcal{V}^{[m]}y^{0}:m\in\{0\}\cup\mathbb{N})} \subset \mathrm{LIM}^{R-\mathcal{D}_{\mathcal{A}}}_{(\mathcal{V}^{[m]}y^{0}:m\in\{0\}\cup\mathbb{N})}$$

and

$$\forall_{\alpha\in\mathcal{A}}\forall_{n\in\{1,2,\ldots,q\}}\left\{J_{\alpha}\left(v,\mathcal{V}^{[n]}v\right) = J_{\alpha}\left(\mathcal{V}^{[n]}v,v\right) = 0\right\}$$

where $\mathcal{J}_{\mathcal{A}} \in \mathbb{J}^{R}_{(\mathcal{C}(I,E),\mathcal{D}_{\mathcal{A}})}$).

(C) Existence of unique solution and convergence property. *If* $(\mathcal{C}(I,E),\mathcal{V})$ *is* $\mathcal{D}_{\mathcal{A}}$*-closed on* $\mathcal{C}(I,E)$, *then there exists* $y \in \mathcal{C}(I,E)$ *such that*

$$\mathcal{Y} = \mathrm{Fix}_{\mathcal{C}(I,E)}(\mathcal{V}) = \{y\}.$$

Moreover, for each $y^{0} \in \mathcal{C}(I,E)$,

$$y \in \mathrm{LIM}^{L-\mathcal{J}_{\mathcal{A}}}_{(\mathcal{V}^{[m]}y^{0}:m\in\{0\}\cup\mathbb{N})} \subset \mathrm{LIM}^{L-\mathcal{D}_{\mathcal{A}}}_{(\mathcal{V}^{[m]}y^{0}:m\in\{0\}\cup\mathbb{N})}$$

and $\forall_{\alpha\in\mathcal{A}}\{J_{\alpha}(y,y) = 0\}$ *where* $\mathcal{J}_{\mathcal{A}} \in \mathbb{J}^{L}_{(\mathcal{C}(I,E),\mathcal{D}_{\mathcal{A}})}$ (*respectively,*

$$y \in \mathrm{LIM}^{R-\mathcal{J}_{\mathcal{A}}}_{(\mathcal{V}^{[m]}y^{0}:m\in\{0\}\cup\mathbb{N})} \subset \mathrm{LIM}^{R-\mathcal{D}_{\mathcal{A}}}_{(\mathcal{V}^{[m]}y^{0}:m\in\{0\}\cup\mathbb{N})}$$

and $\forall_{\alpha\in\mathcal{A}}\{J_{\alpha}(y,y) = 0\}$ *where* $\mathcal{J}_{\mathcal{A}} \in \mathbb{J}^{R}_{(\mathcal{C}(I,E),\mathcal{D}_{\mathcal{A}})}$).

Theorem 14.2 *Let* $(E,\mathcal{P}_{\mathcal{A}})$ *be a Hausdorff sequentially complete locally convex space with the topology defined by the family* $\mathcal{P}_{\mathcal{A}} = \{P_{\alpha} : \alpha \in \mathcal{A}\}$ *of continuous seminorms on* E. *Let* $(\mathcal{C}(I,E),\mathcal{D}_{\mathcal{A}})$ *be a gauge space with the gauge* $\mathcal{D}_{\mathcal{A}} = \{D_{\alpha} : \alpha \in \mathcal{A}\}$ *defined by* (14.4). *Suppose also that*:

(a) $\mathcal{J}_{\mathcal{A}} \in \mathbb{J}^{L}_{(\mathcal{C}(I,E),\mathcal{D}_{\mathcal{A}})}$ (*respectively,* $\mathcal{J}_{\mathcal{A}} \in \mathbb{J}^{R}_{(\mathcal{C}(I,E),\mathcal{D}_{\mathcal{A}})}$).

(b) *There exists* $M \in 2^{\mathcal{C}(I,E)}$ *such that* $(\mathcal{C}(I,E),\mathcal{V})$ *is left* (*respectively, right*) $\mathcal{J}_{\mathcal{A}}$*-admissible on* M.

(c) $\mathcal{S}_{\mathcal{A}} = \{S_{\alpha} : \alpha \in \mathcal{A}\}$ *is an* $\mathcal{S}_{\mathcal{A}}$*-family.*

(d) *The* $\mathcal{S}_{\mathcal{A}}$*-family* $\mathcal{S}_{\mathcal{A}} = \{S_{\alpha} : \alpha \in \mathcal{A}\}$, $(\mathcal{C}(I,E),\mathcal{V})$, M *and* $\mathcal{J}_{\mathcal{A}} = \{J_{\alpha} : \alpha \in \mathcal{A}\} \in \mathbb{J}^{L}_{(\mathcal{C}(I,E),\mathcal{D}_{\mathcal{A}})}$ (*respectively,* $\mathcal{J}_{\mathcal{A}} = \{J_{\alpha} : \alpha \in \mathcal{A}\} \in \mathbb{J}^{R}_{(\mathcal{C}(I,E),\mathcal{D}_{\mathcal{A}})}$) *satisfy*

$$\begin{cases} \forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>1}\exists_{\eta>1}\exists_{r\in\mathbb{N}}\forall_{y^{0}\in M}\forall_{s,l\in\mathbb{N}}\{S_{\alpha}(J_{\alpha}(\mathcal{V}^{[s]}y^{0},\mathcal{V}^{[l]}y^{0})) \\ \quad < \varepsilon\cdot\eta \Rightarrow S_{\alpha}(J_{\alpha}(\mathcal{V}^{[s+r]}y^{0},\mathcal{V}^{[l+r]}y^{0})) < \varepsilon\}. \end{cases}$$

Then the following hold:

(A) Convergence of property. *For each point* $y^{0} \in M$, *we have*

$$\varnothing \neq \mathrm{LIM}^{L-\mathcal{J}_{\mathcal{A}}}_{(\mathcal{V}^{[m]}y^{0}:m\in\{0\}\cup\mathbb{N})} \subset \mathrm{LIM}^{L-\mathcal{D}_{\mathcal{A}}}_{(\mathcal{V}^{[m]}y^{0}:m\in\{0\}\cup\mathbb{N})}$$

where $\mathcal{J}_{\mathcal{A}} \in \mathbb{J}^{L}_{(\mathcal{C}(I,E),\mathcal{D}_{\mathcal{A}})}$ (*respectively,*

$$\varnothing \neq \mathrm{LIM}^{R-\mathcal{J}_{\mathcal{A}}}_{(\mathcal{V}^{[m]}y^{0}:m\in\{0\}\cup\mathbb{N})} \subset \mathrm{LIM}^{R-\mathcal{D}_{\mathcal{A}}}_{(\mathcal{V}^{[m]}y^{0}:m\in\{0\}\cup\mathbb{N})}$$

where $\mathcal{J}_{\mathcal{A}} \in \mathbb{J}^{R}_{(\mathcal{C}(I,E),\mathcal{D}_{\mathcal{A}})}$).

(B) Existence of solutions and convergence property. *If there exist* $q \in \mathbb{N}$ *and* $y^0 \in M$ *such that the single-valued dynamic system* $(\mathcal{C}(I,E),\mathcal{V}^{[q]})$ *is left* (*respectively, right*) $\mathcal{J}_{\mathcal{A}}$*-closed in* y^0, *then*

$$\emptyset \neq \mathrm{Fix}_{\mathcal{C}(I,E)}\big(\mathcal{V}^{[q]}\big) \subset \mathcal{Y}.$$

Moreover, there exists $u \in \mathcal{Y}$ (*respectively,* $v \in \mathcal{Y}$) *such that*

$$u \in \mathrm{LIM}^{L-\mathcal{J}_{\mathcal{A}}}_{(\mathcal{V}^{[m]}y^0 : m \in \{0\} \cup \mathbb{N})} \subset \mathrm{LIM}^{L-\mathcal{D}_{\mathcal{A}}}_{(\mathcal{V}^{[m]}y^0 : m \in \{0\} \cup \mathbb{N})}$$

and

$$\forall_{\alpha \in \mathcal{A}} \forall_{n \in \{1,2,\ldots,q\}} \big\{J_\alpha\big(u, \mathcal{V}^{[n]}(u)\big) = J_\alpha\big(\mathcal{V}^{[n]}(u), u\big) = 0\big\}$$

where $\mathcal{J}_{\mathcal{A}} \in \mathbb{J}^{L}_{(\mathcal{C}(I,E),\mathcal{D}_{\mathcal{A}})}$ (*respectively,*

$$v \in \mathrm{LIM}^{R-\mathcal{J}_{\mathcal{A}}}_{(\mathcal{V}^{[m]}y^0 : m \in \{0\} \cup \mathbb{N})} \subset \mathrm{LIM}^{R-\mathcal{D}_{\mathcal{A}}}_{(\mathcal{V}^{[m]}y^0 : m \in \{0\} \cup \mathbb{N})}$$

and

$$\forall_{\alpha \in \mathcal{A}} \forall_{n \in \{1,2,\ldots,q\}} \big\{J_\alpha\big(v, \mathcal{V}^{[n]}(v)\big) = J_\alpha\big(\mathcal{V}^{[n]}(v), v\big) = 0\big\}$$

where $\mathcal{J}_{\mathcal{A}} \in \mathbb{J}^{R}_{(\mathcal{C}(I,E),\mathcal{D}_{\mathcal{A}})}$).

Now, we formulate and prove the following special cases of Theorems 14.1 and 14.2.

Theorem 14.3 *Let* $(E,\mathcal{P}_{\mathcal{A}})$ *be a Hausdorff sequentially complete locally convex space with the topology defined by the family* $\mathcal{P}_{\mathcal{A}} = \{P_\alpha : \alpha \in \mathcal{A}\}$ *of continuous seminorms on* E. *Let* $(\mathcal{C}(I,E),\mathcal{D}_{\mathcal{A}})$ *be a gauge space with the gauge* $\mathcal{D}_{\mathcal{A}} = \{D_\alpha : \alpha \in \mathcal{A}\}$ *defined by* (14.4). *Suppose also that*:

(a) $\mathcal{S}_{\mathcal{A}} = \{S_\alpha : \alpha \in \mathcal{A}\}$ *is an* $\mathcal{S}_{\mathcal{A}}$*-family.*

(b) *The* $\mathcal{S}_{\mathcal{A}}$*-family* $\mathcal{S}_{\mathcal{A}} = \{S_\alpha : \alpha \in \mathcal{A}\}$ *and* $(\mathcal{C}(I,E),\mathcal{V})$ *satisfy*

$$\left\{\begin{array}{l} \forall_{\alpha \in \mathcal{A}} \forall_{\varepsilon > 1} \exists_{\eta > 1} \exists_{r \in \mathbb{N}} \forall_{x,y \in \mathcal{C}(I,E)} \forall_{s,l \in \mathbb{N}} \\ \quad \{S_\alpha(\sup_{t,\tau,\mu \in I} P_\alpha[K(t,\tau,(\mathcal{V}^{[s]}x)(\mu)) \\ \quad - K(t,\tau,(\mathcal{V}^{[l]}y)(\mu))]) < \varepsilon \cdot \eta \\ \quad \Rightarrow S_\alpha(\sup_{t,\tau,\mu \in I} P_\alpha[K(t,\tau,(\mathcal{V}^{[s+r]}x)(\mu)) \\ \quad - K(t,\tau,(\mathcal{V}^{[l+r]}y)(\mu))]) < \varepsilon\}. \end{array}\right. \tag{14.5}$$

Then the following hold:

(A) Convergence property. *For each* $y^0 \in \mathcal{C}(I,E)$, *there exists* $y \in \mathcal{C}(I,E)$ *such that a sequence* $(y^m = \mathcal{V}^{[m]}y^0 : m \in \{0\} \cup \mathbb{N})$ *is* $\mathcal{D}_{\mathcal{A}}$*-convergent to* y.

(B) Existence of solutions and convergence property. *If there exists* $q \in \mathbb{N}$ *such that* $(\mathcal{C}(I,E),\mathcal{V}^{[q]})$ *is* $\mathcal{D}_{\mathcal{A}}$*-closed on* $\mathcal{C}(I,E)$, *then*:

(B1) $\emptyset \neq \mathrm{Fix}_{\mathcal{C}(I,E)}(\mathcal{V}^{[q]}) \subset \mathcal{Y}$.

(B2) *For each $y^0 \in \mathcal{C}(I,E)$, there exists $y \in \mathrm{Fix}_{\mathcal{C}(I,E)}(\mathcal{V}^{[q]})$ such that sequence $(y^m = \mathcal{V}^{[m]}y^0 : m \in \{0\} \cup \mathbb{N})$ is $\mathcal{D}_{\mathcal{A}}$-convergent to y.*

(C) Existence of unique solution and convergence property. *If $(\mathcal{C}(I,E),\mathcal{V})$ is $\mathcal{D}_{\mathcal{A}}$-closed on $\mathcal{C}(I,E)$, then*:

(C1) *There exists $y \in \mathcal{C}(I,E)$ such that $\mathcal{Y} = \mathrm{Fix}_{\mathcal{C}(I,E)}(\mathcal{V}) = \{y\}$.*

(C2) *For each $y^0 \in \mathcal{C}(I,E)$, the sequence $(y^m = \mathcal{V}^{[m]}y^0 : m \in \{0\} \cup \mathbb{N})$ is $\mathcal{D}_{\mathcal{A}}$-convergent to y.*

Theorem 14.4 *Let $(E,\mathcal{P}_{\mathcal{A}})$ be a Hausdorff sequentially complete locally convex space with the topology defined by the family $\mathcal{P}_{\mathcal{A}} = \{P_\alpha : \alpha \in \mathcal{A}\}$ of continuous seminorms on E. Let $(\mathcal{C}(I,E),\mathcal{D}_{\mathcal{A}})$ be a gauge space with the gauge $\mathcal{D}_{\mathcal{A}} = \{D_\alpha : \alpha \in \mathcal{A}\}$ defined by (14.4). Suppose also that*:

(a) *$\mathcal{S}_{\mathcal{A}} = \{S_\alpha : \alpha \in \mathcal{A}\}$ is an $\mathcal{S}_{\mathcal{A}}$-family.*

(b) *The $\mathcal{S}_{\mathcal{A}}$-family $\mathcal{S}_{\mathcal{A}} = \{S_\alpha : \alpha \in \mathcal{A}\}$, $M \in 2^{\mathcal{C}(I,E)}$ and $(\mathcal{C}(I,E),\mathcal{V})$ satisfy*

$$\begin{cases} \forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>1}\exists_{\eta>1}\exists_{r\in\mathbb{N}}\forall_{y^0\in M}\forall_{s,l\in\mathbb{N}} \\ \quad \{S_\alpha(\sup_{t,\tau,\mu\in I} P_\alpha[K(t,\tau,(\mathcal{V}^{[s]}y^0)(\mu)) \\ \quad - K(t,\tau,(\mathcal{V}^{[l]}y^0)(\mu))]) < \varepsilon\cdot\eta \\ \quad \Rightarrow S_\alpha(\sup_{t,\tau,\mu\in I} P_\alpha[K(t,\tau,(\mathcal{V}^{[s+r]}y^0)(\mu)) \\ \quad - K(t,\tau,(\mathcal{V}^{[l+r]}y^0)(\mu))]) < \varepsilon\}. \end{cases} \tag{14.6}$$

Then the following hold:

(A) Convergence property. *For each $y^0 \in M$, there exists $y \in \mathcal{C}(I,E)$ such that a sequence $(y^m = \mathcal{V}^{[m]}y^0 : m \in \{0\} \cup \mathbb{N})$ is $\mathcal{D}_{\mathcal{A}}$-convergent to y.*

(B) Existence of solutions and convergence property. *If there exist $q \in \mathbb{N}$ and $y^0 \in M$ such that the dynamic system $(\mathcal{C}(I,E),\mathcal{V}^{[q]})$ is $\mathcal{D}_{\mathcal{A}}$-closed in y^0, then*:

(B1) $\emptyset \neq \mathrm{Fix}_{\mathcal{C}(I,E)}(\mathcal{V}^{[q]}) \subset \mathcal{Y}$.

(B2) *There exists $y \in \mathrm{Fix}_{\mathcal{C}(I,E)}(\mathcal{V}^{[q]})$ such that a sequence $(y^m = \mathcal{V}^{[m]}y^0 : m \in \{0\} \cup \mathbb{N})$ is $\mathcal{D}_{\mathcal{A}}$-convergent to y and*

$$\forall_{\alpha\in\mathcal{A}}\forall_{n\in\{1,2,\ldots,q\}}\{D_\alpha(y,\mathcal{V}^{[n]}(y)) = 0\}.$$

Proof of Theorem 14.3 We first note that $(\mathcal{C}(I,E),\mathcal{P}_{\mathcal{A}})$, $\mathcal{P}_{\mathcal{A}} = \mathcal{D}_{\mathcal{A}}$, $C = \{C_\alpha\}_{\alpha\in\mathcal{A}}$, $\forall_{\alpha\in\mathcal{A}}\{C_\alpha = 1\}$, is a triangular space and that assumptions of Theorem 14.3 can be rerun with the following modifications: $\mathcal{D}_{\mathcal{A}}$ is symmetric on $\mathcal{C}(I,E)$, $\mathcal{D}_{\mathcal{A}} = \{D_\alpha : \alpha \in \mathcal{A}\}$ is separating on $\mathcal{C}(I,E)$ since $\mathcal{P}_{\mathcal{A}} = \{P_\alpha : \alpha \in \mathcal{A}\}$ is separating on $\mathcal{C}(I,E)$, the space $(\mathcal{C}(I,E),\mathcal{D}_{\mathcal{A}})$ is $\mathcal{D}_{\mathcal{A}}$-sequentially complete since the space $(E,\mathcal{P}_{\mathcal{A}})$ is $\mathcal{P}_{\mathcal{A}}$-sequentially complete, and a single-valued dynamic system $(\mathcal{C}(I,E),\mathcal{V})$ defined by (14.2) is $\mathcal{J}_{\mathcal{A}} = \mathcal{P}_{\mathcal{A}} = \mathcal{D}_{\mathcal{A}}$-admissible on each $M \in 2^{\mathcal{C}(I,E)}$ since the space $(\mathcal{C}(I,E),\mathcal{D}_{\mathcal{A}})$ is $\mathcal{D}_{\mathcal{A}}$-sequentially complete.

We prove that (14.5) implies

$$\begin{aligned} &\forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>1}\exists_{\eta>1}\exists_{r\in\mathbb{N}}\forall_{x,y\in\mathcal{C}(I,E)}\forall_{s,l\in\mathbb{N}} \\ &\quad \{S_\alpha(D_\alpha[\mathcal{V}^{[s+1]}x,\mathcal{V}^{[l+1]}y]) < \varepsilon\cdot\eta \\ &\quad \Longrightarrow S_\alpha(D_\alpha[\mathcal{V}^{[s+r+1]}x,\mathcal{V}^{[l+r+1]}y]) < \varepsilon\}. \end{aligned} \tag{14.7}$$

To establish this, let $\alpha \in \mathcal{A}$, $t \in I$ and $x, y \in \mathcal{C}(I, E)$ be arbitrary and fixed. Then, in view of (14.1)–(14.4),

$$\begin{aligned}
&P_\alpha\bigl[\bigl(\mathcal{V}^{[s+1]}x\bigr)(t) - \bigl(\mathcal{V}^{[l+1]}y\bigr)(t)\bigr] \\
&\quad = P_\alpha\biggl\{\int_{I(t)} \bigl[K\bigl(t,\tau,\bigl(\mathcal{V}^{[s]}x\bigr)\bigl(h(\tau)\bigr)\bigr) - K\bigl(t,\tau,\bigl(\mathcal{V}^{[l]}y\bigr)\bigl(h(\tau)\bigr)\bigr)\bigr]\,d\tau\biggr\} \\
&\quad \le \int_{I(t)} P_\alpha\bigl\{K\bigl(t,\tau,\bigl(\mathcal{V}^{[s]}x\bigr)\bigl(h(\tau)\bigr)\bigr) - K\bigl(t,\tau,\bigl(\mathcal{V}^{[l]}y\bigr)\bigl(h(\tau)\bigr)\bigr)\bigr\}\,d\tau \\
&\quad \le \sup_{t,\tau\in I} P_\alpha\bigl\{K\bigl(t,\tau,\bigl(\mathcal{V}^{[s]}x\bigr)\bigl(h(\tau)\bigr)\bigr) - K\bigl(t,\tau,\bigl(\mathcal{V}^{[l]}y\bigr)\bigl(h(\tau)\bigr)\bigr)\bigr\} \\
&\quad \le \sup_{t,\tau,\mu\in I} P_\alpha\bigl\{K\bigl(t,\tau,\bigl(\mathcal{V}^{[s]}x\bigr)(\mu)\bigr) - K\bigl(t,\tau,\bigl(\mathcal{V}^{[l]}y\bigr)(\mu)\bigr)\bigr\}.
\end{aligned}$$

Consequently, we have $D_\alpha[\mathcal{V}^{[s+1]}x, \mathcal{V}^{[l+1]}y] = \sup_{t\in I} P_\alpha[(\mathcal{V}^{[s+1]}x)(t) - (\mathcal{V}^{[l+1]}y)(t)] \le \sup_{t,\tau,\mu\in I} P_\alpha[K(t,\tau,(\mathcal{V}^{[s]}x)(\mu)) - K(t,\tau,(\mathcal{V}^{[l]}y)(\mu))]$ which, by (14.5), implies $S_\alpha(D_\alpha[\mathcal{V}^{[s+1]}x, \mathcal{V}^{[l+1]}y]) \le S_\alpha(\sup_{t,\tau,\mu\in I} P_\alpha[K(t,\tau,(\mathcal{V}^{[s]}x)(\mu)) - K(t,\tau,(\mathcal{V}^{[l]}y)(\mu))]) < \varepsilon \cdot \eta$. Similarly, in view of (14.5), we find $S_\alpha(D_\alpha[\mathcal{V}^{[s+r+1]}x, \mathcal{V}^{[l+r+1]}y]) \le S_\alpha(\sup_{t,\tau,\mu\in I} P_\alpha[K(t,\tau,(\mathcal{V}^{[s+r]}x)(\mu)) - K(t,\tau,(\mathcal{V}^{[l+r]}y)(\mu))]) < \varepsilon$. Therefore, in view of (14.5), (14.7) holds.

The statements of Theorem 14.3 are now immediate consequences of Theorem 9.3. □

Proof of Theorem 14.4 Using (14.3), (14.4) and (14.6) we obtain

$$\begin{aligned}
&\forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>1}\exists_{\eta>1}\exists_{r\in\mathbb{N}}\forall_{y^0\in\mathcal{C}(I,E)}\forall_{s,l\in\mathbb{N}} \\
&\quad \bigl\{S_\alpha\bigl(D_\alpha\bigl[\mathcal{V}^{[s+1]}y^0, \mathcal{V}^{[l+1]}y^0\bigr]\bigr) < \varepsilon\cdot\eta \\
&\quad \Longrightarrow S_\alpha\bigl(D_\alpha\bigl[\mathcal{V}^{[s+r+1]}y^0, \mathcal{V}^{[l+r+1]}y^0\bigr]\bigr) < \varepsilon\bigr\}.
\end{aligned}$$

Next, using a similar argument as in the proofs of Theorems 9.4 and 14.3, we have the assertions. □

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