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Fixed point theorems in a new type of modular metric spaces

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Abstract

In this paper, considering both a modular metric space and a generalized metric space in the sense of Jleli and Samet (Fixed Point Theory Appl. 2015:61, 2015), we introduce a new concept of generalized modular metric space. Then we present some examples showing that the generalized modular metric space includes some kind of metric structures. Finally, we provide some fixed point results for both contraction and quasicontraction type mappings on generalized modular metric spaces.

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1 Introduction

In 1990, the fixed point theory in modular function spaces was initiated by Khamsi, Kozlowski, and Reich [10]. Modular function spaces are a special case of the theory of modular vector spaces introduced by Nakano [13]. Modular metric spaces were introduced in [2, 3]. Fixed point theory in modular metric spaces was studied by Abdou and Khamsi [1]. Their approach was fundamentally different from the one studied in [2, 3]. In this paper, we follow the same approach as the one used in [1].

Generalizations of standard metric spaces are interesting because they allow for some deep understanding of the classical results obtained in metric spaces. One has always to be careful when coming up with a new generalization. For example, if we relax the triangle inequality, some of the classical known facts in metric spaces may become impossible to obtain. This is the case with the generalized metric distance introduced by Jleli and Samet in [6]. The authors showed that this generalization encompasses metric spaces, b-metric spaces, dislocated metric spaces, and modular vector spaces.

In this paper, considering both a modular metric space and a generalized metric space in the sense of Jleli and Samet [6], we introduce a new concept of generalized modular metric space. Then we proceed to proving the Banach contraction principle (BCP) and Ćirić's fixed point theorem for quasicontraction mappings in this new space. To prove Ćirić's fixed point theorem in this new space, we take the contraction constant $k < \frac{1}{C}$, where C is as given in Definition 1.1. For readers interested in metric fixed point theory, we recommend the book by Khamsi and Kirk [8], and for more details, see [5, 7, 9, 11, 12].

First, we give the definition of generalized modular metric spaces.

Definition 1.1 Let X be an abstract set. A function $D : (0,\infty) \times X \times X \to [0,\infty]$ is said to be a regular generalized modular metric (GMM) on X if it satisfies the following three axioms:

(GMM_1) If $D_\lambda(x,y) = 0$ for some $\lambda > 0$, then $x = y$ for all $x, y \in X$;
(GMM_2) $D_\lambda(x,y) = D_\lambda(y,x)$ for all $\lambda > 0$ and $x, y \in X$;
(GMM_3) There exists $C > 0$ such that, if $(x,y) \in X \times X$, $\{x_n\} \subset X$ with $\lim_{n\to\infty} D_\lambda(x_n,x) = 0$ for some $\lambda > 0$, then

$$D_\lambda(x,y) \le C \limsup_{n\to\infty} D_\lambda(x_n,y).$$

The pair (X,D) is said to be a generalized modular metric space ($GMMS$).

It is easy to check that if there exist $x, y \in X$ such that there exists $\{x_n\} \subset X$ with $\lim_{n\to\infty} D_\lambda(x_n,x) = 0$ for some $\lambda > 0$, and $D_\lambda(x,y) < \infty$, then we must have $C \ge 1$. In fact, throughout this work, we assume $C \ge 1$.

Let D be a GMM on X. Fix $x_0 \in X$. The sets

$$\begin{cases} X_D = X_D(x_0) = \{x \in X : D_\lambda(x,x_0) \to 0 \text{ as } \lambda \to \infty\} \\ X_D^\star = \{x \in X : \exists \lambda = \lambda(x) > 0 \text{ such that } D_\lambda(x,x_0) < \infty\} \end{cases}$$

are called *generalized modular sets*. Next, we give some examples that inspired our definition of a $GMMS$.

Example 1.1 (Modular vector spaces (MVS) [13]) Let X be a linear vector space over the field $\mathbb{R}$. A function $\rho : X \to [0,\infty]$ is called regular modular if the following hold:

(1) $\rho(x) = 0$ if and only if $x = 0$,
(2) $\rho(\alpha x) = \rho(x)$ if $|\alpha| = 1$,
(3) $\rho(\alpha x + (1-\alpha)y) \le \rho(x) + \rho(y)$ for any $\alpha \in [0,1]$,

for any $x, y \in X$. Let ρ be regular modular defined on a vector space X. The set

$$X_\rho = \left\{x \in X; \lim_{\alpha\to 0} \rho(\alpha x) = 0\right\}$$

is called a MVS. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X_ρ and $x \in X_\rho$. If $\lim_{n\to\infty} \rho(x_n - x) = 0$, then $\{x_n\}_{n\in\mathbb{N}}$ is said to ρ-converge to x. ρ is said to satisfy the Δ_2-condition if there exists $K \ne 0$ such that

$$\rho(2x) \le K\rho(x)$$

for any $x \in X_\rho$. Moreover, ρ is said to satisfy the Fatou property(FP) if

$$\rho(x-y) \le \liminf_{n\to\infty} \rho(x_n - y),$$

whenever $\{x_n\}$ ρ-converges to x for any $x, y, x_n \in X_\rho$. Next, we show that a MVS may be embedded with a GMM structure. Indeed, let (X,ρ) be a MVS. Define $D : (0,+\infty) \times X \times X \to [0,+\infty]$ by

$$D_\lambda(x,y) = \rho\left(\frac{x-y}{\lambda}\right).$$

Then the following hold:

(i) If $D_\lambda(x,y) = 0$ for some $\lambda > 0$ and any $x, y \in X$, then $x = y$;

(ii) $D_\lambda(x,y) = D_\lambda(y,x)$ for any $\lambda > 0$ and $x, y \in X$;

(iii) If ρ satisfies the FP, then for any $\lambda > 0$ and $\{x_n\}$ such that $\{x_n/\lambda\}$ ρ-converges to x/λ, we have

$$\rho\left(\frac{x-y}{\lambda}\right) \le \liminf_{n\to\infty} \rho\left(\frac{x_n - y}{\lambda}\right) \le \limsup_{n\to\infty} \rho\left(\frac{x_n - y}{\lambda}\right),$$

which implies

$$D_\lambda(x,y) \le \liminf_{n\to\infty} D_\lambda(x_n,y) \le \limsup_{n\to\infty} D_\lambda(x_n,y)$$

for any $x, y, x_n \in X_\rho$.

Therefore, (X, D) satisfies all the properties of Definition 1.1 as claimed. Note that the constant C which appears in the property (GMM_3) is equal to 1 provided the FP is satisfied by ρ.

In the next example, we discuss the case of modular metric spaces.

Example 1.2 (Modular metric spaces (MMS) [2, 3]) Let X be an abstract set. For a function $\omega : (0,+\infty) \times X \times X \to [0,\infty]$, we will write

$$\omega(\lambda, x, y) = \omega_\lambda(x,y).$$

The function $\omega : (0,\infty) \times X \times X \to [0,\infty]$ is said to be a regular modular metric(MM) on X if it satisfies the following axioms:

(i) $x = y$ if and only if $\omega_\lambda(x,y) = 0$ for some $\lambda > 0$;

(ii) $\omega_\lambda(x,y) = \omega_\lambda(y,x)$ for all $\lambda > 0$ and $x, y \in M$;

(iii) $\omega_{\lambda+\mu}(x,y) \le \omega_\lambda(x,z) + \omega_\mu(z,y)$ for all $\lambda, \mu > 0$ and $x, y, z \in X$.

Let ω be regular modular on X. Fix $x_0 \in X$. The two sets

$$\begin{cases} X_\omega = X_\omega(x_0) = \{x \in X : \omega_\lambda(x,x_0) \to 0 \text{ as } \lambda \to \infty\} \\ X_\omega^* = X_\omega^*(x_0) = \{x \in X : \exists\lambda = \lambda(x) > 0 \text{ such that } \omega_\lambda(x,x_0) < \infty\} \end{cases}$$

are called modular spaces (around arbitrarily chosen x_0). It is clear that $X_\omega \subset X_\omega^*$, but this inclusion may be proper in general. Let X_ω be a *MMS*. If $\lim_{n\to\infty} \omega_\lambda(x_n,x) = 0$ for some $\lambda > 0$, then we may not have $\lim_{n\to\infty} \omega_\lambda(x_n,x) = 0$ for all $\lambda > 0$. Therefore, as it is done in *MVS*, we will say that ω satisfies the Δ_2-condition if this is the case, i.e., $\lim_{n\to\infty} \omega_\lambda(x_n,x) = 0$ for some $\lambda > 0$ implies $\lim_{n\to\infty} \omega_\lambda(x_n,x) = 0$ for all $\lambda > 0$. We will say that the sequence $\{x_n\}_{n\in\mathbb{N}}$ in X_ω is ω-convergent to $x \in X_\omega$ if $\lim_{n\to\infty} \omega_\lambda(x_n,x) = 0$ for some $\lambda > 0$. The modular function ω is said to satisfy the FP if $\{x_n\}$ is such that $\lim_{n\to\infty} \omega_\lambda(x_n,x) = 0$ for some $\lambda > 0$, we have

$$\omega_\lambda(x,y) \le \liminf_{n\to\infty} \omega_\lambda(x_n,y)$$

for any $y \in X_\omega$. Let X_ω be a *MMS*, where ω is a regular modular. Define $D : (0, +\infty) \times X_\omega \times X_\omega \to [0, +\infty]$ by

$$D_\lambda(x, y) = \omega_\lambda(x, y).$$

Then the following hold:

(i) If $D_\lambda(x, y) = 0$ for some $\lambda > 0$ and $x, y \in X_\omega$, then $x = y$;

(ii) $D_\lambda(x, y) = D_\lambda(y, x)$ for any $\lambda > 0$ and $x, y \in X_\omega$;

(iii) If ω satisfies the FP, then for any $x \in X_\omega$ and $\{x_n\} \subset X_\omega$ such that $\lim_{n\to\infty} D_\lambda(x_n, x) = 0$ for some $\lambda > 0$, we have

$$\omega_\lambda(x, y) \leq \liminf_{n\to\infty} \omega_\lambda(x_n, y) \leq \limsup_{n\to\infty} \omega_\lambda(x_n, y)$$

for any $y \in X_\omega$, which implies

$$D_\lambda(x, y) \leq \liminf_{n\to\infty} D_\lambda(x_n, y) \leq \limsup_{n\to\infty} D_\lambda(x_n, y).$$

In other words, (X_ω, D) is a GMMS.

Example 1.3 (Generalized metric spaces (GMS) [6]) Throughout the paper X is an abstract set. For a function $\mathcal{D} : X \times X \to [0, \infty]$ and $x \in X$, we will introduce the set

$$\mathcal{C}(\mathcal{D}, X, x) = \Big\{ \{x_n\} \subset X; \lim_{n\to\infty} \mathcal{D}(x_n, x) = 0 \Big\}.$$

According to [6], the function $\mathcal{D} : X \times X \to [0, \infty]$ is said to define a generalized metric (GM) on X if it satisfies the following axioms:

($\mathcal{D}_1$) For every $(x, y) \in X \times X$, we have $\mathcal{D}(x, y) = 0 \Rightarrow x = y$;

($\mathcal{D}_2$) For every $(x, y) \in X \times X$, we have $\mathcal{D}(x, y) = \mathcal{D}(y, x)$,

($\mathcal{D}_3$) There exists $C > 0$ such that, if $(x, y) \in X \times X$, $\{x_n\} \in \mathcal{C}(\mathcal{D}, X, x)$, we have

$$\mathcal{D}(x, y) \leq C \limsup_{n\to\infty} \mathcal{D}(x_n, y).$$

The pair $(X, \mathcal{D})$ is then called a *GMS*. Let us show that such a structure may be seen as a *GMMS*. Indeed, let $(X, \mathcal{D})$ be a *GMS*. Define $D : (0, +\infty) \times X \times X \to [0, +\infty]$ by

$$D_\lambda(x, y) = \frac{\mathcal{D}(x, y)}{\lambda}.$$

Clearly, if $\{x_n\} \in \mathcal{C}(\mathcal{D}, X, x)$ for some $x \in X$, then we have

$$\lim_{n\to\infty} D_\lambda(x_n, x) = 0$$

for any $\lambda > 0$. Then the following hold:

(i) If $D_\lambda(x, y) = 0$ for some $\lambda > 0$ and $x, y \in X$, then $x = y$;

(ii) $D_\lambda(x, y) = D_\lambda(y, x)$ for any $\lambda > 0$ and $x, y \in X$;

(iii) There exists $C > 0$ such that, if $(x,y) \in X \times X$, $\{x_n\} \in \mathcal{C}(D_\lambda, X, x)$ for some $\lambda > 0$, we have

$$D_\lambda(x,y) \leq C \limsup_{n\to\infty} D_\lambda(x_n,y).$$

These properties show that (X,D) is a *GMMS*.

2 Fixed point theorems (FPT) in GMMS

The following definition is useful to set new fixed point theory on GMMS.

Definition 2.1 Let (X_D,D) be a *GMMS*.

(1) The sequence $\{x_n\}_{n\in\mathbb{N}}$ in X_D is said to be D-convergent to $x \in X_D$ if and only if $D_\lambda(x_n,x) \to 0$, as $n \to \infty$, for some $\lambda > 0$.
(2) The sequence $\{x_n\}_{n\in\mathbb{N}}$ in X_D is said to be D-Cauchy if $D_\lambda(x_m,x_n) \to 0$, as $m,n \to \infty$, for some $\lambda > 0$.
(3) A subset C of X_D is said to be D-closed if for any $\{x_n\}$ from C which D-converges to x, $x \in C$.
(4) A subset C of X_D is said to be D-complete if for any $\{x_n\}$ D-Cauchy sequence in C such that $\lim_{n,m\to\infty} D_\lambda(x_n,x_m) = 0$ for some λ, there exists a point $x \in C$ such that $\lim_{n,m\to\infty} D_\lambda(x_n,x) = 0$.
(5) A subset C of X_D is said to be D-bounded if, for some $\lambda > 0$, we have

$$\delta_{D,\lambda}(C) = \sup\bigl\{D_\lambda(x,y); x,y \in C\bigr\} < \infty.$$

In general, if $\lim_{n\to\infty} D_\lambda(x_n,x) = 0$ for some $\lambda > 0$, then we may not have $\lim_{n\to\infty} D_\lambda(x_n,x) = 0$ for all $\lambda > 0$. Therefore, as it is done in modular function spaces, we will say that D satisfies Δ_2-condition if and only if $\lim_{n\to\infty} D_\lambda(x_n,x) = 0$ for some $\lambda > 0$ implies $\lim_{n\to\infty} D_\lambda(x_n,x) = 0$ for all $\lambda > 0$.

Another question that comes into this setting is the concept of D-limit and its uniqueness.

Proposition 2.1 *Let (X_D,D) be a GMMS. Let $\{x_n\}$ be a sequence in X_D. Let $(x,y) \in X_D \times X_D$ such that $D_\lambda(x_n,x) \to 0$ and $D_\lambda(x_n,y) \to 0$ as $n \to \infty$ for some $\lambda > 0$. Then $x = y$.*

Proof Using the property (GMM_3), we have

$$D_\lambda(x,y) \leq C \limsup_{n\to\infty} D_\lambda(x_n,y) = 0,$$

which implies from the property (GMM_1) that $x = y$. □

3 The main results

3.1 The Banach contraction principle (BCP) in GMMS

Now, we show an extension of the BCP to the setting of *GMMS* presented formerly. From now on, we mean 1 instead of λ for the same reason Abdou and Khmasi used in their work [1].

Definition 3.1 Let (X_D, D) be a *GMMS* and $f : X_D \to X_D$ be a mapping. f is called a D-contraction mapping if there exists $k \in (0,1)$ such that

$$D_1\big(f(x), f(y)\big) \le kD_1(x,y) \quad \text{for any } (x,y) \in X_D \times X_D.$$

x is said to be a fixed point of f if $f(x) = x$.

Proposition 3.1 *Let (X_D, D) be a GMMS. Let $f : X_D \to X_D$ be a D-contraction mapping. If ω_1 and ω_2 are fixed points of f and $D_1(\omega_1, \omega_2) < \infty$, then we have $\omega_1 = \omega_2$.*

Proof Let $\omega_1, \omega_2 \in X_D$ be two fixed points of f such that $D_1(\omega_1, \omega_2) < \infty$. As f is a D-contraction, there exists $k \in (0,1)$ such that

$$D_1(\omega_1, \omega_2) = D_1\big(f(\omega_1), f(\omega_2)\big) \le kD_1(\omega_1, \omega_2).$$

Since $D_1(\omega_1, \omega_2) < \infty$, we conclude that $D_1(\omega_1, \omega_2) = 0$, which implies $\omega_1 = \omega_2$ from (GMM_1). □

Let (X_D, D) be a *GMMS* and $f : X_D \to X_D$ be a mapping. For any $x \in M$, define the *orbit* of x by

$$\mathcal{O}(x) = \big\{x, f(x), f^2(x), \ldots\big\}.$$

Set $\delta_{D,\lambda}(x) = \sup\{D_\lambda(f^n(x), f^t(x)); n, t \in \mathbb{N}\}$, where $\lambda > 0$. The following result may be seen as an extension of the BCP in *GMMS*.

Theorem 3.1 *Let (X_D, D) be a GMMS. Assume that X_D is D-complete. Let $f : X_D \to X_D$ be a D-contraction mapping. Assume that $\delta_{D,1}(x_0)$ is finite for some $x_0 \in X_D$. Then $\{f^n(x_0)\}$ D-converges to a fixed point ω of f. Moreover, if $D_1(x, \omega) < \infty$ for $x \in X_D$, then $\{f^n(x)\}$ D-converges to ω.*

Proof Let $x_0 \in X_D$ be such that $\delta_{D,1}(x_0) < \infty$. Then

$$D_1\big(f^{n+p}(x_0), f^n(x_0)\big) \le k^n D_1\big(f^p(x_0), x_0\big) \le k^n \delta_{D,1}(x_0)$$

for any $n, p \in \mathbb{N}$. Since $k < 1$, $\{f^n(x_0)\}$ is D-Cauchy. As X_D is D-complete, then there exists $\omega \in X_D$ such that $\lim_{n\to\infty} D_1(f^n(x_0), \omega) = 0$. Since

$$D_1\big(f^n(x_0), f(\omega)\big) \le kD_1\big(f^{n-1}(x_0), \omega\big); \quad n = 1, 2, \ldots,$$

we have $\lim_{n\to\infty} D_1(f^n(x_0), f(\omega)) = 0$. Proposition 2.1 implies that $f(\omega) = \omega$, i.e., ω is a fixed point of f. Let $x \in X_D$ be such that $D_1(x, \omega) < \infty$. Then

$$D_1\big(f^n(x), \omega\big) = D_1\big(f^n(x), f^n(\omega)\big) \le k^n D_1(x, \omega)$$

for any $n \ge 1$. Since $k < 1$, we get $\lim_{n\to\infty} D_1(f^n(x), \omega) = 0$, i.e., $\{f^n(x)\}$ D-converges to ω. □

If $D_1(x,y) < \infty$ for any $x,y \in X_D$, then f has at most one fixed point. Moreover, if X_D is D-complete and $\delta_{D,1}(x) < \infty$ for any $x \in X_D$, then all orbits D-converge to the unique fixed point of f. In metric spaces, $d(x,y)$ is always finite. Because of this reason, any contraction will have at most one fixed point. Moreover, the orbits of the contraction are all bounded. Indeed, let $f : M \to M$ be a contraction, where M is a metric space endowed with a metric distance d. We have

$$d\big(f^{n+1}(x),f^n(x)\big) \le k^n d\big(f(x),x\big)$$

for any $n \in \mathbb{N}$ and $x \in M$, which implies by using the triangle inequality

$$d\big(f^{n+p}(x),f^n(x)\big) \le \sum_{k=0}^{p-1} d\big(f^{n+k+1}(x),f^{n+k}(x)\big) \le \sum_{k=0}^{p-1} k^{n+k} d\big(f(x),x\big) \le \frac{1}{1-k} d\big(f(x),x\big),$$

since $k < 1$. Hence

$$\sup\big\{d\big(f^n(x),f^t(x)\big); n,t \in \mathbb{N}\big\} \le \frac{1}{1-k} d\big(f(x),x\big) < \infty$$

for any $x \in M$.

Next, we investigate the extension of Ćirić's FPT [4] for quasicontraction type mappings in *GMMS* and give a correct version of Theorem 4.3 in [6] since its proof is wrong [7].

3.2 Ćirić quasicontraction in generalized modular metric spaces

First, let us introduce the concept of quasicontraction mappings in the setting of *GMMS*.

Definition 3.2 Let (X_D, D) be a *GMMS*. The mapping $f : X_D \to X_D$ is said to be a D-quasicontraction if there exists $k \in (0,1)$ such that

$$D_1\big(f(x),f(y)\big) \le k \max\big\{D_1(x,y), D_1\big(x,f(x)\big), D_1\big(y,f(y)\big), D_1\big(x,f(y)\big), D_1\big(y,f(x)\big)\big\}$$

for any $(x,y) \in X_D \times X_D$.

Proposition 3.2 *Let (X_D, D) be a GMMS. Let $f : X_D \to X_D$ be a D-quasicontraction mapping. If ω is a fixed point of f such that $D_1(\omega,\omega) < \infty$, then we have $D_1(\omega,\omega) = 0$. Moreover, if ω_1 and ω_2 are two fixed points of f such that $D_1(\omega_1,\omega_2) < \infty, D_1(\omega_1,\omega_1) < \infty$, and $D_1(\omega_2,\omega_2) < \infty$, then we have $\omega_1 = \omega_2$.*

Proof Let ω be a fixed point of f, then

$$\begin{aligned} D_1(\omega,\omega) &= D_1\big(f(\omega),f(\omega)\big) \\ &\le k \max\big\{D_1(\omega,\omega), D_1\big(\omega,f(\omega)\big), D_1\big(\omega,f(\omega)\big), D_1\big(\omega,f(\omega)\big), D_1\big(\omega,f(\omega)\big)\big\} \\ &= kD_1(\omega,\omega). \end{aligned}$$

Since $k < 1$ and $D_1(\omega,\omega) < \infty$, then $D_1(\omega,\omega) = 0$. Let $\omega_1, \omega_2 \in X_D$ be two fixed points of f such that $D_1(\omega_1,\omega_2) < \infty, D_1(\omega_1,\omega_1) < \infty$, and $D_1(\omega_2,\omega_2) < \infty$. Since f is a D-

quasicontraction, there exists $k < 1$ such that

$$\begin{aligned} D_1(\omega_1,\omega_2) &= D_1\big(f(\omega_1),f(\omega_2)\big) \\ &\le k\max\big\{D_1(\omega_1,\omega_2), D_1\big(\omega_1,f(\omega_1)\big), D_1\big(\omega_2,f(\omega_2)\big), \\ &\quad D_1\big(\omega_1,f(\omega_2)\big), D_1\big(\omega_2,f(\omega_1)\big)\big\}. \\ &= k\max\big\{D_1(\omega_1,\omega_2), D_1(\omega_1,\omega_1), D_1(\omega_2,\omega_2)\big\}. \end{aligned}$$

Since $D_1(\omega_1,\omega_1) < \infty$ and $D_1(\omega_2,\omega_2) < \infty$, then $D_1(\omega_1,\omega_1) = D_1(\omega_2,\omega_2) = 0$. Now we have

$$D_1(\omega_1,\omega_2) \le kD_1(\omega_1,\omega_2).$$

Since $D_1(\omega_1,\omega_2) < \infty$ and $k < 1$, then $D_1(\omega_1,\omega_2) = 0$. □

The following result may be seen as an extension of Ćirić's FPT [4] for quasicontraction type mappings in *GMMS*.

Theorem 3.2 *Let (X_D,D) be a D-complete GMMS. Let $f : X_D \to X_D$ be a D-quasi-contraction mapping. Assume that $k < \frac{1}{C}$, where C is the constant from (GMM_3), and there exists $x_0 \in X_D$ such that $\delta_{D,1}(x_0) < \infty$. Then $\{f^n(x_0)\}$ D-converges to some $\omega \in X_D$. If $D_1(x_0,f(\omega)) < \infty$ and $D_1(\omega,f(\omega)) < \infty$, then ω is a fixed point of f.*

Proof Let f be a D-quasicontraction, then there exists $k \in (0,1)$ such that, for all $p,r,n \in \mathbb{N}$ and $x \in X_D$, we have

$$\begin{aligned} D_1\big(f^{n+p+1}(x),f^{n+r+1}(x)\big) \le k\max\big\{&D_1\big(f^{n+p}(x),f^{n+r}(x)\big), \\ &D_1\big(f^{n+p}(x),f^{n+p+1}(x)\big), D_1\big(f^{n+r}(x),f^{n+r+1}(x)\big), \\ &D_1\big(f^{n+p}(x),f^{n+r+1}(x)\big), D_1\big(f^{n+r}(x),f^{n+p+1}(x)\big)\big\}. \end{aligned}$$

Hence $\delta_{D,1}(f(x)) \le k\delta_{D,1}(x)$ for any $x \in X_D$. Consequently, we have

$$\delta_{D,1}\big(f^n(x_0)\big) \le k^n\delta_{D,1}(x_0) \tag{1}$$

for any $n \ge 1$. Using the above inequality, we get

$$D_1\big(f^n(x_0),f^{n+t}(x_0)\big) \le \delta_{D,1}\big(f^n(x_0)\big) \le k^n\delta_{D,1}(x_0) \tag{2}$$

for every $n,m \in \mathbb{N}$. Since $\delta_{D,1}(x_0) < \infty$ and $k < 1/C \le 1$, we have

$$\lim_{n,t\to\infty} D_1\big(f^n(x_0),f^{n+t}(x_0)\big) = 0,$$

which implies that $\{f^n(x_0)\}$ is a D-Cauchy sequence. Since X_D is D-complete, there exists $\omega \in X_D$ such that $\lim_{n\to\infty} D_1(f^n(x_0),\omega) = 0$, i.e., $\{f^n(x_0)\}$ D-converges to ω. Next, we assume $D_1(x_0,f(\omega)) < \infty$ and $D_1(\omega,f(\omega)) < \infty$. Using inequality (2) and the property (GMM_3), we get

$$D_1\big(\omega,f^n(x_0)\big) \le C\limsup_{t\to\infty} D_1\big(f^n(x_0),f^{n+t}(x_0)\big) \le Ck^n\delta_{D,1}(x_0) \tag{3}$$

for every $n,m \in \mathbb{N}$.

Hence,

$$D_1\big(f(x_0),f(\omega)\big) \leq k\max\big\{D_1(x_0,\omega), D_1\big(x_0,f(x_0)\big), D_1\big(\omega,f(\omega)\big)$$
$$D_1\big(f(x_0),\omega\big), D_1\big(x_0,f(\omega)\big)\big\}$$

and, using (1), (2), (3), and $k < 1/C \leq 1$, we have

$$D_1\big(f^2(x_0),f(\omega)\big) \leq \max\big\{k^2C\delta_{D,1}(x_0), kD_1\big(\omega,f(\omega)\big), k^2D_1\big(\omega,f(x_0)\big)\big\}.$$

Progressively, by induction, we can get

$$D_1\big(f^n(x_0),f(\omega)\big) \leq \max\big\{k^nC\delta_{D,1}(x_0), kD_1\big(\omega,f(\omega)\big), k^nD_1\big(\omega,f(x_0)\big)\big\}$$

for every $n \geq 1$. Moreover, we have

$$\limsup_{n\to\infty} D_1\big(f^n(x_0),f(\omega)\big) \leq kD_1\big(\omega,f(\omega)\big),$$

when $D_1(x_0,f(\omega)) < \infty$ and $\delta_{D,1}(x_0) < \infty$. Again the property (GMM_3) implies

$$D_1\big(\omega,f(\omega)\big) \leq C\limsup_{n\to\infty} D_1\big(f^n(x_0),f(\omega)\big) \leq kCD_1\big(\omega,f(\omega)\big).$$

Since $kC < 1$ and $D_1(\omega,f(\omega)) < \infty$, then $D_1(\omega,f(\omega)) = 0$, i.e., $f(\omega) = \omega$. □

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Authors' contributions
All authors conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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