Chidume and Nnakwe *Fixed Point Theory and Applications* (2018) 2018:16
https://doi.org/10.1186/s13663-018-0641-4

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH

Open Access

Convergence theorems of subgradient extragradient algorithm for solving variational inequalities and a convex feasibility problem

C.E. Chidume[1*] and M.O. Nnakwe[1]

Dedicated to Professor H. K. Xu for his contributions in nonlinear operator theory.

*Correspondence:
cchidume@aust.edu.ng
[1] African University of Science and Technology, Abuja, Nigeria

Abstract

Let C be a nonempty closed and convex subset of a uniformly smooth and 2-uniformly convex real Banach space E with dual space E^*. In this paper, a Krasnoselskii-type subgradient extragradient iterative algorithm is constructed and used to approximate a common element of solutions of variational inequality problems and fixed points of a countable family of relatively nonexpansive maps. The theorems proved are improvement of the results of Censor *et al.* (J. Optim. Theory Appl. 148:318–335, 2011).

MSC: Primary 47H09; secondary 47H10; 47J25; 47J05; 47J20

Keywords: Subgradient extragradient algorithm; Variational inequality; Relatively nonexpansive maps

1 Introduction

Let E be a real normed space with dual space E^*, and C be a nonempty closed and convex subset of E. The *variational inequality problem* is to find an element $v \in C$ such that

$$\langle y - v, fv \rangle \geq 0, \quad \forall y \in C, \tag{1.1}$$

where $f : E \to E^*$. The solution set of this variational inequality problem will be denoted by $\mathrm{VI}(f, C)$. This problem has numerous applications in many areas of mathematics, such as in partial differential equations, optimal control, optimization, mathematical programming, and some other nonlinear problems (see, for example, [1] and the references contained in them). The map f is called *K-Lipschitz* and *monotone* if

$$\|f(x) - f(y)\| \leq K\|x - y\|, \quad \forall x, y \in E,$$

and

$$\langle x - y, f(x) - f(y) \rangle \geq 0, \quad \forall x, y \in E,$$

respectively, where $K > 0$ is a Lipschitz constant, and is called *η-strongly monotone* if there exists $\eta > 0$ such that

$$\langle x-y, f(x)-f(y)\rangle \geq \eta\|x-y\|^2, \quad \forall x,y \in E.$$

In the case that E is a real Hilbert space H, some authors have proposed and analyzed several iterative methods for solving the variational inequality problem (1.1). The simplest of them is the following *projection method* given by

$$\begin{cases} x_1 \in H, \\ x_{k+1} = P_C(x^k - \tau f(x^k)), \quad \forall k \geq 1, \end{cases} \tag{1.2}$$

where f is Lipschitz and η-strongly monotone with $\tau \in (0, \frac{2\eta}{K^2})$. Yao *et al.* [18] showed that the projection gradient method (1.2) may not converge if the strong monotonicity assumption is relaxed to plain monotonicity. To overcome this difficulty, Korpelevich [14] proposed the following *extragradient method*:

$$\begin{cases} x_1 \in H, \\ y^k = P_C(x^k - \tau f(x^k)), \\ x^{k+1} = P_C(x^k - \tau f(y^k)), \end{cases} \tag{1.3}$$

for each $k \geq 1$, which converges if f is monotone and Lipschitz. However, the weakness of this extragradient method is that one needs to calculate two projections onto C in each iteration process. It is known that if C is a general closed and convex set, this iteration process might require a huge amount of computation time. To overcome this difficulty, Censor *et al.* [6] introduced the *subgradient extragradient method* given by

$$\begin{cases} x_0 \in H, \\ y^k = P_C(x^k - \tau f(x^k)), \\ T_k = \{w \in H : \langle x^k - \tau f(x^k) - y_k, w - y^k\rangle \leq 0\}, \\ x^{k+1} = P_{T_k}(x^k - \tau f(y^k)), \quad \forall k \geq 0, \end{cases} \tag{1.4}$$

replacing one of the projections onto C of the extragradient method by a projection onto a specific constructible subgradient half-space T_k. This projection method has an advantage in computing over the extragradient method proposed by Korpelevich [14] (see, e.g., Censor *et al.* [5], Dong *et al.* [9] and the references contained in them). They proved the following theorem in a real Hilbert space.

Theorem 1.1 (Censor *et al.*, [6]) *Assume that f is monotone, Lipschitz and* $\mathrm{VI}(f,C) \neq \emptyset$, *with $\tau < \frac{1}{K}$. Then any sequences $\{x^k\}_{k=0}^{\infty}$ and $\{y^k\}_{k=0}^{\infty}$ generated by* (1.4) *weakly converge to the same solution $u^* \in \mathrm{VI}(f,C)$ and, furthermore, $u^* = \lim_{k\to\infty} P_{\mathrm{VI}(f,C)}x^k$.*

In addition, they introduced a *modified subgradient extragradient method* as follows:

$$\begin{cases} x_0 \in H, \\ y^k = P_C(x^k - \tau f(x^k)), \\ T_k = \{w \in H : \langle x^k - \tau f(x^k) - y_k, w - y^k \rangle \leq 0\}, \\ x^{k+1} = \alpha_k x^k + (1-\alpha_k) S P_{T_k}(x^k - \tau f(y^k)), \quad \forall k \geq 0, \end{cases} \tag{1.5}$$

and proved the following theorem in a real Hilbert space.

Theorem 1.2 (Censor *et al.*, [6]) *Assume that f is monotone, Lipschitz and* $\mathrm{VI}(f, C) \cap \mathrm{Fix}(S) \neq \emptyset$, *with $\tau < \frac{1}{K}$. Then any sequences $\{x^k\}$ and $\{y^k\}$ generated by* (1.5) *weakly converge to the same solution $u^* \in \mathrm{VI}(f, C) \cap \mathrm{Fix}(S)$ and, furthermore, $u^* = \lim_{k\to\infty} P_{\mathrm{VI}(f,C)\cap \mathrm{Fix}(S)} x^k$.*

Developing algorithms for solving variational inequality problems has continued to attract the interest of numerous researchers in nonlinear operator theory. The reader may see the following important related papers (Gang *et al.* [11], Anh and Hieu [3], Anh and Hieu [4], Dong *et al.* [10] and the references contained in them).
Motivated by the result of Censor *et al.* [6], we propose in this paper a *Krasnoselskii-type subgradient extragradient algorithm* and prove a *weak convergence theorem* for obtaining a common element of solutions of variational inequality problems and common fixed points for a countable family of relatively-nonexpansive maps in a uniformly smooth and 2-uniformly convex real Banach space. Our theorem is an improvement of the result of Censor *et al.* [6], and a host of other results (see Sect. 5 below).

2 Methods

The paper is organized as follows. Section 3 contains the preliminaries to include definitions and lemmas with corresponding references that will be used in the sequel. Section 4 contains the main result of the paper. In Sect. 5, we compare our theorems with important recent results in the literature and, thereafter, conclude our findings.

3 Preliminaries

Let E be a real normed space with dual space E^*. We shall denote $x_k \rightharpoonup x^*$ and $x_k \to x^*$ to indicate that the sequence $\{x_k\}$ converges *weakly* to x^* and converges *strongly* to x^*, respectively.
A map $J : E \to 2^{E^*}$ defined by $J(x) := \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$ is called the *normalized duality map* on E. The following properties of the duality map will be needed in the sequel (see, e.g., Chidume [7], Cioranescu [8] and the references contained in them):

(1) If E is a reflexive, strictly convex, and smooth real Banach space, then J is surjective, injective, and single-valued.
(2) If E is uniformly smooth, then J is uniformly continuous on a bounded subset of E.
(3) If $E = H$, a real Hilbert space, then J is the identity map on H.

Remark 1 J is weakly sequentially continuous if, for any sequence $\{x_k\} \subset E$ such that $x_k \rightharpoonup x^*$ as $k \to \infty$, then $Jx_k \rightharpoonup Jx^*$ as $k \to \infty$. It is known that the normalized duality map on l_p spaces, $1 < p < \infty$, is weakly sequentially continuous.

Let E be a smooth real Banach space and $\phi : E \times E \to \mathbb{R}$ be a map defined by $\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$ for all $x, y \in E$. This map was introduced by Alber [1] and has been extensively studied by a host of other authors. It is easy to see from the definition of ϕ that, if $E = H$, a real Hilbert space, then $\phi(x,y) = \|x - y\|^2$ for all $x, y \in H$. Furthermore, for any $x, y, z \in E$ and $\beta \in (0,1)$, we have the following properties.

(P_1) $(\|x\| - \|y\|)^2 \leq \phi(x,y) \leq (\|x\| + \|y\|)^2, \forall x, y \in E$.

(P_2) $\phi(x,z) = \phi(x,y) + \phi(y,z) + 2\langle y - x, Jz - Jy\rangle$.

(P_3) $\phi(x, J^{-1}(\beta Jy + (1-\beta)Jz) \leq \beta\phi(x,y) + (1-\beta)\phi(x,z)$.

Definition 3.1 Let C be a nonempty closed and convex subset of a real Banach space E and T be a map from C to E.

(a) x^* is called an *asymptotic fixed point of* T if there exists a sequence $\{x_k\} \subset C$ such that $x_k \rightharpoonup x^*$ and $\|Tx_k - x_k\| \to 0$, as $k \to \infty$. We shall denote the set of asymptotic fixed points of T by $\widehat{F}(T)$.

(b) T is called *relatively nonexpansive* if the fixed point set of T is denoted by $F(T) = \widehat{F}(T) \neq \emptyset$ and $\phi(p, Tx) \leq \phi(p,x)$ for all $x \in C, p \in F(T)$.

Definition 3.2 (Rockafellar, [16]) The normal cone of C at $v \in C$ denoted by $N_C(v)$ is given by $N_C(v) := \{w \in E^* : \langle y - v, w\rangle \leq 0, \forall y \in C\}$.

Definition 3.3 A map $T : E \to 2^{E^*}$ is called *monotone* if $\langle \eta_x - \eta_y, x - y\rangle \geq 0, \forall x, y \in E$ and $\eta_x \in Tx, \eta_y \in Tx$. Furthermore, T is *maximal monotone* if it is monotone and the graph $G(T) := \{(x,y) \in E \times E^* : y \in T(x)\}$ is not properly contained in the graph of any other monotone operator.

Definition 3.4 A convex feasibility problem is a problem of finding a point in the intersection of convex sets.

Lemma 3.5 (Rockafellar, [16]) *Let C be a nonempty closed and convex subset of a reflexive Banach space E. Let $f : C \to E^*$ be a monotone and hemicontinuous map and $T \subset E \times E^*$ be a map defined by*

$$Tv = \begin{cases} f(v) + N_C(v) & \text{if } v \in C, \\ \emptyset & \text{if } v \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in \mathrm{VI}(f, C)$.

Remark 2 It is known that a monotone map T is maximal if given $(x,y) \in E \times E^*$ and if $\langle x - u, y - v\rangle \geq 0, \forall (u,v) \in G(T)$, then $y \in Tx$.

Lemma 3.6 (Matsushita and Takahashi, [15]) *Let E be a smooth, strictly convex, and reflexive Banach space and C be a nonempty closed convex subset of E. Then the following hold*:

(1) $\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \leq \phi(x,y), \forall x \in C, y \in E$.

(2) $z = \Pi_C x \iff \langle z - y, Jx - Jz\rangle \geq 0, \forall y \in C$.

Lemma 3.7 (Kamimura and Takahashi, [12]) *Let E be a uniformly convex and uniformly smooth real Banach space and $\{x_n\}_{n=1}^\infty, \{y_n\}_{n=1}^\infty$ be sequences in E such that either $\{x_n\}_{n=1}^\infty$ or $\{y_n\}_{n=1}^\infty$ is bounded. If $\lim_{n\to\infty} \phi(x_n, y_n) = 0$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 3.8 (Xu, [17]) *Let E be a uniformly convex real Banach space. Let $r > 0$. Then there exists a strictly increasing continuous and convex function $g : [0,\infty) \to [0,\infty)$ such that $g(0) = 0$ and the following inequality holds*:

$$\left\|\lambda x + (1-\lambda)y\right\|^2 \le \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)g\big(\|x-y\|\big), \quad \textit{for all } x, y \in B_r(0),$$

where $B_r(0) := \{v \in E : \|v\| \le r\}$ and $\lambda \in [0,1]$.

Lemma 3.9 (Xu, [17]) *Let E be a 2-uniformly convex real Banach space. Then there exists a constant $c_2 > 0$ such that, for every $x, y \in E$,*

$$c_2\|x-y\|^2 \le \langle x-y, jx - jy\rangle \ge 0, \quad \forall jx \in Jx, jy \in Jy.$$

Lemma 3.10 (Xu, [17]) *Let E be a 2-uniformly convex and smooth real Banach space. Then, for any $x, y \in E$ and for some $\alpha > 0$,*

$$\alpha\|x-y\|^2 \le \phi(x,y).$$

Without loss of generality, we may assume $\alpha \in (0,1)$.

Lemma 3.11 (Kohsaka and Takahashi, [13]) *Let C be a closed convex subset of a uniformly convex and uniformly smooth Banach space E. Let $T_i : C \to E, i = 1, 2, \dots,$ be a countable sequence of relatively nonexpansive maps such that $\bigcap_{i=1}^\infty F(T_i) \neq \emptyset$. Suppose that $\{\alpha_i\} \subset (0,1)$ and $\{\beta_i\}_{i=1}^\infty \subset (0,1)$ are sequences such that $\sum_{i=1}^\infty \alpha_i = 1$ and $U : C \to E$ is defined by*

$$Ux := J^{-1}\left(\sum_{i=1}^\infty \alpha_i\big(\beta_i Jx + (1-\beta_i)JT_i x\big)\right) \quad \textit{for each } x \in C,$$

then U is relatively nonexpansive and $F(U) = \bigcap_{i=1}^\infty F(T_i)$.

4 Main result

In the sequel, $\alpha \in (0,1)$ is the constant appearing in Lemma 3.10.

4.1 The Krasnoselskii-type subgradient extragradient algorithm

Let E be a uniformly smooth and 2-uniformly convex real Banach space with dual space E^*. Let C be a nonempty closed and convex subset of E. Let J be the normalized duality maps on E.

Algorithm 1 Let $\{v_k\}$ be a sequence generated iteratively by

$$\begin{cases} v_1 \in E \quad \text{and} \quad \tau > 0, \\ y_k = \Pi_C J^{-1}(Jv_k - \tau f(v_k)), \\ T_k = \{w \in E : \langle w - y_k, (Jv_k - \tau f(v_k)) - Jy_k\rangle \le 0\}, \\ v_{k+1} = \Pi_{T_k} J^{-1}(Jv_k - \tau f(y_k)), \quad \forall k \ge 1. \end{cases} \tag{4.1}$$

If $v_k = y_k$, we stop. Otherwise, replace k by $(k+1)$ and return to algorithm.

We shall make the following assumptions.

C_1 The map f is monotone on E.

C_2 The map f is Lipschitz on E, with constant $K > 0$.

C_3 $\mathrm{VI}(f, C) \neq \emptyset$.

Lemma 4.1 *If $v_k = y_k$ in Algorithm* 1, *then $v_k \in \mathrm{VI}(f, C)$.*

Proof If $v_k = y_k$, then $v_k = \Pi_C J^{-1}(Jv_k - \tau f(v_k)) \in C$. Furthermore, by the characterization of the generalized projection onto C, we obtain that

$$\begin{aligned} &\langle w - v_k, Jv_k - \tau f(v_k) - Jv_k\rangle \leq 0, \quad \forall w \in C \\ &\iff \quad \tau\langle w - v_k, f(v_k)\rangle \geq 0, \quad \forall w \in C, \tau > 0. \end{aligned} \tag{4.2}$$

Hence, $v_k \in \mathrm{VI}(f, C)$. □

The following lemma is crucial for the proof of our main theorem.

Lemma 4.2 *Let $\{v_k\}_{k=1}^{\infty}$ be the sequence defined in Algorithm* 1. *Assume conditions C_1, C_2, and C_3 hold with $\tau \in (0, \frac{\alpha}{K})$. Then, for any $v \in \mathrm{VI}(f, C)$, the following inequality holds*:

$$\phi(v, v_{k+1}) \leq \phi(v, v_k) - \left(1 - \frac{\tau K}{\alpha}\right)\phi(y_k, v_k) - \left(1 - \frac{\tau K}{\alpha}\right)\phi(v_{k+1}, y_k), \quad \forall k \geq 1.$$

Proof Let $v \in \mathrm{VI}(f, C)$. Then we have that

$$\begin{aligned} &\langle y_k - v, f(y_k) - f(v)\rangle \geq 0, \quad \forall k \geq 1 \\ &\implies \quad \langle v - v_{k+1}, f(y_k)\rangle \leq \langle y_k - v_{k+1}, f(y_k)\rangle. \end{aligned} \tag{4.3}$$

Since $v_{k+1} \in T_k$, we have that $\langle v_{k+1} - y_k, Jv_k - \tau f(v_k) - Jy_k\rangle \leq 0, \forall k \geq 1$. From the above inequality, we obtain that

$$\begin{aligned} &\langle v_{k+1} - y_k, Jv_k - \tau f(y_k) - Jy_k\rangle \\ &\quad = \langle v_{k+1} - y_k, Jv_k - \tau f(v_k) - Jy_k\rangle + \tau\langle v_{k+1} - y_k, f(v_k) - f(y_k)\rangle \\ &\quad \leq \tau\langle v_{k+1} - y_k, f(v_k) - f(y_k)\rangle. \end{aligned} \tag{4.4}$$

Set $Jz_k = Jv_k - \tau f(y_k)$. Then we compute as follows:

$$\begin{aligned} \phi(v, v_{k+1}) &\leq \phi(v, z_k) - \phi(v_{k+1}, z_k) \\ &= \|v\|^2 - 2\langle v, Jv_k - \tau f(y_k)\rangle - \|v_{k+1}\|^2 + 2\langle v_{k+1}, Jv_k - \tau f(y_k)\rangle \\ &= \phi(v, v_k) - \|v_k\|^2 + 2\langle v, \tau f(y_k)\rangle - \|v_{k+1}\|^2 + 2\langle v_{k+1}, Jv_k - \tau f(y_k)\rangle \\ &= \phi(v, v_k) - \phi(v_{k+1}, v_k) + 2\tau\langle v - v_{k+1}, f(y_k)\rangle. \end{aligned}$$

From inequality (4.3) and property P_2, it follows that

$$\phi(v, v_{k+1}) \leq \phi(v, v_k) - \phi(v_{k+1}, v_k) + 2\tau\langle y_k - v_{k+1}, f(y_k)\rangle$$

$$= \phi(v, v_k) - \phi(y_k, v_k) - \phi(v_{k+1}, y_k) + 2\langle v_{k+1} - y_k, Jv_k - \tau f(y_k) - Jy_k\rangle.$$

From inequality (4.4), it follows that

$$\phi(v, v_{k+1}) \leq \phi(v, v_k) - \phi(y_k, v_k) - \phi(v_{k+1}, y_k) + 2\tau\langle v_{k+1} - y_k, f(v_k) - f(y_k)\rangle.$$

By condition C_3 and Lemma 3.10 in the above inequality, it follows that

$$\begin{aligned}\phi(v, v_{k+1}) &\leq \phi(v, v_k) - \phi(y_k, v_k) - \phi(v_{k+1}, y_k) + 2\tau K\|v_{k+1} - y_k\|\|v_k - y_k\| \\ &\leq \phi(v, v_k) - \left(1 - \frac{\tau K}{\alpha}\right)\phi(y_k, v_k) - \left(1 - \frac{\tau K}{\alpha}\right)\phi(v_{k+1}, y_k).\end{aligned}$$

This completes the proof. □

Theorem 4.3 *Let E be a uniformly smooth and 2-uniformly convex real Banach space with dual space E^*. Let C be a nonempty closed and convex subset of E and $f : E \to E^*$ be a map satisfying conditions C_1 and C_2 with $\tau \in (0, \frac{\alpha}{K})$. Assume that condition C_3 holds and J is weakly sequentially continuous on E. Then the sequence $\{v_k\}_{k=1}^{\infty}$ generated iteratively by Algorithm 1 converges weakly to some $v^* \in \mathrm{VI}(f, C)$.*

Proof Since $\mathrm{VI}(f, C) \neq \emptyset$, let $v \in \mathrm{VI}(f, C)$. Define $\gamma := 1 - \frac{\tau K}{\alpha}$, then $\gamma \in (0, 1)$. By Lemma 4.2, we have that $\lim_{k\to\infty} \phi(v, v_k)$ exists, $\{\phi(y_k, v_k)\}$ is bounded and

$$\phi(y_k, v_k) \leq \frac{1}{\gamma}\big(\phi(v, v_k) - \phi(v, v_{k+1})\big), \quad \forall k \geq 1.$$

Taking limit of both sides of the above inequality, we have that

$$\lim_{k\to\infty} \phi(y_k, v_k) = 0. \tag{4.5}$$

By Lemma (3.7), $\lim_{n\to\infty} \|y_k - v_k\| = 0$.

Next, we show that $\Omega_\omega(v_k) \subset \mathrm{VI}(f, C)$, where $\Omega_\omega(v_k)$ is the set of weak sub-sequential limit of $\{v_k\}$. Let $x^* \in \Omega_\omega(v_k)$ and $\{v_{k_j}\}_{j=1}^{\infty}$ be a subsequence of $\{v_k\}_{k=1}^{\infty}$ such that

$$v_{k_j} \rightharpoonup x^* \text{ as } j \to \infty. \text{ Consequently, } y_{k_j} \rightharpoonup x^* \text{ as } j \to \infty. \tag{4.6}$$

Let $T : E \to E^*$ be a map defined by

$$Tv = \begin{cases} fv + N_C(v) & \text{if } v \in C, \\ \emptyset & \text{if } v \notin C, \end{cases} \tag{4.7}$$

where $N_C(v)$ is the normal cone to C at $v \in C$. Then T is maximal monotone and $T^{-1}(0) = \mathrm{VI}(f, C)$ (Rockafellar [16]). Let $(v, w) \in G(T)$, where $G(T)$ is the graph of T. Then $w \in Tv = fv + N_C(v)$. Hence, we get that $w - fv \in N_C(v)$. This implies that $\langle v - t, w - fv\rangle \geq 0, \forall t \in C$. In particular,

$$\langle v - y_k, w - f(v)\rangle \geq 0. \tag{4.8}$$

Furthermore, $y_k = \Pi_C J^{-1}(Jv_k - \tau f(v_k)), \forall k \geq 1$. By characterization of the generalized projection map, we obtain that

$$\langle y_k - v, Jv_k - \tau f(v_k) - Jy_k \rangle \geq 0, \quad \forall v \in C. \tag{4.9}$$

This implies that

$$\left\langle v - y_k, \frac{Jy_k - Jv_k}{\tau} + f(v_k) \right\rangle \geq 0, \quad \forall v \in C. \tag{4.10}$$

Using inequalities (4.8) and (4.10) for some $M_0 > 0$, Cauchy–Schwarz inequality, and condition C_2, we have that

$$\begin{aligned}
&\langle v - y_{k_j}, w \rangle \\
&\quad \geq \langle v - y_{k_j}, f(v) \rangle \\
&\quad \geq \langle v - y_{k_j}, f(v) \rangle - \left\langle v - y_{k_j}, \frac{Jy_{k_j} - Jv_{k_j}}{\tau} + f(v_{k_j}) \right\rangle \\
&\quad = \langle v - y_{k_j}, f(v) - f(y_{k_j}) \rangle + \langle v - y_{k_j}, f(y_{k_j}) - f(v_{k_j}) \rangle - \left\langle v - y_{k_j}, \frac{Jy_{k_j} - Jv_{k_j}}{\tau} \right\rangle \\
&\quad \geq -KM_0 \|y_{k_j} - v_{k_j}\| - M_0 \|Jy_{k_j} - Jv_{k_j}\|.
\end{aligned} \tag{4.11}$$

Taking limit of both sides of inequality (4.11) and using the fact that J is uniformly continuous on bounded subset of E, we obtain that

$$\langle v - x^*, w \rangle \geq 0. \tag{4.12}$$

Since T is a maximal monotone operator, it follows that $x^* \in T^{-1}(0) = \mathrm{VI}(f, C)$, which implies that $\Omega_\omega(v_k) \subset \mathrm{VI}(f, C)$.

Now, we show that $v_k \rightharpoonup x^*$ as $k \to \infty$. Define $x_k := \Pi_{\mathrm{VI}(f,C)} v_k$. Then $\{x_k\} \subset \mathrm{VI}(f, C)$. Furthermore, by Lemmas 4.2 and 3.6, we have that

$$\phi(x_k, v_{k+1}) \leq \phi(x_k, v_k) \quad \text{and} \quad \phi(x_{k+1}, v_{k+1}) \leq \phi(x_k, v_{k+1}) - \phi(x_k, x_{k+1}), \tag{4.13}$$

which implies that $\{\phi(x_k, v_k)\}$ converges. From inequality (4.13) and for any $m > k$, we have that

$$\phi(x_k, v_m) \leq \phi(x_k, v_k) \quad \text{and} \quad \phi(x_k, x_m) \leq \phi(x_k, v_m) - \phi(x_m, v_m). \tag{4.14}$$

Furthermore, $\lim_{k \to \infty} \phi(x_k, x_m) = 0$. Hence, by Lemma 3.7, we obtain that $\lim_{k,m \to \infty} \|x_k - x_m\| = 0$, which implies that $\{x_k\}$ is a Cauchy sequence in $\mathrm{VI}(f, C)$. Therefore, there exists $u^* \in \mathrm{VI}(f, C)$ such that $\lim_{k \to \infty} x_k = u^*$.

Now, using the definition of $x_k = \Pi_{\mathrm{VI}(f,C)} v_k, \forall k \geq 0$, it follows from Lemma 3.6 that for any $p \in \mathrm{VI}(f, C)$, we have that

$$\langle x_k - p, Jx_k - Jv_k \rangle \geq 0. \tag{4.15}$$

Let $\{v_{k_i}\}$ be any subsequence of $\{v_k\}$. We may assume without loss of generality that $\{v_{k_i}\}$ converges weakly to some $p^* \in \mathrm{VI}(f, C)$. By inequality (4.15), weak sequential continuity of J, and the fact that $\lim_{k\to\infty} x_k = u^*$, we obtain that

$$\langle u^* - p^*, Jp^* - Ju^*\rangle \geq 0. \tag{4.16}$$

However, from the monotonicity of J, we obtain that

$$\langle u^* - p^*, Ju^* - Jp^*\rangle \geq 0. \tag{4.17}$$

Combining inequalities (4.16) and (4.17), we have that

$$\langle u^* - p^*, Ju^* - Jp^*\rangle = 0. \tag{4.18}$$

By Lemma 3.9, we obtain that

$$\|u^* - p^*\|^2 \leq \frac{1}{c_2}\langle u^* - p^*, Ju^* - Jp^*\rangle = 0,$$

which implies that $u^* = p^*$. Hence, $v_k \rightharpoonup u^* = \lim_{k\to\infty} x_k$. This completes the proof. □

4.2 The modified Krasnoselskii-type subgradient extragradient algorithm

Algorithm 2 Let $\{v_k\}_{k=1}^{\infty}$ be a sequence generated iteratively by

$$\begin{cases} v_1 \in E \quad \text{and} \quad \tau > 0, \\ y_k = \Pi_C J^{-1}(Jv_k - \tau f(v_k)), \\ T_k = \{w \in E : \langle w - y_k, (Jv_k - \tau f(v_k)) - Jy_k\rangle \leq 0\}, \\ v_{k+1} = J^{-1}(\beta Jv_k + (1-\beta)JS\Pi_{T_k}J^{-1}(Jv_k - \tau f(y_k))), \quad \forall k \geq 1. \end{cases} \tag{4.19}$$

We shall make the following assumption.

C_4 $\mathcal{G} := \mathrm{VI}(f, C) \cap F(S) \neq \emptyset$, $F(S)$ is the set of fixed points of S.

The following lemma is crucial for the proof of the next theorem.

Lemma 4.4 *Let E be a uniformly smooth and 2-uniformly convex real Banach space with dual space E^*. Let C be a nonempty closed and convex subset of E. Let $S : E \to E$ be a relatively nonexpansive map and $f : E \to E^*$ be a map satisfying conditions C_1 and C_2 with $\tau \in (0, \frac{\alpha}{K})$, and let $\beta \in (0, 1)$. Assume that condition C_4 holds and J is weakly sequentially continuous on E. Then the sequence $\{v_k\}_{k=1}^{\infty}$ generated iteratively by Algorithm 2 converges weakly to some $v^* \in \mathcal{G}$.*

Proof

Denote $t_k = \Pi_{T_k}J^{-1}(Jv_k - \tau f(y_k)), \forall k \geq 1$, $Jz_k := Jv_k - \tau f(y_k)$, and $\gamma = 1 - \frac{\tau K}{\alpha}$.

Since $\mathcal{G} \neq \emptyset$, let $u \in \mathcal{G}$. Then we have that

$$\phi(u, t_k) \leq \phi(u, z_k) - \phi(t_k, z_k)$$

$$\begin{aligned}
&= \|u\|^2 - 2\langle u, Jv_k - \tau f(y_k)\rangle - \|t_k\|^2 + 2\langle t_k, Jv_k - \tau f(y_k)\rangle \\
&= \phi(u, v_k) - \phi(t_k, v_k) + 2\tau\langle u - t_k, f(y_k)\rangle \\
&= \phi(u, v_k) - \phi(t_k, v_k) + 2\tau\langle u - y_k, f(y_k) - f(u)\rangle + 2\tau\langle y_k - t_k, f(y_k)\rangle \\
&\quad + 2\tau\langle u - y_k, f(u)\rangle.
\end{aligned}$$

By C_1, $\langle u - y_k, f(y_k) - f(u)\rangle \le 0, \forall k \ge 1$. Consequently, $\langle u - y_k, f(u)\rangle \le 0, \forall k \ge 1$. Thus, from the last line of the above inequality and by inequality (4.4), we obtain that

$$\begin{aligned}
\phi(u, t_k) &\le \phi(u, v_k) - \phi(t_k, v_k) + 2\tau\langle y_k - t_k, f(y_k)\rangle \\
&= \phi(u, v_k) - \phi(y_k, v_k) - \phi(t_k, y_k) + 2\langle t_k - y_k, Jv_k - \tau f(y_k) - Jy_k\rangle \\
&\le \phi(u, v_k) - \phi(y_k, v_k) - \phi(t_k, y_k) + 2\tau\langle t_k - y_k, f(v_k) - f(y_k)\rangle. \qquad (4.20)
\end{aligned}$$

By condition C_2 and Lemma 3.10, we have that

$$\begin{aligned}
\phi(u, t_k) &\le \phi(u, v_k) - \phi(y_k, v_k) - \phi(t_k, y_k) + \frac{\tau K}{\alpha}\big(\phi(t_k, y_k) + \phi(y_k, v_k)\big) \\
&= \phi(u, v_k) - \gamma\phi(t_k, y_k) - \gamma\phi(y_k, v_k) \le \phi(u, v_k). \qquad (4.21)
\end{aligned}$$

Applying Lemma 3.8, inequality (4.21), and relative nonexpansivity of S, we obtain that

$$\begin{aligned}
\phi(u, v_{k+1}) &= \phi\big(u, J^{-1}\big(\beta Jv_k + (1-\beta)J(St_k)\big)\big) \\
&\le \beta\phi(u, v_k) + (1-\beta)\phi(u, t_k)\big) - \beta(1-\beta)g\big(\|Jv_k - J(St_k)\|\big) \qquad (4.22) \\
&\le \beta\phi(u, v_k) + (1-\beta)\big(\phi(u, v_k) - \gamma\phi(t_k, y_k) - \gamma\phi(y_k, v_k)\big) \le \phi(u, v_k). \qquad (4.23)
\end{aligned}$$

This implies that $\lim_{k\to\infty}\phi(u, v_k)$ exists. Consequently, $\{v_k\}_{k=1}^{\infty}$ is bounded. From inequality (4.21), $\{t_k\}_{k=1}^{\infty}$ is bounded. Also, from inequality (4.22), we obtain that

$$\begin{aligned}
&\phi(y_k, v_k) \le \frac{1}{\gamma(1-\beta)}\big(\phi(u, v_k) - \phi(u, v_{k+1})\big) \quad \text{and} \\
&\phi(t_k, y_k) \le \frac{1}{\gamma(1-\beta)}\big(\phi(u, v_k) - \phi(u, v_{k+1})\big).
\end{aligned}$$

From these inequalities, we obtain that

$$\lim_{k\to\infty}\phi(y_k, v_k) = 0 \quad \text{and} \quad \lim_{k\to\infty}\phi(t_k, y_k) = 0. \qquad (4.24)$$

By Lemma 3.7, it follows that $\lim\|y_k - v_k\| = 0$ and $\lim\|t_k - y_k\| = 0$. Consequently, we obtain $\lim_{k\to\infty}\|v_k - t_k\| = 0$.

Next, we show that $\Omega_\omega(v_k) \subset \mathcal{G} = F(S) \cap \mathrm{VI}(f, C)$, where $\Omega_\omega(v_k)$ is the set of weak subsequential limit of $\{v_k\}$. Let $x^* \in \Omega_\omega(v_k)$ and $\{v_{k_j}\}_{j=1}^{\infty}$ be a subsequence of $\{v_k\}_{k=1}^{\infty}$ such that

$$v_{k_j} \rightharpoonup x^* \text{ as } j \to \infty. \text{ Consequently, } t_{k_j} \rightharpoonup x^* \text{ as } j \to \infty.$$

By definition of S, $\{St_k\}_{k=1}^{\infty}$ is bounded. From inequalities (4.22) and (4.23), we have that

$$g\big(\|Jv_k - J(St_k)\|\big) \leq \frac{1}{\beta(1-\beta)}\big(\phi(u,v_k) - \phi(u,v_{k+1})\big). \tag{4.25}$$

Applying the property of g, we obtain that

$$\lim_{k\to\infty} \|Jv_k - J(St_k)\| = 0.$$

By the uniform continuity of J^{-1} on a bounded subset of E^*, we get that

$$\lim_{k\to\infty} \|v_k - St_k\| = 0, \tag{4.26}$$

so that

$$\|St_k - t_k\| \leq \|St_k - v_k\| + \|v_k - t_k\| \to 0 \quad \text{as } k \to \infty, \tag{4.27}$$

which implies that $Sx^* = x^*$. Hence, $x^* \in F(S)$.

Next, we show that $x^* \in \mathrm{VI}(f,C)$. Following the same line of argument as in the proof of Theorem 4.3, we have that $x^* \in \mathrm{VI}(f,C)$, and this implies that $\Omega_\omega(v_k) \subset \mathcal{G}$.

Define $x_k := \Pi_{\mathcal{G}} v_k$. Then $\{x_k\} \subset \mathcal{G}$. Now, following the same line of argument as in the proof of Theorem 4.3, we obtain that $u^* = p^*$. Hence, $v_k \rightharpoonup u^* = \lim_{k\to\infty} x_k$.This proof is complete. □

4.3 A convergence theorem for a convex feasibility problem

In what follows, we shall make the following assumption.

C_5 $\mathcal{V} := \bigcap_{i=1}^{\infty} F(T_i) \cap \mathrm{VI}(f,C) \neq \emptyset$, where $F(T_i) := \{x \in E : T_i x = x, \forall i \geq 1\}$.

We now prove the following theorem.

Theorem 4.5 *Let E be a uniformly smooth and 2-uniformly convex real Banach space with dual space E^*. Let C be a nonempty closed and convex subset of E. Let $T_i : E \to E$, $i = 1,2,\dots$, be a countable family of relatively nonexpansive maps and $f : E \to E^*$ be a map satisfying conditions C_1 and C_2 with $\tau \in (0, \frac{\alpha}{K})$, and let $\beta \in (0,1)$. Assume that condition C_5 holds and J is weakly sequentially continuous on E.Then the sequence $\{v_k\}_{k=1}^{\infty}$ generated iteratively by Algorithm 2 converges weakly to some $v^* \in \mathcal{V}$, where*

$$Sx = J^{-1}\left(\sum_{i=1}^{\infty} \delta_i\big(\gamma_i Jx + (1-\gamma_i)JT_i x\big)\right), \qquad \sum_{i=1}^{\infty} \delta_i = 1 \quad \textit{and} \quad \{\gamma_i\}_{i=1}^{\infty} \subset (0,1).$$

Proof By Lemma 3.11, S is relatively nonexpansive and $F(S) = \bigcap_{i=1}^{\infty} F(T_i)$. Also, by Lemma 4.4, the result of Theorem 4.5 follows. □

Corollary 4.6 *Let H be a real Hilbert space, and let C be a nonempty closed and convex subset of H. Let $T_i : H \to H, i = 1,2,\dots$, be a countable family of nonexpansive maps and $f : H \to H$ be a monotone and K-Lipschitz map. Let the sequence $\{v_k\}_{k=1}^{\infty}$ be generated* *iteratively by*

$$\begin{cases} v_1 \in E \quad and \quad \tau > 0, \\ y_k = P_C(v_k - \tau f(v_k)), \\ T_k = \{w \in E : \langle w - y_k, (v_k - \tau f(v_k)) - y_k \rangle \leq 0\}, \\ v_{k+1} = (\beta v_k + (1-\beta) S P_{T_k}(v_k - \tau f(y_k)), \quad \forall k \geq 1. \end{cases} \tag{4.28}$$

Assume that C_1, C_2, *and* C_5 *hold with* $\tau \in (0, \frac{1}{K})$, *and let* $\beta \in (0,1)$. *Then* $\{v_k\}_{k=1}^{\infty}$ *converges weakly to* $v^* \in \mathcal{V} := \bigcap_{i=1}^{\infty} F(T_i) \cap \mathrm{VI}(f, C)$, *where* $Sx = (\sum_{i=1}^{\infty} \delta_i(\gamma_i x + (1-\gamma_i)T_i x))$, $\sum_{i=1}^{\infty} \delta_i = 1$ *and* $\{\gamma_i\}_{i=1}^{\infty} \subset (0,1)$.

Proof In a Hilbert space, J is the identity map and $\phi(y,z) = \|y-z\|^2, \forall y,z \in H$. Thus, the conclusion follows from Theorem 4.5. □

Annotations. The result of Corollary 4.6 is an immediate consequence of Theorem 4.5.

5 Discussion

All the theorems of this paper are applicable in l_p spaces, $1 < p \leq 2$, since these spaces are uniformly smooth and 2-uniformly convex, and on these spaces, the normalized duality map is weakly sequentially continuous. The analytical representations of the duality map in these spaces, where $p^{-1} + q^{-1} = 1$ (see, e.g., Theorem 4.3, Alber and Ryazantseva [2]; p. 36) are:

$$Jx = \|x\|_{l_p}^{2-p} y \in l_p, \quad y = \left\{|x_1|^{p-2}x_1, |x_2|^{p-2}x_2, \ldots\right\}, x = \{x_1, x_2, \ldots\},$$
$$J^{-1}x = \|x\|_{l_q}^{2-q} y \in l_q, \quad y = \left\{|x_1|^{q-2}x_1, |x_2|^{q-2}x_2, \ldots\right\}, x = \{x_1, x_2, \ldots\}.$$

- Theorem 4.3, which approximates a solution of a variational inequality problem, extends Theorem 5.1 of Censor *et al.* [6] from a Hilbert space to the more general uniformly smooth and 2-uniformly convex real Banach space with weakly sequentially continuous duality map.
- Theorem 4.5, which approximates a common solution of a variational inequality problem and a common fixed point of a countable family of relatively nonexpansive maps, extends Theorem 7.1 of Censor *et al.* [6] from a Hilbert space to a uniformly smooth and 2-uniformly convex real Banach space with weakly sequentially continuous duality map, and from a single *nonexpansive map* to a countable family of *relatively nonexpansive maps.*
- The control parameters in Algorithm 2 of Theorem 4.5 are two arbitrarily fixed constants $\beta \in (0,1)$ and $\tau \in (0,1)$ which are to be computed once and then used at each step of the iteration process, while the parameters in equation (1.5) studied by Censor *et al.* [6] are $\alpha_k \in (0,1)$ and $\tau \in (0,1)$, and α_k is to be computed at each step of the iteration process. Consequently, the sequence of Algorithm 2 is of *Krasnoselskii type* and the sequence defined by equation (1.5) is of *Mann type*. It is well known that a Krasnoselskii-type sequence converges as fast as a geometric progression, which is slightly better than the convergence rate obtained from any Mann-type sequence.

6 Conclusion

In this paper, we considered Krasnoselskii-type subgradient extragradient algorithms for approximating a common element of solutions of variational inequality problems and fixed points of a countable family of relatively nonexpansive maps in a uniformly smooth and 2-uniformly convex real Banach space. A weak convergence of the sequence generated by our algorithm is proved. Furthermore, results obtained are applied in l_p-spaces, $1 < p \leq 2$.

Acknowledgements
Not applicable.

Funding
This work is supported from ACBF Research Grant Funds to AUST.

Abbreviations
Not applicable.

Availability of data and materials
Data sharing is not applicable to this article.

Competing interests
The authors declare that they have no conflict of interest.

Authors' contributions
All the authors contributed evenly in the writing of this paper. They read and approved the final manuscript.

Publisher's Note

Received: 29 December 2017 Accepted: 18 May 2018 Published online: 18 June 2018

References

1. Alber, Y.: Metric and generalized projection operators in Banach spaces: properties and applications. In: Kartsatos, A.G. (ed.) Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, pp. 15–50. Dekker, New York (1996)
2. Alber, Y., Ryazantseva, I.: Nonlinear Ill Posed Problems of Monotone Type. Springer, London (2006)
3. Anh, P.K., Hieu, D.V.: Parallel and sequential hybrid methods for a finite family of asymptotically quasi-ϕ-nonexpansive maps in a uniformly smooth and uniformly convex real Banach space. J. Appl. Math. Comput. (2014). https://doi.org/10.1007/s12190-014-0801-6
4. Anh, P.K., Hieu, D.V.: Parallel hybrid methods for variational inequalities, equilibrium problems and common fixed point problems. Vietnam J. Math. (2014). https://doi.org/10.1007/s10013-015-0129-z
5. Censor, Y., Gibali, A., Reich, S.: Two extensions of Korpelevich's extragradient method for solving the variational inequality problem in Euclidean space. Technical report, (2010)
6. Censor, Y., Gibali, A., Reich, S.: The subgradient extragradient method for solving variational inequalities in Hilbert space. J. Optim. Theory Appl. **148**, 318–335 (2011)
7. Chidume, C.E.: Geometric Properties of Banach Spaces and Nonlinear Iterations. Lecture Notes in Mathematics, vol. 1965. Springer, London (2009)
8. Cioranescu, I.: Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems, vol. 62. Kluwer Academic, Norwell (1990)
9. Dong, Q.L., Cho, Y.J., Zhong, L.L., Rassias, T.M.: Inertial projection and contraction algorithms for variational inequalities. J. Glob. Optim. (2017). https://doi.org/10.1007/s10898-017-0506-0
10. Dong, Q.L., Hieu, D.V.: Modified subgradient extragradient method for variational inequality problems. Numer. Algor. https://doi.org/10.1007/s11075-017-0452-4
11. Gang, C., Gibali, A., Olaniyi, S.I., Shehu, Y.: A new double-projection method for solving variational inequalities Banach spaces. J. Optim. Theory Appl. (2018). https://doi.org/10.1007/s10957-018-1228-2
12. Kamimura, S., Takahashi, W.: Strong convergence of a proximal-type algorithm in a Banach space. SIAM J. Optim. **13**(3), 938–945 (2002)
13. Kohsaka, F., Takahashi, W.: The set of common fixed points of an infinite family of relatively nonexpansive mappings in Banach and function spaces. In: Proceedings of the International Symposium on Banach and Function Spaces II, pp. 361–373. Yokohama Publishers, Yokohama (2008)
14. Korpelevich, G.M.: The extragradient method for finding saddle points and other problems. Ekon. Mat. Metody **12**, 747–756 (1976)
15. Matsushita, S.Y., Takahashi, W.: A strong convergence theorem for relatively nonexpansive mappings in Banach space. J. Approx. Theory **134**, 257–912 (2005)
16. Rockafellar, R.T.: On the maximality of sums of nonlinear monotone operators. Trans. Am. Math. Soc. **149**, 75–88 (1970)

17. Xu, H.K.: Inequalities in Banach spaces with applications. Nonlinear Anal., Theory Methods Appl. **16**(12), 1127–1138 (1991)
18. Yao, Y., Marino, G., Muglia, L.: A modified Korpelevich's method convergent to the minimum-norm solution of a variational inequality. Optimization **63**, 559–569 (2014)