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The split common fixed point problem for infinite families of demicontractive mappings

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Abstract

In this paper, we propose a new algorithm for solving the split common fixed point problem for infinite families of demicontractive mappings. Strong convergence of the proposed method is established under suitable control conditions. We apply our main results to study the split common null point problem, the split variational inequality problem, and the split equilibrium problem in the framework of a real Hilbert space. A numerical example supporting our main result is also given.

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Keywords: Fixed point problem; Demicontractive mappings; Null point problem; Variational inequality problem; Equilibrium problem

1 Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let I denote the identity mapping. Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator with adjoint operator A^*.

The *split feasibility problem* (SFP), which was first introduced by Censor and Elfving [1], is to find

$$v^* \in C \quad \text{such that} \quad Av^* \in Q. \tag{1}$$

Let P_C and P_Q be the orthogonal projections onto the sets C and Q, respectively. Assume that (1) has a solution. It known that $v^* \in H_1$ solves (1) if and only if it solves the fixed point equation

$$v^* = P_C\bigl(I + \gamma A^*(P_Q - I)A\bigr)v^*,$$

where $\gamma > 0$ is any positive constant.

SFP has been used to model significant real-world inverse problems in sensor networks, radiation therapy treatment planning, antenna design, immaterial science, computerized tomography, etc. (see [2–4]).

The *split common fixed point problem* (SCFP) for mappings T and S, which was first introduced by Censor and Segal [5], is to find

$$v^* \in F(T) \quad \text{such that} \quad Av^* \in F(S), \tag{2}$$

where $T : H_1 \to H_1$ and $S : H_2 \to H_2$ are two mappings satisfying $F(T) = \{x \in H_1 : Tx = x\} \neq \emptyset$ and $F(S) = \{x \in H_2 : Sx = x\} \neq \emptyset$, respectively. Since each closed and convex subset may be considered as a fixed point set of a projection onto the subset, the SCFP is a generalization of the SFP. Recently, the SFP and SCFP have been studied by many authors; see, for example, [6–11].

In 2010, Moudafi [11] introduced the following algorithm for solving (2) for two demi-contractive mappings:

$$\begin{cases} x_1 \in H_1 \text{ choose arbitrarily,} \\ u_n = x_n + \gamma\alpha A^*(S-I)Ax_n, \\ x_{n+1} = (1-\beta_n)u_n + \beta_n Tu_n, \quad n \in \mathbb{N}. \end{cases} \tag{3}$$

He proved that $\{x_n\}$ converges weakly to some solution of SCFP.

The *multiple set split feasibility problem* (MSSFP), which was first introduced by Censor et al. [4], is to find

$$v^* \in \bigcap_{i=1}^{m} C_i \quad \text{such that} \quad Av^* \in \bigcap_{i=1}^{r} Q_i, \tag{4}$$

where $\{C_i\}_{i=1}^m$ and $\{Q_i\}_{i=1}^r$ are families of nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. We see that if $m = r = 1$, then problem (4) reduces to problem (1).

Recently, Eslamian [12] considered the problem of finding a point

$$v^* \in \bigcap_{i=1}^{m} F(U_i) \quad \text{such that} \quad A_1 v^* \in \bigcap_{i=1}^{m} F(S_i) \quad \text{and} \quad A_2 v^* \in \bigcap_{i=1}^{m} F(T_i), \tag{5}$$

where $A_1, A_2 : H_1 \to H_2$ are bounded linear operators, and $U_i : H_1 \to H_1$, $T_i : H_2 \to H_2$ and $S_i : H_2 \to H_2$, $i = 1, 2, \ldots, m$. He also presented a new algorithm to solve (5) for finite families of quasi-nonexpansive mappings:

$$\begin{cases} x_1 \in H_1 \text{ choose arbitrarily,} \\ u_n = x_n + \sum_{i=1}^m \frac{1}{m}\eta\beta A_1^*(S_i - I)A_1 x_n, \\ y_n = u_n + \sum_{i=1}^m \frac{1}{m}\eta'\beta' A_2^*(T_i - I)A_2 u_n, \\ z_n = \alpha_{n,0} y_n + \sum_{i=1}^m \alpha_{n,i} U_i y_n, \\ x_{n+1} = \theta_n \gamma f(x_n) + (I - \theta_n B)z_n, \quad n \in \mathbb{N}. \end{cases} \tag{6}$$

He proved that $\{x_n\}$ converges strongly to some solution of (5) under some control conditions.

Question. Can we modify algorithm (6) to a simple one for solving the problem of finding

$$v^* \in \bigcap_{i=1}^{\infty} F(U_i) \quad \text{such that} \quad A_1 v^* \in \bigcap_{i=1}^{\infty} F(S_i) \quad \text{and} \quad A_2 v^* \in \bigcap_{i=1}^{\infty} F(T_i), \tag{7}$$

where $A_1, A_2 : H_1 \to H_2$ are bounded linear operators, and $\{U_i : H_1 \to H_1 : i \in \mathbb{N}\}$, $\{T_i : H_2 \to H_2 : i \in \mathbb{N}\}$ and $\{S_i : H_2 \to H_2 : i \in \mathbb{N}\}$ are infinite families of k_3-, k_2-, and k_1-demicontractive mappings, respectively.

In this work, we introduce a new algorithm for solving problem (7) for infinite families of demicontractive mappings and prove its strong convergence to a solution of problem (7).

2 Preliminaries

Throughout this paper, we adopt the following notations.

(i) "$\to$" and "$\rightharpoonup$" denote the strong and weak convergence, respectively.

(ii) $\omega_\omega(x_n)$ denotes the set of the cluster points of $\{x_n\}$ in the weak topology, that is, $\exists\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup x$.

(iii) Γ is the solution set of problem (7), that is,

$$\Gamma = \left\{ v^* \in \bigcap_{i=1}^{\infty} F(U_i) : A_1 v^* \in \bigcap_{i=1}^{\infty} F(S_i) \text{ and } A_2 v^* \in \bigcap_{i=1}^{\infty} F(T_i) \right\}.$$

A mapping P_C is said to be a *metric projection* of H onto C if for every $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - z\|, \quad \forall z \in C.$$

It is known that P_C is a firmly nonexpansive mapping. Moreover, P_C is characterized by the following property: $\langle x - P_C x, y - P_C x\rangle \leq 0$ for all $x \in H$, $y \in C$. A bounded linear operator $B : H \to H$ is said to be *strongly positive* if there is a constant $\xi > 0$ such that

$$\langle Bx, x\rangle \geq \xi \|x\|^2 \quad \text{for all } x \in H.$$

Definition 2.1 The mapping $T : H \to H$ is said to be

(i) *L-Lipschitzian* if there exists $L > 0$ such that

$$\|Tu - Tv\| \leq L\|u - v\| \quad \text{for all } u, v \in H;$$

(ii) *α-contraction* if T is α-Lipschitzian with $\alpha \in [0, 1)$, that is,

$$\|Tu - Tv\| \leq \alpha\|u - v\| \quad \text{for all } u, v \in H;$$

(iii) *nonexpansive* if T is 1-Lipschitzian;

(iv) *quasi-nonexpansive* if $F(T) \neq \emptyset$ and

$$\|Tu - v\| \leq \|u - v\| \quad \text{for all } u \in H, v \in F(T);$$

(v) *firmly nonexpansive* if

$$\|Tu - Tv\|^2 \leq \|u - v\|^2 - \|(u - v) - (Tu - Tv)\|^2 \quad \text{for all } u, v \in H;$$

or equivalently, for all $u, v \in H$,

$$\|Tu - Tv\|^2 \leq \langle Tu - Tv, u - v\rangle;$$

(vi) λ-*inverse strongly monotone* if there exists $\lambda > 0$ such that

$$\langle u - v, Tu - Tv\rangle \geq \lambda \|Tu - Tv\|^2 \quad \text{for all } u, v \in H;$$

(vii) k-*demicontractive* if $F(T) \neq \emptyset$ and there exists $k \in [0,1)$ such that

$$\|Tu - v\|^2 \leq \|u - v\|^2 + k\|u - Tu\|^2 \quad \text{for all } u \in H, v \in F(T).$$

The following example is an infinite family of k-demicontractive mappings in $\mathbb{R}^2$.

Example 2.2 For $i \in \mathbb{N}$, let $U_i : \mathbb{R}^2 \to \mathbb{R}^2$ be defined for all $x_1, x_2 \in \mathbb{R}$ by

$$U_i(x_1, x_2) = \left(\frac{-2i}{i+1}x_1, x_2\right),$$

and $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^2$. Observe that $F(U_i) = 0 \times \mathbb{R}$ for all $i \in \mathbb{N}$, that is, if $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}$ and $p = (0, p_2) \in F(U_i)$, then

$$\begin{aligned}
\|U_i x - p\|^2 &= \left\|\left(\frac{-2i}{i+1}x_1, x_2\right) - (0, p_2)\right\|^2 \\
&= \left(\frac{-2i}{i+1}\right)^2 |x_1|^2 + |x_2 - p_2|^2 \\
&\leq 4|x_1|^2 + |x_2 - p_2|^2 \\
&= |x_1|^2 + \frac{3}{4}(1+1)^2|x_1|^2 + |x_2 - p_2|^2 \\
&\leq \|x - p\|^2 + \frac{3}{4}\left(1 + \frac{2i}{i+1}\right)^2 |x_1|^2 \\
&= \|x - p\|^2 + \frac{3}{4}\|U_i x - x\|^2.
\end{aligned}$$

So, U_i are $\frac{3}{4}$-demicontractive mappings for all $i \in \mathbb{N}$.

Definition 2.3 The mapping $T : H \to H$ is said to be *demiclosed at zero* if for any sequence $\{u_n\} \subset H$ with $u_n \rightharpoonup u$ and $Tu_n \to 0$, we have $Tu = 0$.

Lemma 2.4 ([13]) *Assume that B is a self-adjoint strongly positive bounded linear operator on a Hilbert space H with coefficient $\xi > 0$ and $0 < \mu \leq \|B\|^{-1}$. Then $\|I - \mu B\| \leq 1 - \xi\mu$.*

Lemma 2.5 ([14]) *Let H be a real Hilbert space. Then the following results hold*:

(i) $\|u + v\|^2 = \|u\|^2 + 2\langle u, v\rangle + \|v\|^2 \ \forall u, v \in H$;

(ii) $\|u + v\|^2 \leq \|u\|^2 + 2\langle v, u + v\rangle \ \forall u, v \in H$.

Lemma 2.6 ([15]) *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation*:

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n, \quad n \in \mathbb{N},$$

where

(i) $\{\gamma_n\} \subset (0,1)$, $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(ii) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \le 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.7 ([16]) *Let $\{\kappa_n\}$ be a sequence of real numbers that does not decrease at infinity, that is, there exists at a subsequence $\{\kappa_{n_i}\}$ of $\{\kappa_n\}$ that satisfies $\kappa_{n_i} < \kappa_{n_i+1}$ for all $i \in \mathbb{N}$. For every $n \ge n_o$, define the integer sequence $\{\tau(n)\}$ as follows*:

$$\tau(n) = \max\{l \in \mathbb{N} : l \le n, \kappa_l < \kappa_{l+1}\},$$

where $n_o \in \mathbb{N}$ is such that $\{l \le n_o : \kappa_l < \kappa_{l+1}\} \neq \emptyset$. Then:
(i) $\tau(n_o) \le \tau(n_o+1) \le \cdots$, *and* $\tau(n) \to \infty$;
(ii) *for all* $n \ge n_o$, $\max\{\kappa_n, \kappa_{\tau(n)}\} \le \kappa_{\tau(n)+1}$.

3 Results and discussion

In this section, we propose a new algorithm, which is a modification of (6) and prove its strong convergence under some suitable conditions. We start with the following important lemma.

Lemma 3.1 *For two real Hilbert spaces H_1 and H_2, let $A : H_1 \to H_2$ be a bounded linear operator with adjoint operator A^*. If $T : H_2 \to H_2$ is a k-demicontractive mapping, then*

$$\|x + \delta A^*(T-I)Ax - x^*\|^2 \le \|x - x^*\|^2 - \delta_n\big(1 - k - \delta\|A\|^2\big)\|(T-I)Ax\|^2$$

for all $x^* \in H_1$ such that $Ax^* \in F(T)$.

Proof Suppose that $T : H_2 \to H_2$ is a k-demicontractive mapping and let $x^* \in H_1$ be such that $Ax^* \in F(T)$. Then we have

$$\begin{aligned}\|x - x^* + \delta A^*(T-I)Ax\|^2 &\le \|x - x^*\|^2 + 2\delta\langle x - x^*, A^*(T-I)Ax\rangle \\ &\quad + \delta^2\|A\|^2\|(T-I)Ax\|^2. \end{aligned} \tag{8}$$

Since A is a bounded linear operator with adjoint operator A^* and T is a k-demicontractive mapping, by Lemma 2.5(ii) we deduce that

$$\begin{aligned}\langle x - x^*, A^*(T-I)Ax\rangle &= \langle Ax - Ax^*, (T-I)Ax\rangle \\ &= \langle TAx - Ax^*, TAx - Ax\rangle - \|(T-I)Ax\|^2 \\ &= \frac{1}{2}\big[\|TAx - Ax^*\|^2 + \|TAx - Ax\|^2 - \|Ax - Ax^*\|^2\big] \\ &\quad - \|(T-I)Ax\|^2 \\ &\le \frac{1}{2}\big[\|Ax - Ax^*\|^2 + k\|TAx - Ax\|^2 \\ &\quad + \|TAx - Ax\|^2 - \|Ax - Ax^*\|^2\big] - \|(T-I)Ax\|^2 \\ &= \frac{k-1}{2}\|(T-I)Ax\|^2. \end{aligned} \tag{9}$$

From (8) and (9) we get

$$\|x + \delta A^*(T - I)Ax - x^*\|^2 \leq \|x - x^*\|^2 - \delta\big(1 - k - \delta\|A\|^2\big)\|(T - I)Ax\|^2.$$

This completes the proof. □

Lemma 3.2 *For two real Hilbert spaces H_1 and H_2, let $A : H_1 \to H_2$ be a bounded linear operator with adjoint operator A^*, and let $\{T_i : H_2 \to H_2 : i \in \mathbb{N}\}$ be an infinite family of k-demicontractive mappings. Let $\{x_n\}$ be sequence in H_1, and let*

$$u_n = x_n + \sum_{i=1}^{n} \alpha_{n,i}\delta_n A^*(T_i - I)Ax_n, \quad \forall n \in \mathbb{N}, \tag{10}$$

where $\{\alpha_{n,i}\}$ is a real sequence in $[0, 1]$ satisfying $\sum_{i=1}^{n} \alpha_{n,i} = 1$. Then we have

$$\|u_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \sum_{i=1}^{n} \alpha_{n,i}\delta_n\big(1 - k - \delta_n\|A\|^2\big)\|(T_i - I)Ax_n\|^2$$

for all $x^* \in H_1$ such that $Ax^* \in \bigcap_{i=1}^{\infty} F(T_i)$.

Proof Let $x^* \in H_1$ be such that $Ax^* \in \bigcap_{i=1}^{\infty} F(T_i)$. From (10) and Lemma 3.1 we obtain

$$\begin{aligned}\|u_n - x^*\|^2 &\leq \sum_{i=1}^{n} \alpha_{n,i}\|x_n - x^* + \delta_n A^*(T_i - I)Ax_n\|^2 \\ &\leq \sum_{i=1}^{n} \alpha_{n,i}\big[\|x_n - x^*\|^2 - \delta_n\big(1 - k - \delta_n\|A\|^2\big)\|(T_i - I)Ax_n\|^2\big] \\ &= \|x_n - x^*\|^2 - \sum_{i=1}^{n} \alpha_{n,i}\delta_n\big(1 - k - \delta_n\|A\|^2\big)\|(T_i - I)Ax_n\|^2.\end{aligned}$$

This completes the proof. □

Lemma 3.3 *Let $\{T_i : H_1 \to H_1 : i \in \mathbb{N}\}$ be an infinite family of k-demicontractive mappings from a Hilbert space H_1 to itself. Let $\{x_n\}$ be sequence in H_1, and let*

$$u_n = x_n + \sum_{i=1}^{n} \alpha_{n,i}\delta_n(T_i - I)x_n, \quad \forall n \in \mathbb{N}, \tag{11}$$

where $\{\alpha_{n,i}\}$ is a real sequence in $[0, 1]$ satisfying $\sum_{i=1}^{n} \alpha_{n,i} = 1$. Then we have

$$\|u_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \sum_{i=1}^{n} \alpha_{n,i}\delta_n(1 - k - \delta_n)\|(T_i - I)x_n\|^2$$

for all $x^* \in \bigcap_{i=1}^{\infty} F(T_i)$.

Proof The statement directly follows from Lemma 3.2 by putting $H_1 = H_2$ and $A = I$. □

Now, we introduce a new algorithm for solving problem (7) for an infinite family of demicontractive mappings and then prove its strong convergence.

Theorem 3.4 *Let H_1 and H_2 be two real Hilbert spaces, and let $A_1, A_2 : H_1 \to H_2$ be two bounded linear operators with adjoint operators A_1^* and A_2^*, respectively. Let $f : H_1 \to H_1$ be a ρ-contraction mapping, and let B be a self-adjoint strongly positive bounded linear operator on H_1 with coefficient $\xi > 2\rho$ and $\|B\| = 1$. Let $\{S_i : H_2 \to H_2 : i \in \mathbb{N}\}$, $\{T_i : H_2 \to H_2 : i \in \mathbb{N}\}$, and $\{U_i : H_1 \to H_1 : i \in \mathbb{N}\}$ be infinite families of k_1-, k_2-, and k_3-demicontractive mappings such that $S_i - I$, $T_i - I$, and $U_i - I$ are demiclosed at zero, respectively. Suppose that $\Gamma = \{v^* \in \bigcap_{i=1}^\infty F(U_i) : A_1 v^* \in \bigcap_{i=1}^\infty F(S_i)$ and $A_2 v^* \in \bigcap_{i=1}^\infty F(T_i)\} \neq \emptyset$. For arbitrary $x_1 \in H_1$, let $\{u_n\}$, $\{v_n\}$, $\{y_n\}$, and $\{x_n\}$ be generated by*

$$\begin{cases} u_n = x_n + \sum_{i=1}^n \alpha_{n,i}\delta_n A_1^*(S_i - I)A_1 x_n, \\ v_n = u_n + \sum_{i=1}^n \beta_{n,i}\theta_n A_2^*(T_i - I)A_2 u_n, \\ y_n = v_n + \sum_{i=1}^n \gamma_{n,i}\tau_n (U_i - I)v_n, \\ x_{n+1} = \sigma_n f(y_n) + (I - \sigma_n B)y_n, \quad n \in \mathbb{N}, \end{cases} \tag{12}$$

where $\{\delta_n\}$, $\{\theta_n\}$, $\{\tau_n\}$, $\{\sigma_n\}$, $\{\alpha_{n,i}\}$, $\{\beta_{n,i}\}$, and $\{\gamma_{n,i}\}$ are sequences in $[0,1]$ satisfying the following conditions:

(C1) $\sum_{i=1}^n \alpha_{n,i} = \sum_{i=1}^n \beta_{n,i} = \sum_{i=1}^n \gamma_{n,i} = 1$ *for all* $n \in \mathbb{N}$;
(C2) $\liminf_{n\to\infty} \alpha_{n,i} > 0$, $\liminf_{n\to\infty} \beta_{n,i} > 0$, *and* $\liminf_{n\to\infty} \gamma_{n,i} > 0$ *for all* $i \in \mathbb{N}$;
(C3) $\lim_{n\to\infty} \sigma_n = 0$ *and* $\sum_{n=1}^\infty \sigma_n = \infty$;
(C4) $0 < a_1 \leq \delta_n \leq a_2 < \frac{1-k_1}{\|A_1\|^2}$;
(C5) $0 < b_1 \leq \theta_n \leq b_2 < \frac{1-k_2}{\|A_2\|^2}$;
(C6) $0 < c_1 \leq \tau_n \leq c_2 < 1 - k_3$.

Then the sequence $\{x_n\}$ converges strongly to $x^* = P_\Gamma(f + I - B)x^*$.

Proof For any $u, v \in H_1$, by Lemma 2.4 we have

$$\begin{aligned} \|P_\Gamma(f + I - B)u - P_\Gamma(f + I - B)v\| &\leq \|(f + I - B)u - (f + I - B)v\| \\ &\leq \|f(u) - f(v)\| + \|I - B\|\|u - v\| \\ &\leq \rho\|u - v\| + (1 - \xi)\|u - v\| \\ &\leq (1 - \rho)\|u - v\|, \end{aligned}$$

that is, the mapping $P_\Gamma(f + I - B)$ is a contraction. So, by the Banach contraction principle there is a unique element $x^* \in H_1$ such that $x^* = P_\Gamma(f + I - B)x^*$.

Let $x^* = P_\Gamma(f + I - B)x^*$, that is, $x^* \in \bigcap_{i=1}^\infty F(U_i)$ is such that $A_1 x^* \in \bigcap_{i=1}^\infty F(S_i)$ and $A_2 x^* \in \bigcap_{i=1}^\infty F(T_i)$. From Lemmas 3.2 and 3.3 and from (12) we obtain

$$\|u_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \sum_{i=1}^n \alpha_{n,i}\delta_n\bigl(1 - k_1 - \delta_n\|A_1\|^2\bigr)\|(S_i - I)A_1 x_n\|^2, \tag{13}$$

$$\|v_n - x^*\|^2 \leq \|u_n - x^*\|^2 - \sum_{i=1}^n \beta_{n,i}\theta_n\bigl(1 - k_2 - \theta_n\|A_2\|^2\bigr)\|(T_i - I)A_2 u_n\|^2, \tag{14}$$

and

$$\|y_n - x^*\|^2 \le \|v_n - x^*\|^2 - \sum_{i=1}^{n} \lambda_{n,i}\tau_n(1-k_3-\tau_n)\|(U_i - I)v_n\|^2. \tag{15}$$

Therefore

$$\begin{aligned}\|y_n - x^*\|^2 \le{}& \|x_n - x^*\|^2 - \sum_{i=1}^{n} \alpha_{n,i}\delta_n\bigl(1-k_1-\delta_n\|A_1\|^2\bigr)\|(S_i - I)A_1x_n\|^2 \\ &- \sum_{i=1}^{n} \beta_{n,i}\theta_n\bigl(1-k_2-\theta_n\|A_2\|^2\bigr)\|(T_i - I)A_2u_n\|^2 \\ &- \sum_{i=1}^{n} \lambda_{n,i}\tau_n(1-k_3-\tau_n)\|(U_i - I)v_n\|^2. \end{aligned}\tag{16}$$

By conditions (C4), (C5), and (C6) we have

$$\|y_n - x^*\| \le \|x_n - x^*\|. \tag{17}$$

By condition (C3) we may assume that $\sigma_n \in (0, \|B\|^{-1})$ for all $n \in \mathbb{N}$. By Lemma 2.4 we get $\|I - \sigma_n B\| \le 1 - \sigma_n\xi$. From (12) and (17) we get

$$\begin{aligned}\|x_{n+1} - x^*\| &= \|\sigma_n f(y_n) + (I - \sigma_n B)y_n - x^*\| \\ &= \|\sigma_n\bigl(f(y_n) - Bx^*\bigr) + (I - \sigma_n B)\bigl(y_n - x^*\bigr)\| \\ &\le \sigma_n\bigl[\|f(y_n) - f(x^*)\| + \|f(x^*) - Bx^*\|\bigr] + \|I - \sigma_n B\|\|y_n - x^*\| \\ &\le \sigma_n\rho\|y_n - x^*\| + \sigma_n\|f(x^*) - Bx^*\| + (1-\sigma_n\xi)\|y_n - x^*\| \\ &\le \bigl(1 - \sigma_n(\xi - \rho)\bigr)\|x_n - x^*\| + \sigma_n(\xi-\rho)\frac{\|f(x^*) - Bx^*\|}{\xi - \rho} \\ &\le \max\left\{\|x_n - x^*\|, \frac{\|f(x^*) - Bx^*\|}{\xi - \rho}\right\} \\ &\vdots \\ &\le \max\left\{\|x_1 - x^*\|, \frac{\|f(x^*) - Bx^*\|}{\xi - \rho}\right\}. \end{aligned}\tag{18}$$

Therefore $\{x_n\}$ is bounded, and we also have that $\{y_n\}$ and $\{f(y_n)\}$ are bounded. To this end, we consider the following two cases.

Case 1. Suppose that $\{\|x_n - x^*\|\}_{n=n_o}^{\infty}$ is nonincreasing for some $n_o \in \mathbb{N}$. Then we get that $\lim_{n\to\infty}\|x_n - x^*\|$ exists. By (16), (17), and Lemma 2.5(i) we get

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|\sigma_n\bigl(f(y_n) - Bx^*\bigr) + (I - \sigma_n B)\bigl(y_n - x^*\bigr)\|^2 \\ &\le \sigma_n\|f(y_n) - Bx^*\|^2 + (1 - \sigma_n\xi)\|y_n - x^*\|^2 \\ &\quad + 2\sigma_n(1-\sigma_n\xi)\|f(y_n) - Bx^*\|\|y_n - x^*\| \\ &\le \sigma_n M + \|x_n - x^*\|^2 \end{aligned}$$

$$
\begin{aligned}
&- \sum_{i=1}^{n} \alpha_{n,i}\delta_n\bigl(1-k_1-\delta_n\|A_1\|^2\bigr)\bigl\|(S_i-I)A_1x_n\bigr\|^2 \\
&- \sum_{i=1}^{n} \beta_{n,i}\theta_n\bigl(1-k_2-\theta_n\|A_2\|^2\bigr)\bigl\|(T_i-I)A_2u_n\bigr\|^2 \\
&- \sum_{i=1}^{n} \lambda_{n,i}\tau_n(1-k_3-\tau_n)\bigl\|(U_i-I)v_n\bigr\|^2,
\end{aligned}
$$

where

$$
M = \sup_n\bigl\{\bigl\|f(y_n)-Bx^*\bigr\|^2 + 2\bigl\|f(y_n)-Bx^*\bigr\|\,\bigl\|x_n-x^*\bigr\|\bigr\}.
$$

This implies, for $j = 1,2,\ldots,n$,

$$
\begin{aligned}
&\alpha_{n,j}\delta_n\bigl(1-k_1-\delta_n\|A_1\|^2\bigr)\bigl\|(S_j-I)A_1x_n\bigr\|^2 \\
&\quad\le \sum_{i=1}^{n} \alpha_{n,i}\delta_n\bigl(1-k_1-\delta_n\|A_1\|^2\bigr)\bigl\|(S_i-I)A_1x_n\bigr\|^2 \\
&\quad\le \sigma_n M + \bigl\|x_n-x^*\bigr\|^2 - \bigl\|x_{n+1}-x^*\bigr\|^2, \qquad (19)\\
&\beta_{n,j}\theta_n\bigl(1-k_2-\theta_n\|A_2\|^2\bigr)\bigl\|(T_j-I)A_2u_n\bigr\|^2 \\
&\quad\le \sum_{i=1}^{n} \beta_{n,i}\theta_n\bigl(1-k_2-\theta_n\|A_2\|^2\bigr)\bigl\|(T_i-I)A_2u_n\bigr\|^2 \\
&\quad\le \sigma_n M + \bigl\|x_n-x^*\bigr\|^2 - \bigl\|x_{n+1}-x^*\bigr\|^2, \qquad (20)
\end{aligned}
$$

and

$$
\begin{aligned}
\lambda_{n,j}\tau_n(1-k_3-\tau_n)\bigl\|(U_j-I)v_n\bigr\|^2 &\le \sum_{i=1}^{n} \lambda_{n,i}\tau_n(1-k_3-\tau_n)\bigl\|(U_i-I)v_n\bigr\|^2 \\
&\le \sigma_n M + \bigl\|x_n-x^*\bigr\|^2 - \bigl\|x_{n+1}-x^*\bigr\|^2. \qquad (21)
\end{aligned}
$$

From (19), (20), (21), and conditions (C2)–(C6) we obtain

$$
\lim_{n\to\infty}\bigl\|(S_j-I)A_1x_n\bigr\| = \lim_{n\to\infty}\sum_{i=1}^{n}\alpha_{n,i}\bigl\|(S_i-I)A_1x_n\bigr\|^2 = 0, \qquad (22)
$$

$$
\lim_{n\to\infty}\bigl\|(T_j-I)A_2u_n\bigr\| = \lim_{n\to\infty}\sum_{i=1}^{n}\beta_{n,i}\bigl\|(T_i-I)A_2u_n\bigr\|^2 = 0, \qquad (23)
$$

and

$$
\lim_{n\to\infty}\bigl\|(U_j-I)v_n\bigr\| = \lim_{n\to\infty}\sum_{i=1}^{n}\lambda_{n,i}\bigl\|(U_i-I)v_n\bigr\|^2 = 0. \qquad (24)
$$

Next, we show that

$$
\limsup_{n\to\infty}\bigl\langle f\bigl(x^*\bigr)-Bx^*, x_n-x^*\bigr\rangle \le 0, \quad \text{where } x^* = P_\Gamma(f+I-B)x^*.
$$

To see this, choose a subsequence $\{x_{n_p}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle f(x^*) - Bx^*, x_n - x^*\rangle = \lim_{p\to\infty}\langle f(x^*) - Bx^*, x_{n_p} - x^*\rangle.$$

Since the sequence $\{x_{n_p}\}$ is bounded, there exists a subsequence $\{x_{n_{p_j}}\}$ of $\{x_{n_p}\}$ such that $x_{n_{p_j}} \rightharpoonup z \in H_1$. Without loss of generality, we may assume that $x_{n_p} \rightharpoonup z \in H_1$. Since A_1 is a bounded linear operator, this yields that $A_1 x_{n_p} \rightharpoonup A_1 z$. By the demiclosedness principle of $S_i - I$ at zero and (22) we get $A_1 z \in \bigcap_{i=1}^{\infty} F(S_i)$. By (12) and (22) we have

$$\begin{aligned}\|u_n - x_n\|^2 &= \left\| x_n + \sum_{i=1}^{n} \alpha_{n,i}\delta_n A_1^*(S_i - I)A_1 x_n - x_n \right\|^2 \\ &\le \sum_{i=1}^{n} \alpha_{n,i}\delta_n \|A_1\|^2 \big\|(S_i - I)A_1 x_n\big\|^2 \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Similarly, we also have $\|v_n - u_n\| \to 0$ as $n \to \infty$. Using the fact that $x_{n_p} \rightharpoonup z$ and $\|u_n - x_n\| \to 0$, we conclude that $u_{n_p} \rightharpoonup z$. Since A_2 is a bounded linear operator, we get that $A_2 u_{n_p} \rightharpoonup A_2 z$. By the demiclosedness principle of $T_i - I$ at zero and (23) we get $A_2 z \in \bigcap_{i=1}^{\infty} F(T_i)$. Again, since $u_{n_p} \rightharpoonup z$ and $\|v_n - u_n\| \to 0$, we conclude that $v_{n_p} \rightharpoonup z$. By the demiclosedness principle of $U_i - I$ at zero and (24) we also have $z \in \bigcap_{i=1}^{\infty} F(U_i)$. Therefore $z \in \Gamma$.

Since $x^* = P_\Gamma(f + I - B)x^*$ and $z \in \Gamma$, we get

$$\begin{aligned}\limsup_{n\to\infty}\langle f(x^*) - Bx^*, x_n - x^*\rangle &= \lim_{p\to\infty}\langle f(x^*) - Bx^*, x_{n_p} - x^*\rangle \\ &= \langle f(x^*) - Bx^*, z - x^*\rangle \le 0.\end{aligned} \tag{25}$$

Using Lemma 2.5 and (17), we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|\sigma_n(f(y_n) - Bx^*) + (I - \sigma_n B)(y_n - x^*)\|^2 \\ &\le (1 - \sigma_n\xi)\|y_n - x^*\|^2 + 2\sigma_n\langle f(y_n) - Bx^*, x_{n+1} - x^*\rangle \\ &\le (1 - \sigma_n\xi)\|x_n - x^*\|^2 + 2\rho\sigma_n\|y_n - x^*\|\|x_{n+1} - x^*\| \\ &\quad + 2\sigma_n\langle f(x^*) - Bx^*, x_{n+1} - x^*\rangle \\ &\le (1 - \sigma_n\xi)\|x_n - x^*\|^2 + \rho\sigma_n\big[\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2\big] \\ &\quad + 2\sigma_n\langle f(x^*) - Bx^*, x_{n+1} - x^*\rangle \\ &= \big(1 - \sigma_n(\xi - \rho)\big)\|x_n - x^*\|^2 + \rho\sigma_n\|x_{n+1} - x^*\|^2 \\ &\quad + 2\sigma_n\langle f(x^*) - Bx^*, x_{n+1} - x^*\rangle.\end{aligned}$$

This implies that

$$\|x_{n+1} - x^*\|^2 \le \left[1 - \frac{\sigma_n(\xi - \rho)}{1 - \sigma_n\rho}\right]\|x_n - x^*\|^2 + \frac{2\sigma_n}{1 - \sigma_n\rho}\langle f(x^*) - Bx^*, x_{n+1} - x^*\rangle. \tag{26}$$

By (25), (26), and Lemma 2.6 we conclude that $x_n \to x^*$ as $n \to \infty$.

Case 2. Suppose that there exists an integer m_o such that

$$\|x_{m_o} - x^*\| \le \|x_{m_o+1} - x^*\|.$$

Put $\kappa_n = \|x_n - x^*\|$ for all $n \ge m_o$. Then we have $\kappa_{m_o} \le \kappa_{m_o+1}$. Let $\{\mu(n)\}$ be the sequence defined by

$$\mu(n) = \max\{l \in \mathbb{N} : l \le n, \kappa_l \le \kappa_{l+1}\}$$

for all $n \ge m_o$. By Lemma 2.7 we obtain that $\{\mu(n)\}$ is a nondecreasing sequence such that

$$\lim_{n\to\infty} \mu(n) = \infty \quad \text{and} \quad \kappa_{\mu(n)} \le \kappa_{\mu(n)+1} \quad \text{for all } n \ge m_o.$$

By the same argument as in case 1 we obtain

$$\lim_{n\to\infty} \|(S_i - I)A_1 x_{\mu(n)}\| = 0, \qquad \lim_{n\to\infty} \|(T_i - I)A_2 u_{\mu(n)}\| = 0,$$

and

$$\lim_{n\to\infty} \|(U_i - I)v_{\mu(n)}\| = 0.$$

By the demiclosedness principle of $S_i - I$, $T_i - I$, and $U_i - I$ at zero, we have $\omega_\omega(x_{\mu(n)}) \subset \Gamma$. This implies that

$$\limsup_{n\to\infty} \langle f(x^*) - Bx^*, x_{\mu(n)} - x^*\rangle \le 0.$$

By a similar argument from (26) we also have

$$\kappa^2_{\mu(n)+1} \le \left[1 - \frac{\sigma_{\mu(n)}(\xi - \rho)}{1 - \sigma_{\mu(n)}\rho}\right]\kappa^2_{\mu(n)} + \frac{2\sigma_{\mu(n)}}{1 - \sigma_{\mu(n)}\rho}\langle f(x^*) - Bx^*, x_{\mu(n)+1} - x^*\rangle.$$

So, we get $\lim_{n\to\infty} \kappa_{\mu(n)} = 0$ and also have $\lim_{n\to\infty} \kappa_{\mu(n)+1} = 0$. By Lemma 2.7 we have

$$0 \le \kappa_n \le \max\{\kappa_n, \kappa_{\mu(n)}\} \le \kappa_{\mu(n)+1}.$$

Therefore $x_n \to x^*$ as $n \to \infty$. This completes the proof. □

By setting $T_i = I$ for all $i \in \mathbb{N}$ in Theorem 3.4 we obtain the following result.

Corollary 3.5 *Let H_1 and H_2 be two real Hilbert spaces, let $A_1 : H_1 \to H_2$ be a bounded linear operator with adjoint operator A_1^*. Let $f : H_1 \to H_1$ be a ρ-contraction mapping, and let B be a self-adjoint strongly positive bounded linear operator on H_1 with coefficient $\xi > 2\rho$ and $\|B\| = 1$. Let $\{S_i : H_2 \to H_2 : i \in \mathbb{N}\}$ and $\{U_i : H_1 \to H_1 : i \in \mathbb{N}\}$ be infinite families of k_1- and k_3-demicontractive mappings such that $S_i - I$ and $U_i - I$ are demiclosed at zero, respectively. Suppose that $\Omega = \{v^* \in \bigcap_{i=1}^\infty F(U_i) : A_1 v^* \in \bigcap_{i=1}^\infty F(S_i)\} \neq \emptyset$. For arbitrary*

$x_1 \in H_1$, let $\{u_n\}$, $\{y_n\}$, and $\{x_n\}$ be generated by

$$\begin{cases} u_n = x_n + \sum_{i=1}^{n} \alpha_{n,i}\delta_n A_1^*(S_i - I)A_1 x_n, \\ y_n = u_n + \sum_{i=1}^{n} \gamma_{n,i}\tau_n(U_i - I)u_n, \\ x_{n+1} = \sigma_n f(y_n) + (I - \sigma_n B)y_n, \quad n \in \mathbb{N}, \end{cases} \tag{27}$$

where $\{\delta_n\}$, $\{\tau_n\}$, $\{\sigma_n\}$, $\{\alpha_{n,i}\}$, and $\{\gamma_{n,i}\}$ are sequences in $[0,1]$ satisfying the following conditions:

(C1) $\sum_{i=1}^{n} \alpha_{n,i} = \sum_{i=1}^{n} \gamma_{n,i} = 1$ *for all* $n \in \mathbb{N}$;

(C2) $\liminf_{n\to\infty} \alpha_{n,i} > 0$ *and* $\liminf_{n\to\infty} \gamma_{n,i} > 0$ *for all* $i \in \mathbb{N}$;

(C3) $\lim_{n\to\infty} \sigma_n = 0$ *and* $\sum_{n=1}^{\infty} \sigma_n = \infty$;

(C4) $0 < a_1 \leq \delta_n \leq a_2 < \frac{1-k_1}{\|A_1\|^2}$;

(C5) $0 < c_1 \leq \tau_n \leq c_2 < 1 - k_3$.

Then the sequence $\{x_n\}$ converges strongly to $x^* = P_\Omega(f + I - B)x^*$.

Remark 3.6 By the same setting as in Corollary 3.5, Eslamian [17] used another algorithm for solving the same problem as in Corollary 3.5; see [17], Theorem 3.3. Note that each step of our algorithm is much easier for computation than that of Eslamian [17] because our algorithm concerns only the finite sum.

By setting $f(y) = \nu$ for all $y \in H_1$ and $B = I$ in Theorem 3.4 we obtain the following result.

Corollary 3.7 *Let H_1 and H_2 be two real Hilbert spaces, and let $A_1, A_2 : H_1 \to H_2$ be two bounded linear operators with adjoint operators A_1^* and A_2^*, respectively. Let $\{S_i : H_2 \to H_2 : i \in \mathbb{N}\}$, $\{T_i : H_2 \to H_2 : i \in \mathbb{N}\}$, and $\{U_i : H_1 \to H_1 : i \in \mathbb{N}\}$ be infinite families of k_1-, k_2-, and k_3-demicontractive mappings such that $S_i - I$, $T_i - I$, and $U_i - I$ are demiclosed at zero, respectively. Suppose that $\Gamma = \{\nu^* \in \bigcap_{i=1}^{\infty} F(U_i) : A_1\nu^* \in \bigcap_{i=1}^{\infty} F(S_i)$ and $A_2\nu^* \in \bigcap_{i=1}^{\infty} F(T_i)\} \neq \emptyset$. For arbitrary $x_1 \in H_1$, let $\{u_n\}$, $\{v_n\}$, $\{y_n\}$, and $\{x_n\}$ be generated by*

$$\begin{cases} u_n = x_n + \sum_{i=1}^{n} \alpha_{n,i}\delta_n A_1^*(S_i - I)A_1 x_n, \\ v_n = u_n + \sum_{i=1}^{n} \beta_{n,i}\theta_n A_2^*(T_i - I)A_2 u_n, \\ y_n = v_n + \sum_{i=1}^{n} \gamma_{n,i}\tau_n(U_i - I)v_n, \\ x_{n+1} = \sigma_n \nu + (1 - \sigma_n)y_n, \quad n \in \mathbb{N}, \end{cases} \tag{28}$$

where $\{\delta_n\}$, $\{\theta_n\}$, $\{\tau_n\}$, $\{\sigma_n\}$, $\{\alpha_{n,i}\}$, $\{\beta_{n,i}\}$, and $\{\gamma_{n,i}\}$ are sequences in $[0,1]$ satisfying the following conditions:

(C1) $\sum_{i=1}^{n} \alpha_{n,i} = \sum_{i=1}^{n} \beta_{n,i} = \sum_{i=1}^{n} \gamma_{n,i} = 1$ *for all* $n \in \mathbb{N}$;

(C2) $\liminf_{n\to\infty} \alpha_{n,i} > 0$, $\liminf_{n\to\infty} \beta_{n,i} > 0$, *and* $\liminf_{n\to\infty} \gamma_{n,i} > 0$ *for all* $i \in \mathbb{N}$;

(C3) $\lim_{n\to\infty} \sigma_n = 0$ *and* $\sum_{n=1}^{\infty} \sigma_n = \infty$;

(C4) $0 < a_1 \leq \delta_n \leq a_2 < \frac{1-k_1}{\|A_1\|^2}$;

(C5) $0 < b_1 \leq \theta_n \leq b_2 < \frac{1-k_2}{\|A_2\|^2}$;

(C6) $0 < c_1 \leq \tau_n \leq c_2 < 1 - k_3$.

Then the sequence $\{x_n\}$ converges strongly to $x^* = P_\Gamma(\nu)$.

It is known that every quasi-nonexpansive mapping is 0-demicontractive mapping, so the following result is directly obtained by Theorem 3.2.

Corollary 3.8 *Let H_1 and H_2 be two real Hilbert spaces, and let $A_1, A_2 : H_1 \to H_2$ be two bounded linear operators with adjoint operators A_1^* and A_2^*, respectively. Let $\{S_i : H_2 \to H_2 : i \in \mathbb{N}\}$, $\{T_i : H_2 \to H_2 : i \in \mathbb{N}\}$, and $\{U_i : H_1 \to H_1 : i \in \mathbb{N}\}$ be infinite families of quasi-nonexpansive mappings such that $S_i - I$, $T_i - I$, and $U_i - I$ are demiclosed at zero, respectively. Suppose that $\Gamma = \{v^* \in \bigcap_{i=1}^\infty F(U_i) : A_1 v^* \in \bigcap_{i=1}^\infty F(S_i)$ and $A_2 v^* \in \bigcap_{i=1}^\infty F(T_i)\} \neq \emptyset$. For arbitrary $x_1 \in H_1$, let $\{u_n\}$, $\{v_n\}$, $\{y_n\}$, and $\{x_n\}$ be generated by*

$$\begin{cases} u_n = x_n + \sum_{i=1}^n \alpha_{n,i}\delta_n A_1^*(S_i - I)A_1 x_n, \\ v_n = u_n + \sum_{i=1}^n \beta_{n,i}\theta_n A_2^*(T_i - I)A_2 u_n, \\ y_n = v_n + \sum_{i=1}^n \gamma_{n,i}\tau_n (U_i - I)v_n, \\ x_{n+1} = \sigma_n v + (1 - \sigma_n)y_n, \quad n \in \mathbb{N}, \end{cases} \tag{29}$$

where $\{\delta_n\}$, $\{\theta_n\}$, $\{\tau_n\}$, $\{\sigma_n\}$, $\{\alpha_{n,i}\}$, $\{\beta_{n,i}\}$, and $\{\gamma_{n,i}\}$ are sequences in $[0,1]$ satisfying the following conditions:

(C1) $\sum_{i=1}^n \alpha_{n,i} = \sum_{i=1}^n \beta_{n,i} = \sum_{i=1}^n \gamma_{n,i} = 1$ *for all* $n \in \mathbb{N}$;
(C2) $\liminf_{n\to\infty} \alpha_{n,i} > 0$, $\liminf_{n\to\infty} \beta_{n,i} > 0$, *and* $\liminf_{n\to\infty} \gamma_{n,i} > 0$ *for all* $i \in \mathbb{N}$;
(C3) $\lim_{n\to\infty} \sigma_n = 0$ *and* $\sum_{n=1}^\infty \sigma_n = \infty$;
(C4) $0 < a_1 \le \delta_n \le a_2 < \frac{1}{\|A_1\|^2}$;
(C5) $0 < b_1 \le \theta_n \le b_2 < \frac{1}{\|A_2\|^2}$;
(C6) $0 < c_1 \le \tau_n \le c_2 < 1$.

Then the sequence $\{x_n\}$ converges strongly to $x^* = P_\Gamma(v)$.

4 Applications

4.1 The split common null point problem

Let M be the set-valued mapping of H into 2^H. The *effective domain* of M is denoted by $D(M)$, that is, $D(M) = \{x \in H : Mx \neq \emptyset\}$. The mapping M is said to be *monotone* if

$$\langle x - y, u - v\rangle \ge 0, \quad \forall x, y \in D(M), u \in Mx, v \in My.$$

A monotone mapping M is said to be *maximal* if the graph $G(M)$ is not properly contained in the graph of any other monotone map, where $G(M) = \{(x,y) \in H \times H : y \in Mx\}$. It is known that M is maximal if and only if for $(x,u) \in H \times H$, $\langle x - y, u - v\rangle \ge 0$ for every $(y,v) \in G(M)$ implies $u \in Mx$. For the maximal monotone operator M, we can associate its *resolvent* J_δ^M defined by

$$J_\delta^M \equiv (I + \delta M)^{-1} : H \to D(M), \quad \text{where } \delta > 0.$$

It is known that if M is a maximal monotone operator, then the resolvent J_δ^M is firmly nonexpansive, and $F(J_\delta^M) = M^{-1}0 \equiv \{x \in H : 0 \in Mx\}$ for every $\delta > 0$.

Let H_1 and H_2 be two real Hilbert spaces. Let $M_i : H_1 \to 2^{H_1}$, $O_i : H_2 \to 2^{H_2}$, and $P_i : H_2 \to 2^{H_2}$ be multivalued mappings. The *split common null point problem* (SCNPP) [18] is to find a point $u^* \in H_1$ such that

$$0 \in \bigcap_{i=1}^p M_i u^* \tag{30}$$

and the points $v_j^* = A_j u^* \in H_2$ satisfy

$$0 \in \bigcap_{j=1}^{q} O_j v_j^*, \tag{31}$$

where $A_j : H_1 \to H_2$ ($1 \le j \le q$) are bounded linear operators.

Now, we apply Theorem 3.4 to solve the problem of finding a point $u^* \in H_1$ such that

$$0 \in \bigcap_{i=1}^{\infty} M_i u^* \tag{32}$$

and the points $v^* = A_1 u^* \in H_2$ and $s^* = A_2 u^* \in H_2$ satisfy

$$0 \in \bigcap_{i=1}^{\infty} O_i v^* \quad \text{and} \quad 0 \in \bigcap_{i=1}^{\infty} P_i s^*, \tag{33}$$

where $A_1, A_2 : H_1 \to H_2$ are bounded linear operators.

Since every firmly nonexpansive mapping is a 0-demicontractive mapping, we obtain the following theorem for problem (32)–(33).

Theorem 4.1 *Let H_1 and H_2 be two real Hilbert spaces, and let $A_1, A_2 : H_1 \to H_2$ be two bounded linear operators with adjoint operators A_1^* and A_2^*, respectively. Let $f : H_1 \to H_1$ be a ρ-contraction mapping, and let B be a self-adjoint strongly positive bounded linear operator on H_1 with coefficient $\xi > 2\rho$ and $\|B\| = 1$. Let $\{M_i : H_1 \to 2^{H_1} : i \in \mathbb{N}\}$, $\{O_i : H_2 \to 2^{H_2} : i \in \mathbb{N}\}$, and $\{P_i : H_2 \to 2^{H_2} : i \in \mathbb{N}\}$ be maximal monotone mappings. Suppose that $\Omega = \{v^* \in \bigcap_{i=1}^{\infty} M_i^{-1}0 : A_1 v^* \in \bigcap_{i=1}^{\infty} O_i^{-1}0$ and $A_2 v^* \in \bigcap_{i=1}^{\infty} P_i^{-1}0\} \neq \emptyset$. For arbitrary $x_1 \in H_1$, let $\{u_n\}$, $\{v_n\}$, $\{y_n\}$, and $\{x_n\}$ be generated by*

$$\begin{cases} u_n = x_n + \sum_{i=1}^{n} \alpha_{n,i}\delta_n A_1^*(J_{r_1}^{O_i} - I)A_1 x_n, \\ v_n = u_n + \sum_{i=1}^{n} \beta_{n,i}\theta_n A_2^*(J_{r_2}^{P_i} - I)A_2 u_n, \\ y_n = v_n + \sum_{i=1}^{n} \gamma_{n,i}\tau_n (J_{r_3}^{M_i} - I)v_n, \\ x_{n+1} = \sigma_n f(y_n) + (I - \sigma_n B)y_n, \quad n \in \mathbb{N}, \end{cases} \tag{34}$$

where $r_1, r_2, r_3 > 0$ and $\{\delta_n\}$, $\{\theta_n\}$, $\{\tau_n\}$, $\{\sigma_n\}$, $\{\alpha_{n,i}\}$, $\{\beta_{n,i}\}$, $\{\gamma_{n,i}\}$ are sequences in $[0,1]$ satisfying the following conditions:

(C1) $\sum_{i=1}^{n} \alpha_{n,i} = \sum_{i=1}^{n} \beta_{n,i} = \sum_{i=1}^{n} \gamma_{n,i} = 1$ *for all* $n \in \mathbb{N}$;
(C2) $\liminf_{n\to\infty} \alpha_{n,i} > 0$, $\liminf_{n\to\infty} \beta_{n,i} > 0$, *and* $\liminf_{n\to\infty} \gamma_{n,i} > 0$ *for all* $i \in \mathbb{N}$;
(C3) $\lim_{n\to\infty} \sigma_n = 0$ *and* $\sum_{n=1}^{\infty} \sigma_n = \infty$;
(C4) $0 < a_1 \le \delta_n \le a_2 < \frac{1}{\|A_1\|^2}$;
(C5) $0 < b_1 \le \theta_n \le b_2 < \frac{1}{\|A_2\|^2}$;
(C6) $0 < c_1 \le \tau_n \le c_2 < 1$.

Then the sequence $\{x_n\}$ converges strongly to $x^* = P_\Omega(f + I - B)x^*$.

4.2 The split variational inequality problem

Let C and Q be nonempty closed convex subsets of two real Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator, $g : H_1 \to H_1$, and $h : H_2 \to H_2$.

The *split variational inequality problem* (SVIP) is to find a point $u^* \in C$ such that

$$\langle g(u^*), x - u^* \rangle \geq 0, \quad \forall x \in C, \tag{35}$$

and the point $v^* = Au^* \in Q$ satisfy

$$\langle h(v^*), y - v^* \rangle \geq 0, \quad \forall y \in Q. \tag{36}$$

We denote the solution set of the SVIP by $\Omega = \mathrm{SVIP}(C, Q, g, h, A)$. The set of all solutions of *variational inequality problem* (35) is denoted by $\mathrm{VIP}(C, g)$, and it is known that $\mathrm{VIP}(C, g) = F(P_C(I - \lambda g))$ for all $\lambda > 0$.

Let $A_1, A_2 : H_1 \to H_2$ be two bounded linear operators, $g_i : H_1 \to H_1$, and $h_i, l_i : H_2 \to H_2$. In this section, we apply Theorem 3.4 to solve the problem of finding a point $u^* \in \bigcap_{i=1}^{\infty} C_i$ such that

$$\langle g_i(u^*), x - u^* \rangle \geq 0, \quad \forall x \in \bigcap_{i=1}^{\infty} C_i, \tag{37}$$

and the point $v^* = A_1 u^* \in \bigcap_{i=1}^{\infty} Q_i$, $s^* = A_2 u^* \in \bigcap_{i=1}^{\infty} K_i$ satisfy

$$\langle h_i(v^*), y - v^* \rangle \geq 0, \quad \forall y \in \bigcap_{i=1}^{\infty} Q_i, \quad \text{and} \quad \langle l_i(s^*), z - s^* \rangle \geq 0, \quad \forall z \in \bigcap_{i=1}^{\infty} K_i, \tag{38}$$

where $\{C_i\}_{i\in\mathbb{N}}$ is a family of nonempty closed convex subsets of a real Hilbert space H_1, and $\{Q_i\}_{i\in\mathbb{N}}$ and $\{K_i\}_{i\in\mathbb{N}}$ are two families of nonempty closed convex subsets of a real Hilbert space H_2. We now prove a strong convergence theorem for problem (37)–(38).

Theorem 4.2 *Let $\{C_i\}_{i\in\mathbb{N}}$ be the family of nonempty closed convex subsets of a real Hilbert space H_1, let $\{Q_i\}_{i\in\mathbb{N}}$ and $\{K_i\}_{i\in\mathbb{N}}$ be two families of nonempty closed convex subsets of a real Hilbert space H_2, and let $A_1, A_2 : H_1 \to H_2$ be two bounded linear operators with adjoint operators A_1^* and A_2^*, respectively. Let $f : H_1 \to H_1$ be a ρ-contraction mapping, and let B be a self-adjoint strongly positive bounded linear operator on H_1 with coefficient $\xi > 2\rho$ and $\|B\| = 1$. Let $\{g_i : H_1 \to H_1 : i \in \mathbb{N}\}$, $\{h_i : H_2 \to H_2; i \in \mathbb{N}\}$, and $\{l_i : H_2 \to H_2; i \in \mathbb{N}\}$ be r_1-, r_2-, and r_3-inverse strongly monotone mappings, respectively. Let $r = \min\{r_1, r_2, r_3\}$ and $\mu \in (0, 2r)$. Suppose that $\Omega = \{v^* \in \bigcap_{i=1}^{\infty} \mathrm{VIP}(C_i, g_i) : A_1 v^* \in \bigcap_{i=1}^{\infty} \mathrm{VIP}(Q_i, h_i)$ and $A_2 v^* \in \bigcap_{i=1}^{\infty} \mathrm{VIP}(K_i, l_i)\} \neq \emptyset$. For arbitrary $x_1 \in H_1$, let $\{u_n\}$, $\{v_n\}$, $\{y_n\}$, and $\{x_n\}$ be generated by*

$$\begin{cases} u_n = x_n + \sum_{i=1}^{n} \alpha_{n,i} \delta_n A_1^*(P_{Q_i}(I - \mu h_i) - I) A_1 x_n, \\ v_n = u_n + \sum_{i=1}^{n} \beta_{n,i} \theta_n A_2^*(P_{K_i}(I - \mu l_i) - I) A_2 u_n, \\ y_n = v_n + \sum_{i=1}^{n} \gamma_{n,i} \tau_n (P_{C_i}(I - \mu g_i) - I) v_n, \\ x_{n+1} = \sigma_n f(y_n) + (I - \sigma_n B) y_n, \quad n \in \mathbb{N}, \end{cases} \tag{39}$$

where $\{\delta_n\}$, $\{\theta_n\}$, $\{\tau_n\}$, $\{\sigma_n\}$, $\{\alpha_{n,i}\}$, $\{\beta_{n,i}\}$, $\{\gamma_{n,i}\}$ are sequences in $[0, 1]$ satisfying the following conditions:

(C1) $\sum_{i=1}^{n} \alpha_{n,i} = \sum_{i=1}^{n} \beta_{n,i} = \sum_{i=1}^{n} \gamma_{n,i} = 1$ *for all* $n \in \mathbb{N}$;
(C2) $\liminf_{n\to\infty} \alpha_{n,i} > 0$, $\liminf_{n\to\infty} \beta_{n,i} > 0$, *and* $\liminf_{n\to\infty} \gamma_{n,i} > 0$ *for all* $i \in \mathbb{N}$;
(C3) $\lim_{n\to\infty} \sigma_n = 0$ *and* $\sum_{n=1}^{\infty} \sigma_n = \infty$;

(C4) $0 < a_1 \leq \delta_n \leq a_2 < \frac{1}{\|A_1\|^2}$;
(C5) $0 < b_1 \leq \theta_n \leq b_2 < \frac{1}{\|A_2\|^2}$;
(C6) $0 < c_1 \leq \tau_n \leq c_2 < 1$.
Then the sequence $\{x_n\}$ *converges strongly to* $x^* = P_\Omega(f + I - B)x^*$.

Proof It is known that $S_i := P_{Q_i}(I - \mu h_i)$, $T_i =: P_{K_i}(I - \mu l_i)$, and $U_i := P_{C_i}(I - \mu g_i)$ are nonexpensive mappings for all $\mu \in (0, 2r)$, and hence they are 0-demicontractive mappings. We obtain the desired result from Theorem 3.4. □

4.3 The split equilibrium problem

Let H_1 and H_2 be two real Hilbert spaces, and let C and Q be nonempty closed convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator, and let $g : C \times C \to \mathbb{R}$ and $h : Q \times Q \to \mathbb{R}$ be two bifunctions. The *split equilibrium problem* (SEP) is to find a point $u^* \in C$ such that

$$g\bigl(u^*, x\bigr) \geq 0, \quad \forall x \in C, \tag{40}$$

and $Au^* \in Q$ satisfy

$$h\bigl(Au^*, y\bigr) \geq 0, \quad \forall y \in Q. \tag{41}$$

The set of all solutions of *equilibrium problem* (40) is denoted by $\mathrm{EP}(g)$.

Lemma 4.3 ([19]) *Let C be a nonempty closed convex subset of H, and let g be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying the following conditions*:
(A1) $g(x, x) = 0$ *for all* $x \in C$;
(A2) *g is monotone, that is,* $g(x, y) + g(y, x) \leq 0$ *for all* $x, y \in C$;
(A3) *for all* $x, y, z \in C$,

$$\limsup_{t \downarrow 0} g\bigl(tz + (1 - t)x, y\bigr) \leq g(x, y);$$

(A4) *$g(x, \cdot)$ is convex and lower semicontinuous for all $x \in C$.*
Let $g : C \times C \to \mathbb{R}$ be a bifunction satisfying conditions (A1)–(A4), *and let $r > 0$ and $x \in H$. Then there exists $z \in C$ such that*

$$g(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \quad \textit{for all } y \in C.$$

Lemma 4.4 ([20]) *Let C be a nonempty closed convex subset of H, and let g be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying conditions* (A1)–(A4). *For $r > 0$ and $x \in H$, define the mapping $T_r^g : H \to C$ of g by*

$$T_r^g x = \left\{z \in C : g(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\right\}, \quad \forall x \in H.$$

Then the following hold:
(i) *T_r^g is single-valued*;
(ii) *T_r^g is firmly nonexpansive*;

(iii) $F(T_r^g) = \mathrm{EP}(g)$;
(iv) $\mathrm{EP}(g)$ *is closed and convex*.

Let $A_1, A_2 : H_1 \to H_2$ be two bounded linear operators, and let $g_i : C_i \times C_i \to \mathbb{R}$ and $h_i, l_i : Q_i \times Q_i \to \mathbb{R}$ be bifunctions for all $i \in \mathbb{N}$. In this section, we apply Theorem 3.4 to solve the problem of finding a point

$$u^* \in \bigcap_{i=1}^{\infty} \mathrm{EP}(g_i) \quad \text{such that} \quad A_1 v^* \in \bigcap_{i=1}^{\infty} \mathrm{EP}(h_i) \quad \text{and} \quad A_2 v^* \in \bigcap_{i=1}^{\infty} \mathrm{EP}(l_i). \tag{42}$$

By Lemma 4.4(iii) we have that $T_{r_1}^{h_i}$, $T_{r_2}^{l_i}$, and $T_{r_3}^{g_i}$ are firmly nonexpansive mappings, and hence they are 0-demicontractive mappings. We obtain the following result from Theorem 3.4.

Theorem 4.5 *Let $\{C_i\}_{i\in\mathbb{N}}$ be a family of nonempty closed convex subsets of a real Hilbert space H_1, let $\{Q_i\}_{i\in\mathbb{N}}$ and $\{K_i\}_{i\in\mathbb{N}}$ be two families of nonempty closed convex subsets of a real Hilbert space H_2, and let $A_1, A_2 : H_1 \to H_2$ be two bounded linear operators with adjoint operators A_1^* and A_2^*, respectively. Let $f : H_1 \to H_1$ be a ρ-contraction mapping, and let B be a self-adjoint strongly positive bounded linear operator on H_1 with coefficient $\xi > 2\rho$ and $\|B\| = 1$. Let $g_i : C_i \times C_i \to \mathbb{R}$ and $h_i, l_i : Q_i \times Q_i \to \mathbb{R}$ be bifunctions satisfying conditions* (A1)–(A4) *for all $i \in \mathbb{N}$. Suppose that $\Omega = \{v^* \in \bigcap_{i=1}^{\infty} \mathrm{EP}(g_i) : A_1 v^* \in \bigcap_{i=1}^{\infty} \mathrm{EP}(h_i)$ and $A_2 v^* \in \bigcap_{i=1}^{\infty} \mathrm{EP}(l_i)\} \neq \emptyset$. For arbitrary $x_1 \in H_1$, let $\{u_n\}$, $\{v_n\}$, $\{y_n\}$, and $\{x_n\}$ be generated by*

$$\begin{cases} u_n = x_n + \sum_{i=1}^n \alpha_{n,i}\delta_n A_1^*(T_{r_1}^{h_i} - I)A_1 x_n, \\ v_n = u_n + \sum_{i=1}^n \beta_{n,i}\theta_n A_2^*(T_{r_2}^{l_i} - I)A_2 u_n, \\ y_n = v_n + \sum_{i=1}^n \gamma_{n,i}\tau_n(T_{r_3}^{g_i} - I)v_n, \\ x_{n+1} = \sigma_n f(y_n) + (I - \sigma_n B)y_n, \quad n \in \mathbb{N}, \end{cases} \tag{43}$$

where $r_1, r_2, r_3 > 0$ and $\{\delta_n\}$, $\{\theta_n\}$, $\{\tau_n\}$, $\{\sigma_n\}$, $\{\alpha_{n,i}\}$, $\{\beta_{n,i}\}$, $\{\gamma_{n,i}\}$ are sequences in $[0,1]$ satisfying the following conditions:
(C1) $\sum_{i=1}^n \alpha_{n,i} = \sum_{i=1}^n \beta_{n,i} = \sum_{i=1}^n \gamma_{n,i} = 1$ *for all* $n \in \mathbb{N}$;
(C2) $\liminf_{n\to\infty} \alpha_{n,i} > 0$, $\liminf_{n\to\infty} \beta_{n,i} > 0$, *and* $\liminf_{n\to\infty} \gamma_{n,i} > 0$ *for all* $i \in \mathbb{N}$;
(C3) $\lim_{n\to\infty} \sigma_n = 0$ *and* $\sum_{n=1}^{\infty} \sigma_n = \infty$;
(C4) $0 < a_1 \le \delta_n \le a_2 < \frac{1}{\|A_1\|^2}$;
(C5) $0 < b_1 \le \theta_n \le b_2 < \frac{1}{\|A_2\|^2}$;
(C6) $0 < c_1 \le \tau_n \le c_2 < 1$.
Then the sequence $\{x_n\}$ converges strongly to $x^* = P_\Omega(f + I - B)x^*$.

5 Numerical example for the main result

We now give a numerical example of the studied method. Let $H_1 = H_2 = (\mathbb{R}^2, \|\cdot\|_2)$. Define the mappings $S_i : \mathbb{R}^2 \to \mathbb{R}^2$, $U_i : \mathbb{R}^2 \to \mathbb{R}^2$, and $T_i : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$S_i(x_1, x_2) = \frac{-3i}{i+1}(x_1, x_2), \qquad U_i(x_1, x_2) = \left(\frac{-2i}{i+1}x_1, x_2\right), \quad i \in \mathbb{N},$$

and

$$T_i(x_1, x_2) = \begin{cases} (x_1, \frac{x_2}{3i}\sin\frac{1}{x_2}) & \text{if } x_2 \neq 0, \\ (x_1, 0) & \text{if } x_2 = 0, \end{cases} \quad i \in \mathbb{N}$$

for all $x_1, x_2 \in \mathbb{R}$. Then S_i are $\frac{12}{25}$-demicontractive mappings for all $i \in \mathbb{N}$ and $\bigcap_{i=1}^{\infty} F(S_i) = \{(0,0)\}$, U_i are $\frac{3}{4}$-demicontractive mappings for all $i \in \mathbb{N}$ and $\bigcap_{i=1}^{\infty} F(U_i) = 0 \times \mathbb{R}$, and T_i are 0-demicontractive mappings for all $i \in \mathbb{N}$ and $\bigcap_{i=1}^{\infty} F(T_i) = \mathbb{R} \times 0$. Next, we define the mappings $f : \mathbb{R}^2 \to \mathbb{R}^2$, $B : \mathbb{R}^2 \to \mathbb{R}^2$, $A_1 : \mathbb{R}^2 \to \mathbb{R}^2$, and $A_2 : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$f(x_1, x_2) = \left(\frac{x_1}{8}, \frac{x_2}{8}\right), \qquad B(x_1, x_2) = \left(x_1, \frac{x_2}{2}\right), \qquad A_1(x_1, x_2) = (x_1, 2x_1),$$

and

$$A_2(x_1, x_2) = (x_2 - x_1, 2x_1)$$

for all $x_1, x_2 \in \mathbb{R}$. Then f is a $\frac{1}{8}$-contraction, B is a self-adjoint strongly positive bounded linear operator with coefficient $\xi = \frac{1}{2}$, and A_1, A_2 are bounded linear operators. Define the real sequence $\{\alpha_{n,i}\}$, $\{\beta_{n,i}\}$, and $\{\gamma_{n,i}\}$ as follows:

$$\alpha_{n,i} = \begin{cases} 1 & \text{if } n = i = 1, \\ \frac{1}{2^i}(\frac{n}{n+1}) & \text{if } n > i, \\ 1 - \sum_{i=1}^{n-1} \frac{1}{2^i}(\frac{n}{n+1}) & \text{if } n = i > 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$\beta_{n,i} = \begin{cases} 1 & \text{if } n = i = 1, \\ \frac{1}{3^i}(\frac{n}{n+1}) & \text{if } n > i, \\ 1 - \sum_{i=1}^{n-1} \frac{1}{3^i}(\frac{n}{n+1}) & \text{if } n = i > 1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\gamma_{n,i} = \begin{cases} 1 & \text{if } n = i = 1, \\ \frac{1}{4^{i+1}}(\frac{n}{2n+1}) & \text{if } n > i, \\ 1 - \sum_{i=1}^{n-1} \frac{1}{4^{i+1}}(\frac{n}{2n+1}) & \text{if } n = i > 1, \\ 0 & \text{otherwise,} \end{cases}$$

that is,

$$\alpha_{n,i} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1/3 & 2/3 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 3/8 & 3/16 & 7/16 & 0 & 0 & 0 & 0 & 0 & \dots \\ 2/5 & 1/5 & 1/10 & 3/10 & 0 & 0 & 0 & 0 & \dots \\ 5/12 & 5/24 & 5/48 & 5/96 & 7/32 & 0 & 0 & 0 & \dots \\ 3/7 & 3/14 & 3/28 & 3/56 & 3/112 & 19/112 & 0 & 0 & \dots \\ 7/16 & 7/32 & 7/64 & 7/128 & 7/256 & 7/512 & 71/512 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix},$$

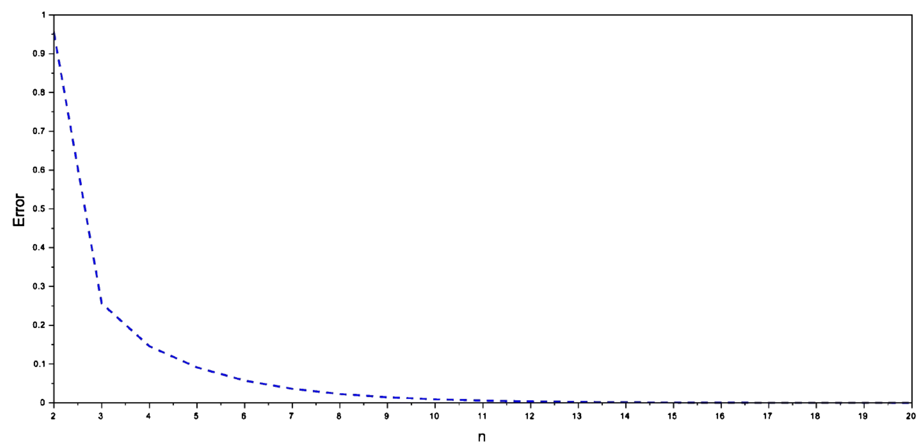

Figure 1 Graph for errors

$$\beta_{n,i} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 2/9 & 7/9 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1/4 & 1/12 & 2/3 & 0 & 0 & 0 & 0 & 0 & \dots \\ 4/15 & 4/45 & 4/135 & 83/135 & 0 & 0 & 0 & 0 & \dots \\ 5/18 & 5/54 & 5/162 & 5/486 & 143/243 & 0 & 0 & 0 & \dots \\ 2/7 & 2/21 & 2/63 & 2/189 & 1/284 & 325/567 & 0 & 0 & \dots \\ 7/24 & 7/72 & 7/216 & 7/648 & 1/278 & 1/833 & 58/103 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix},$$

and

$$\gamma_{n,i} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1/40 & 39/40 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 3/112 & 3/448 & 433/448 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1/36 & 1/144 & 1/576 & 185/192 & 0 & 0 & 0 & 0 & \dots \\ 5/176 & 5/704 & 1/563 & 1/2253 & 51/53 & 0 & 0 & 0 & \dots \\ 3/104 & 3/416 & 1/555 & 1/2219 & 1/8875 & 976/1015 & 0 & 0 & \dots \\ 7/240 & 7/960 & 1/549 & 1/2194 & 1/8777 & 1/35{,}109 & 618/643 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

We see that $\lim_{n\to\infty} \alpha_{n,i} = \frac{1}{2^i}$, $\lim_{n\to\infty} \beta_{n,i} = \frac{1}{3^i}$, and $\lim_{n\to\infty} \gamma_{n,i} = \frac{1}{2^{2i+3}}$ for $i \in \mathbb{N}$. Now, we start with the initial point $x_1 = (1,1)$ and let $\{x_n\}$ be the sequence generated by (12). Suppose that x_n is of the form $x_n = (a_n, b_n)$. where $a_n, b_n \in \mathbb{R}$. The criterion for stopping our testing method is taken as $\|x_{n-1} - x_n\|_2 < 10^{-6}$. Choose $\delta_n = \frac{n}{11n-1}$, $\theta_n = \frac{n}{30n-1}$, $\tau_n = \frac{n}{2n-1}$, and $\sigma_n = \frac{1}{n^{0.01}}$ for all $n \in \mathbb{N}$. Figure 1 shows the errors $\|x_{n-1} - x_n\|_2$ of our proposed method. The values of x_n and $\|x_{n-1} - x_n\|_2$ are shown in Table 1.

We observe from Table 1 that $x_n \to (0,0) \in \Gamma$. We also note that the error is bounded by $\|x_{30} - x_{31}\|_2 < 10^{-6}$, and we can use $x_{31} = (0.00000003, 0.00000117)$ to approximate the solution of (7) with accuracy at least 6 D.P.

6 Conclusion

We introduce a new algorithm for solving the split common fixed point problem (7) of the infinite families of demicontractive mappings in Hilbert spaces. Strong convergence of the proposed algorithm is obtained under some suitable control conditions. The main

Table 1 Numerical experiment for x_n

n	a_n	b_n	$\|x_{n-1} - x_n\|_2$
1	1.00000000	1.00000000	–
2	0.12500000	0.62500000	0.95197164
3	0.01751567	0.39224395	0.25637524
4	0.00414010	0.24675959	0.14609793
5	0.00202951	0.15549870	0.09128529
6	0.00140947	0.09811767	0.05738438
7	0.00107109	0.06197693	0.03614232
8	0.00063002	0.03918347	0.02279773
9	0.00047832	0.02479206	0.01439221
10	0.00030270	0.01569709	0.00909667
11	0.00022553	0.00994467	0.00575293
12	0.00014616	0.00630378	0.00364176
13	0.00008740	0.00399788	0.00230665
14	0.00005861	0.00253664	0.00146152
⋮	⋮	⋮	⋮
28	0.00000009	0.00000450	0.00000257
29	0.00000007	0.00000287	0.00000163
30	0.00000007	0.00000183	0.00000104
31	0.00000003	0.00000117	0.00000066

results of this paper can be considered as an extension of work by Eslamian [12] by providing an algorithm for finding a solution of problem (7), which is a generalization of problem (5).

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Abbreviations
SFP, The split feasibility problem; SCFP, The split common fixed point problem; MSSFP, The multiple set split feasibility problem; SCNPP, The split common null point problem; SVIP, The split variational inequality problem; SEP, The split equilibrium problem.

Availability of data and materials
Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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