Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

Inertial algorithm for approximating a common fixed point for a countable family of relatively nonexpansive maps

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Abstract

In this paper, we study an inertial algorithm for approximating a common fixed point for a countable family of relatively nonexpansive maps in a uniformly convex and uniformly smooth real Banach space. We prove a strong convergence theorem. This theorem is an improvement of the result of Matsushita and Takahashi (J. Approx. Theory 134:257–266, 2005) and the result of Dong *et al.* (Optim. Lett. 12:87–102, 2018). Finally, we give some applications of our theorem.

MSC: 47H09; 47H05; 47J25; 47J05

Keywords: Inertial algorithm; Relatively nonexpansive maps; Generalised projection; Strong convergence

1 Introduction

An inertial-type algorithm was first proposed by Polyak [3] as an acceleration process in solving a smooth convex minimisation problem. An inertial-type algorithm is a two-step iterative method in which the next iterate is defined by making use of the previous two iterates. It is well known that incorporating an inertial term in an algorithm speeds up or accelerates the rate of convergence of the sequence generated by the algorithm. Consequently, a lot of research interest is now devoted to the inertial-type algorithm (see e.g. [2, 4, 5] and the references contained in them).

Let E be a real Banach space and E^* be its dual space. Let C be a nonempty, closed and convex subset of E, and let T be a map from C into itself. The fixed point set of T is defined as follows: $F(T) := \{x \in C : Tx = x\}$. The normalised duality map J from E to 2^{E^*} is defined by $Jx = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2, \forall x \in E\}$, where $\langle \cdot, \cdot\rangle$ denotes the duality pairing. The following properties of the normalised duality map will be needed in the sequel (see e.g. Ibaraki and Takahashi [6]):

1. If E is a reflexive, strictly convex and smooth real Banach space, then J is surjective, injective and single-valued. If E is uniformly smooth, then J is uniformly continuous on bounded sets.
2. In a real Hilbert space H, the duality map J is the identity map on H.

The Lyapunov functional $\psi : E \times E \to [0, \infty)$ is defined by $\psi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$, $\forall x, y \in E$, where J is the normalised duality map from E to E^*. It was first introduced by

Alber and has been extensively studied by many authors (see e.g. Alber [7], Chidume *et al.* [8, 9], Kamimura and Takahashi [10], Reich [11, 12], Takahashi and Zembayashi [13], Zegeye [14] and a host of other authors). It is easy to see from the definition of ψ that, in a real Hilbert space H, $\psi(x,y) = \|x-y\|^2$, $\forall x,y \in H$. Furthermore, for any $x,y,z \in E$ and $\alpha \in (0,1)$, we have the following properties:

(N_0) $(\|x\| - \|y\|)^2 \leq \psi(x,y) \leq (\|x\| + \|y\|)^2$,
(N_1) $\psi(x,y) = \psi(x,z) + \psi(z,y) + 2\langle z-x, Jy - Jz\rangle$,
(N_2) $\psi(x, J^{-1}(\alpha Jy + (1-\alpha)Jz)) \leq \alpha\psi(x,y) + (1-\alpha)\psi(x,z)$,
(N_3) $\psi(x,y) \leq \|x\|\|Jx - Jy\| + \|y\|\|x-y\|$.

Let E be a smooth, strictly convex and reflexive real Banach space, and let $C \subseteq E$ be nonempty, closed and convex. The map $\Pi_C : E \to C$ defined by $\Pi_C x := u^0 \in C$ such that $\psi(u^0,x) = \arg\inf_{y\in C}\psi(y,x)$ is called the *generalised projection*, where $\arg\inf_{y\in C}\psi(y,x)$ is the set of all the minimisers of ψ. We remark that in a real Hilbert space H, the generalised projection Π_C coincides with the metric projection P_C from H onto C.

A point $x^* \in C$ is called an *asymptotic fixed point* of T if there exists a sequence $\{u^n\} \subseteq C$ such that $u^n \rightharpoonup x^*$ and $\|u^n - Tu^n\| \to 0$, as $n \to \infty$ (see e.g. Chang *et al.* [15]). Here we shall denote the set of asymptotic fixed points of T by $\widehat{F}(T)$.

A real Banach space E is called an *Opial* space (see e.g. Chidume [16]) if whenever $\{u^n\}$ is a sequence in E such that $u^n \rightharpoonup x \in E$, then

$$\liminf_{n\to\infty}\|u^n - x\| < \liminf_{n\to\infty}\|u^n - y\|, \quad \forall y \in E, y \neq x.$$

Definition 1.1 A map $T : C \to C$ is said to be *relatively nonexpansive* if:
(i) $\widehat{F}(T) = F(T) \neq \emptyset$, and
(ii) $\psi(p,Tx) \leq \psi(p,x)$, $\forall x \in C, p \in F(T)$.

Remark 1 Every real Hilbert space is an Opial space, and so if $\{u^n\}$ is a sequence in H such that $u^n \rightharpoonup x^*$ and $\|u^n - Tu^n\| \to 0$, it is well known that if T is nonexpansive, then $Tx^* = x^*$ and $\widehat{F}(T) = F(T)$. Moreover, since $\psi(x,y) = \|x-y\|^2$, $\forall x,y \in H$, it follows that T is relatively nonexpansive.

Let $B := \{x \in E : \|x\| = 1\}$ be the unit sphere of E. A Banach space E is said to be *strictly convex* if, for all $x,y \in B$, $x \neq y \Rightarrow \frac{\|x+y\|}{2} < 1$. The space E is said to have the *Kadec–Klee* property if whenever $\{u^n\}$ is a sequence in E that converges weakly to $u^0 \in E$ and $\|u^n\| \to \|u^0\|$, as $n \to \infty$, then $\{u^n\}$ converges strongly to u^0. A space E is said to be *uniformly convex* if, for each $\epsilon \in (0,2]$, there exists $\delta > 0$ such that $\|x-y\| \geq \epsilon \Rightarrow \|\frac{x+y}{2}\| < 1-\delta$, $\forall x,y \in B$. It is well known that a uniformly convex real Banach space is reflexive, strictly convex and has the Kadec–Klee property [17, 18].

A function $\delta : (0,2] \to [0,1]$ called the *modulus of convexity* of E is defined as follows: $\delta(\epsilon) := \inf\{1 - \|\frac{x+y}{2}\| : x,y \in E, \|x\| = \|y\| = 1, \|x-y\| \geq \epsilon\}$. It follows that E is uniformly convex if $\delta(\epsilon) > 0$, $\forall\epsilon \in (0,2]$. A real Banach space E is said to be *smooth* if the limit

$$\lim_{t\to 0}\frac{\|x+ty\| - \|x\|}{t} \text{ exists for all } x,y \in B. \tag{1.1}$$

It is said to be *uniformly smooth* if the limit in (1.1) above is attained uniformly, for all $x,y \in B$ (see e.g. Chidume [16]).

In 2003, Nakajo and Takahashi [19] studied the following *CQ* iterative algorithm for approximating a fixed point of a nonexpansive map in a real Hilbert space: $u^0 \in C$ and

$$\begin{cases} v^n = \alpha_n u^n + (1-\alpha_n)Tu^n, \\ C_n = \{z \in C : \|v^n - z\| \leq \|u^n - z\|\}, \\ Q_n = \{z \in C : \langle u^n - z, u^n - u^0\rangle \leq 0\}, \\ u^{n+1} = \mathbb{P}_{C_n \cap Q_n} u^0, \quad n \geq 1, \end{cases} \tag{1.2}$$

where $\alpha_n \in [0,1]$, C is a nonempty, closed and convex subset of a real Hilbert space, T is a nonexpansive map from C into itself and $\mathbb{P}_{C_n \cap Q_n}$ is the metric projection from C onto $C_n \cap Q_n$. They proved that the sequence generated by algorithm (1.2) converges strongly to a fixed point of T.

In 2005, Matsushita and Takahashi [1] studied the following iterative algorithm for approximating a fixed point of a relatively nonexpansive map in a uniformly convex and uniformly smooth real Banach space: $u^0 \in C$ and

$$\begin{cases} v^n = J^{-1}(\alpha_n Ju^n + (1-\alpha_n)JTu^n), \\ C_n = \{z \in C : \psi(z, v^n) \leq \psi(z, u^n)\}, \\ Q_n = \{z \in C : \langle u^n - z, Ju^n - Ju^0\rangle \leq 0\}, \\ u^{n+1} = \Pi_{C_n \cap Q_n} u^0, \quad n \geq 1. \end{cases} \tag{1.3}$$

They proved that the sequence $\{u^n\}$ generated by algorithm (1.3) converges strongly to a fixed point of T.

Recently, Dong *et al.* [2] studied the following *inertial* CQ algorithm for nonexpansive maps in a *real Hilbert space*. They proved the following theorem:

Theorem 1.2 (Dong *et al.*, [2, Theorem 4.1]) *Let $T : H \to H$ be a nonexpansive map such that $F(T) \neq \emptyset$. Let $\{\alpha_n\} \subset [\alpha_1, \alpha_2]$, $\alpha_1 \in (-\infty, 0]$, $\alpha_2 \in [0, \infty)$, $\{\beta_n\} \subset [\beta, 1]$, $\beta \in (0, 1]$. Set $u^0, u^1 \in H$ arbitrarily. Define a sequence $\{u^n\}$ by the following algorithm*:

$$\begin{cases} w^n = u^n + \alpha_n(u^n - u^{n-1}), \\ v^n = (1-\beta_n)w^n + \beta_n T(w^n), \\ C_n = \{z \in H : \|v^n - z\| \leq \|w^n - z\|\}, \\ Q_n = \{z \in H : \langle u^n - z, u^n - u^0\rangle \leq 0\}, \\ u^{n+1} = \mathbb{P}_{C_n \cap Q_n} u^0, \quad n \geq 0. \end{cases} \tag{1.4}$$

Then the iterative sequence $\{u^n\}$ generated by algorithm (1.4) *converges in norm to $P_{F(T)}u^0$.*

In this paper, motivated by the results of Matsushita and Takahashi [1] and Dong *et al.* [2], we study an *inertial algorithm in a uniformly convex and uniformly smooth real Banach space* and prove a strong convergence theorem for the sequence generated by our algorithm. As a consequence of this result, we obtain a strong convergence theorem for approximating a *common fixed point for a countable family of relatively nonexpansive maps*. Our theorem is an improvement of the results of Dong *et al.* [2], Matsushita and Takahashi [1], Nakajo and Takahashi [19] and a host of other results.

2 Preliminaries

Lemma 2.1 (Alber [7]) *Let C be a nonempty closed and convex subset of a strictly convex and reflexive real Banach space E. If $x \in E$ and $u^0 \in C$, then*

$$\begin{aligned} &u^0 = \Pi_C x \quad \Longleftrightarrow \quad \langle u^0 - y, Ju^0 - Jx \rangle \leq 0, \quad \forall y \in C, \quad \text{and} \\ &\psi(y, \Pi_C x) + \psi(\Pi_C x, x) \leq \psi(y, x), \quad \forall y \in C, x \in E. \end{aligned} \tag{2.1}$$

Lemma 2.2 (Kamimura and Takahashi [10]) *Let E be a smooth and uniformly convex real Banach space, and let $\{u^n\}$ and $\{v^n\}$ be two sequences of E. If either $\{u^n\}$ or $\{v^n\}$ is bounded and $\psi(u^n, v^n) \to 0$, as $n \to \infty$, then $\|u^n - v^n\| \to 0$, as $n \to \infty$.*

Remark 2 Using (N_3), it is easy to see that the converse of Lemma 2.2 is also true whenever $\{u^n\}$ and $\{v^n\}$ are both bounded.

Lemma 2.3 (Matsushita and Takahashi [1]) *Let E be a smooth and strictly convex real Banach space, and let C be a nonempty, closed and convex subset of E. Let T be a map from C into itself such that $F(T) \neq \emptyset$ and $\psi(y, Tx) \leq \psi(y, x)$, $\forall (y, x) \in F(T) \times C$. Then $F(T)$ is closed and convex.*

Lemma 2.4 (Kohsaka and Takahashi [20, Theorem 3.3]) *Let C be a nonempty, closed and convex subset of a uniformly convex and uniformly smooth real Banach space E, and let $T_i : C \to E$, $i = 1, 2, 3, \ldots$, be a countable family of relatively nonexpansive maps such that $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Suppose $\{\alpha_i\} \subset (0, 1)$ and $\{\beta_i\} \subset (0, 1)$ are sequences such that $\sum_{i=1}^{\infty} \alpha_i = 1$ and $T : C \to E$ is defined by*

$$Tx = J^{-1}\left(\sum_{i=1}^{\infty} \alpha_i \big(\beta_i Jx + (1 - \beta_i) JT_i x\big)\right) \quad \text{for each } x \in C.$$

Then T is relatively nonexpansive and $F(T) = \bigcap_{i=1}^{\infty} F(T_i)$.

3 Main results

We first prove the following lemma which will be central for the proof of our main theorem.

Lemma 3.1 *Let E be a uniformly convex and uniformly smooth real Banach space. Let $T : E \longrightarrow E$ be a relatively nonexpansive map such that $F(T) \neq \emptyset$. Let $\{u^n\}$ be generated by the following algorithm*: $u^0, u^1 \in E$ *and*

$$\begin{cases} C_0 = E, \\ w^n = u^n + \alpha_n (u^n - u^{n-1}), \\ v^n = J^{-1}[(1 - \beta) Jw^n + \beta JTw^n], \\ C_{n+1} = \{z \in C_n : \psi(z, v^n) \leq \psi(z, w^n)\}, \\ u^{n+1} = \Pi_{C_{n+1}} u^0, \quad n \geq 1, \end{cases} \tag{3.1}$$

where $\alpha_n \in (0, 1)$, $\beta \in (0, 1)$. Then $\{u^n\}$ converges strongly to a point $p = \Pi_{F(T)} u^0$.

Proof We partition our proof into four steps.

Step 1. We show that $\{u^n\}$ is well defined and $F(T) \subseteq C_n$, $\forall n \geq 0$.

Let $z \in C_{n+1}$, then

$$\begin{aligned} &\psi(z, v^n) \leq \psi(z, w^n) \\ &\quad \Leftrightarrow \quad \|z\|^2 - 2\langle z, Jv^n\rangle + \|v^n\|^2 \leq \|z\|^2 - 2\langle z, Jw^n\rangle + \|w^n\|^2 \\ &\quad \Leftrightarrow \quad 2\langle z, Jw^n - Jv^n\rangle \leq \|w^n\|^2 - \|v^n\|^2. \end{aligned} \tag{3.2}$$

Using (3.2) above, we observe that C_n is closed and convex, $\forall n \geq 0$. We now show that $F(T) \subseteq C_n$, $\forall n \geq 0$. For n=0, $F(T) \subseteq C_0 = E$. Assume $F(T) \subseteq C_n$, let $p \in F(T)$. Then, by (N_2) and the fact that T is relatively nonexpansive, we have that

$$\begin{aligned} \psi(p, v^n) &= \psi(p, J^{-1}((1-\beta)Jw^n + \beta JT(w^n)) \\ &\leq (1-\beta)\psi(p, w^n) + \beta\psi(p, Tw^n) \\ &\leq (1-\beta)\psi(p, w^n) + \beta\psi(p, w^n) = \psi(p, w^n), \end{aligned} \tag{3.3}$$

which implies that $p \in C_{n+1}$. So, by induction, $F(T) \subseteq C_n$ for all $n \geq 0$. Thus, $\{u^n\}$ is well defined.

Step 2. We show that $\{u^n\}$, $\{v^n\}$ and $\{w^n\}$ are bounded.

We observe that $u^n = \Pi_{C_n} u^0$ and $C_{n+1} \subseteq C_n$ for all $n \geq 0$. So, employing Lemma 2.1, we have that $\psi(u^n, u^0) \leq \psi(u^{n+1}, u^0)$. Hence, $\{\psi(u^n, u^0)\}$ is nondecreasing. Furthermore, we obtain that $\psi(u^n, u^0) = \psi(\Pi_{C_n} u^0, u^0) \leq \psi(p, u^0) - \psi(p, u^n) \leq \psi(p, u^0)$, which implies that $\{\psi(u^n, u^0)\}$ is bounded; and hence by (N_0), $\{u^n\}$ is bounded; and consequently, $\{\psi(u^n, u^0)\}$ is convergent. From Lemma 2.1, we have that

$$\psi(u^m, u^n) = \psi(u^m, \Pi_{C_n} u^0) \leq \psi(u^m, u^0) - \psi(u^n, u^0) \to 0, \quad \text{as } n, m \to \infty.$$

Hence, $\{u^n\}$ is Cauchy and this implies that $\|u^{n+1} - u^n\| \to 0$, as $n \to \infty$.

Using the definition of w^n, we have that $\|u^n - w^n\| = \|\alpha_n(u^{n-1} - u^n)\| \leq \|u^{n-1} - u^n\| \to 0$, as $n \to \infty$. Since $\{w^n\}$ is bounded, by Remark 2, we have that $\psi(u^n, w^n) \to 0$, as $n \to \infty$. Since $u^{n+1} \in C_n$, it follows that $0 \leq \psi(u^{n+1}, v^n) \leq \psi(u^{n+1}, w^n) \to 0$, as $n \to \infty$, which implies that $\|u^n - v^n\| \to 0$, as $n \to \infty$; and consequently, $\{v^n\}$ is bounded.

Step 3. We show that $\|w^n - Tw^n\| \to 0$, as $n \to \infty$.

Using Remark 2, we deduce that $\psi(u^n, v^n) \to 0$, as $n \to \infty$. By (N_1) and the uniform continuity of J on bounded sets, we have that

$$\begin{aligned} \psi(w^n, v^n) &= \psi(w^n, u^n) + \psi(u^n, v^n) + 2\langle u^n - w^n, Jv^n - Ju^n\rangle \\ &\leq \psi(w^n, u^n) + \psi(u^n, v^n) + 2\|u^n - w^n\| \|Ju^n - Jv^n\| \to 0, \quad \text{as } n \to \infty. \end{aligned} \tag{3.4}$$

Next, from the definition of v^n, we observe that

$$\|Jv^n - Jw^n\| = \beta\|JTw^n - Jw^n\|.$$

Since $\|v^n - w^n\| \to 0$, by the uniform continuity of J and J^{-1} on bounded sets, we have

$$\|w^n - Tw^n\| \to 0, \quad \text{as } n \to \infty.$$

Step 4. We show that $u^n \to \Pi_{F(T)}u^0$.

Since $\{w^n\}$ is bounded, there exists $\{w^{n_k}\}$, a subsequence of $\{w^n\}$, such that $w^{n_k} \rightharpoonup x^*$, as $k \to \infty$. Using Step 3, we obtain that $\|w^{n_k} - Tw^{n_k}\| \to 0$, as $k \to \infty$. Since our map is relatively nonexpansive, we have that $x^* \in F(T)$. Thus, it follows from Step 2 that there exists $\{u^{n_k}\}$, a subsequence of $\{u^n\}$, such that $u^{n_k} \rightharpoonup x^*$, as $k \to \infty$. We now show that $x^* = \Pi_{F(T)}u^0$. Set $v = \Pi_{F(T)}u^0$.

From $u^n = \Pi_{C_n}u^0$ and $F(T) \subseteq C_n$, $\forall n \geq 1$, we have $\psi(u^n, u^0) \leq \psi(v, u^0)$. Employing the weak lower semi-continuity of norm, we obtain

$$\begin{aligned}\psi(x^*, u^0) &= \|x^*\|^2 - 2\langle x^*, Ju^0\rangle + \|u^0\|^2 \\ &\leq \liminf\bigl(\|u^{n_k}\|^2 - 2\langle u^{n_k}, Ju^0\rangle + \|u^0\|^2\bigr) \\ &= \liminf \psi(u^{n_k}, u^0) \leq \limsup \psi(u^{n_k}, u^0) \leq \psi(v, u^0).\end{aligned} \tag{3.5}$$

However,

$$\begin{aligned}&\psi(v, u^0) \leq \psi(z, u^0), \quad \text{for all } z \in F(T) \\ &\quad\Rightarrow \quad \psi(v, u^0) \leq \psi(x^*, u^0) \leq \psi(v, u^0) \\ &\quad\Rightarrow \quad \psi(v, u^0) = \psi(x^*, u^0).\end{aligned} \tag{3.6}$$

By the uniqueness of $\Pi_{F(T)}u^0$, $v = x^*$. So, we deduce that $x^* = \Pi_{F(T)}u^0$. Next, we show that $u^{n_k} \to x^*$, as $k \to \infty$. Using (3.5) and (3.6), we obtain that $\psi(u^{n_k}, u^0) \to \psi(x^*, u^0)$, as $k \to \infty$. As a result of this, we obtain that $\|u^{n_k}\| \to \|x^*\|$, as $k \to \infty$. By the Kadec–Klee property of E, we conclude that $u^{n_k} \to x^*$, as $k \to \infty$. Consequently, since $\{u^n\}$ is convergent, we obtain that $u^n \to x^*$, as $n \to \infty$. Therefore, $u^n \to \Pi_{F(T)}u^0$. This completes the proof. □

Using Lemma 3.1, we now prove our main theorem of this paper.

Theorem 3.2 *Let E be a uniformly convex and uniformly smooth real Banach space. Let $T_i : E \to E$, $i = 1, 2, 3, \dots$, be a countable family of relatively nonexpansive maps such that $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Suppose $\{\alpha_i\} \subset (0,1)$ and $\{\beta_i\} \subset (0,1)$ are sequences such that $\sum_{i=1}^{\infty} \alpha_i = 1$ and $T : E \to E$ is defined by $Tx = J^{-1}(\sum_{i=1}^{\infty} \alpha_i(\beta_i Jx + (1 - \beta_i)JT_i x))$ for each $x \in E$. Let $\{u^n\}$ be generated by the following algorithm: $u^0, u^1 \in E$ and*

$$\begin{cases} C_0 = E, \\ w^n = u^n + \alpha_n(u^n - u^{n-1}), \\ v^n = J^{-1}[(1-\beta)Jw^n + \beta JTw^n], \\ C_{n+1} = \{z \in C_n : \psi(z, v^n) \leq \psi(z, w^n)\}, \\ u^{n+1} = \Pi_{C_{n+1}}u^0, \quad n \geq 0, \end{cases} \tag{3.7}$$

where $\alpha_n \in (0,1)$, $\beta \in (0,1)$, then $\{u^n\}$ converges strongly to a point $p = \Pi_{F(T)}u^0$.

Proof From Lemma 2.4, T is relatively nonexpansive and $F(T) = \bigcap_{i=1}^{\infty} F(T_i)$. The conclusion follows from Lemma 3.1. □

Corollary 3.3 *Let E be uniformly convex and uniformly smooth real Banach spaces L_p (or l_p or $W_p^m(\Omega)$), $1 < p < \infty$. Let $T_i : E \to E$, $i = 1, 2, 3, \dots$, be a countable family of relatively nonexpansive maps such that $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Suppose that $\{\alpha_i\} \subset (0,1)$ and $\{\beta_i\} \subset (0,1)$ are sequences such that $\sum_{i=1}^{\infty} \alpha_i = 1$ and $T : E \to E$ is defined by $Tx = J^{-1}(\sum_{i=1}^{\infty} \alpha_i(\beta_i Jx + (1-\beta_i)JT_i x))$ for each $x \in E$. Let $\{u^n\}$ be generated by the following algorithm: $u^0, u^1 \in E$ and*

$$\begin{cases} C_0 = E, \\ w^n = u^n + \alpha_n(u^n - u^{n-1}), \\ v^n = J^{-1}[(1-\beta)Jw^n + \beta JTw^n], \\ C_{n+1} = \{z \in C_n : \psi(z, v^n) \leq \psi(z, w^n)\}, \\ u^{n+1} = \Pi_{C_{n+1}} u^0, \quad n \geq 1, \end{cases} \tag{3.8}$$

where $\alpha_n \in (0,1)$, $\beta \in (0,1)$. Then $\{u^n\}$ converges strongly to $\Pi_{F(T)} u^0$.

Corollary 3.4 *Let H be a real Hilbert space. Let $T_i : H \to H$, $i = 1, 2, 3, \dots$, be a countable family of nonexpansive maps such that $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Suppose $\{\alpha_i\} \subset (0,1)$ and $\{\beta_i\} \subset (0,1)$ are sequences such that $\sum_{i=1}^{\infty} \alpha_i = 1$ and $T : H \to H$ is defined by $Tx = (\sum_{i=1}^{\infty} \alpha_i(\beta_i x + (1-\beta_i)T_i x))$ for each $x \in H$. Let $\{u^n\}$ be generated by the following algorithm: $u^0, u^1 \in H$ and*

$$\begin{cases} C_0 = H, \\ w^n = u^n + \alpha_n(u^n - u^{n-1}), \\ v^n = (1-\beta)w^n + \beta Tw^n, \\ C_{n+1} = \{z \in C_n : \|z - v^n\| \leq \|z - w^n\|\}, \\ u^{n+1} = \Pi_{C_{n+1}} u^0, \quad n \geq 1, \end{cases} \tag{3.9}$$

where $\alpha_n \in (0,1)$, $\beta \in (0,1)$. Then $\{u^n\}$ converges strongly to $\Pi_{F(T)} u^0$.

Proof Using Remark 1, we have that T_i is relatively nonexpansive for each $i \geq 1$. Thus, the conclusion follows from Theorem 3.2. $\square$

4 Conclusion

Theorem 3.2 is an improvement of the result of Matsushita and Takahashi [1] and the result of Dong *et al.* [2] in the following sense:

- The algorithms studied in Matsushita and Takahashi [1] and Dong *et al.* [2] require at *each step of the iteration process* the computation of two subsets C_n and Q_n of C; their intersection $C_n \cap Q_n$, and the projection of the initial vector onto this intersection. In our iteration process, the subset Q_n has been dispensed with. Furthermore, the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ used in the algorithms of Matsushita and Takahashi [1] and Dong *et al.* [2], which are also to be computed at each step of the iteration process, have been replaced by a fixed constant β in our algorithm. This β is to be computed once and used at each step of the iteration process. Consequently, our algorithm reduces computational cost.

- In Matsushita and Takahashi [1], the authors proved a strong convergence theorem for a relatively nonexpansive map $T : C \to C$. In our Theorem 3.2, a strong convergence theorem is proved for a *countable family of* relatively nonexpansive maps $T_i : E \to E$, $i \in \mathbb{N}$. Furthermore, unlike the algorithm of Matsushita and Takahashi, our algorithm has an *inertial term* which is known to improve the speed of convergence over algorithms without an inertial term (see e.g. [2–5, 21] and the references contained in them).
- In Dong *et al.* [2, Theorem 4.1], the authors proved a strong convergence theorem in a real Hilbert space for one nonexpansive map. Our theorem is proved in the much more general uniformly convex and uniformly smooth real Banach spaces and for a countable family of relatively nonexpansive maps.

Acknowledgements
The authors thank the African Capacity Building Foundation (ACBF) for sponsoring this research work.

Funding
Research supported from ACBF Research Grant Funds to AUST.

Competing interests
The authors declare that they have no conflict of interest.

Authors' contributions
CEC formulated the problem and suggested the method of proof of the Theorem to SII and AA. The computations using the method suggested by CEC were carried out by SII and AA. The analysis of the computations to arrive at the proof of the Theorem was done jointly by CEC, SII, and AA. All authors read and approved the final manuscript.

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Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 9 November 2017 Accepted: 16 February 2018 Published online: 12 March 2018

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