Fixed Point Theory and Applications
a SpringerOpen Journal

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On solving the split generalized equilibrium problem and the fixed point problem for a countable family of nonexpansive multivalued mappings

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Abstract

We consider the split generalized equilibrium problem and the fixed point problem for a countable family of nonexpansive multivalued mappings in real Hilbert spaces. Then, using the shrinking projection method, we prove a strong convergence theorem for finding a common solution of the considered problems. A numerical example is presented to illustrate the convergence result. Our results improve and extend the corresponding results in the literature.

MSC: 47H09; 47H10; 65K10; 65K15

Keywords: Fixed point problems; Split generalized equilibrium problems; Nonexpansive multivalued mapping; Hilbert spaces

1 Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H, $\varphi : C \times C \to \mathbb{R}$, and let $F : C \times C \to \mathbb{R}$ be two bifunctions. The *generalized equilibrium problem* is to find $x \in C$ such that

$$F(x,y) + \varphi(x,y) \geq 0, \quad \forall y \in C. \tag{1.1}$$

The solution set of generalized equilibrium problem is denoted by $\text{GEP}(F, \varphi)$. In particular, if $\varphi = 0$, then this problem reduces to the equilibrium problem to find $x \in C$ such that $F(x,y) \geq 0$ for all $y \in C$. The solution set of the equilibrium problem is denoted by $\text{EP}(F)$.

The generalized equilibrium problem is very general in the sense that it includes, as particular cases, optimization problems, variational inequality problems, minimization problems, fixed point problems, mixed equilibrium problem, Nash equilibrium problems in noncooperative games, and others; see, for example, [1–6].

In 2013, Kazmi and Rizvi [7] introduced and studied the following split generalized equilibrium problem. Let $C \subseteq H_1$ and $Q \subseteq H_2$, let $F_1, \varphi_1 : C \times C \to \mathbb{R}$ and $F_2, \varphi_2 : Q \times Q \to \mathbb{R}$ be nonlinear bifunctions, and let $A : H_1 \to H_2$ be a bounded linear operator. The *split*

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generalized equilibrium problem is to find $x^* \in C$ such that

$$F_1(x^*, x) + \varphi_1(x^*, x) \geq 0, \quad \forall x \in C, \tag{1.2}$$

and such that

$$y^* = Ax^* \in Q \quad \text{solves} \quad F_2(y^*, y) + \varphi_2(y^*, y) \geq 0, \quad \forall y \in Q. \tag{1.3}$$

The solution set of the split generalized equilibrium problem is denoted by

$$\mathrm{SGEP}(F_1, \varphi_1, F_2, \varphi_2) := \{x^* \in C : x^* \in \mathrm{GEP}(F_1, \varphi_1) \text{ and } Ax^* \in \mathrm{GEP}(F_2, \varphi_2)\}.$$

The authors also gave an iterative algorithm to find a common element of the solution sets of the split generalized equilibrium problem in real Hilbert spaces; for more details, we refer to [7–9]. If $\varphi_1 = 0$ and $\varphi_2 = 0$, then the split generalized equilibrium problem reduces to the split equilibrium problem; see [10]. If $F_2 = 0$ and $\varphi_2 = 0$, the split generalized equilibrium problem reduces to the equilibrium problem considered by Cianciaruso et al. [11].

In 2008, Takahashi et al. [12] introduced the following iterative algorithm, which is known as the *shrinking projection method*, for finding a fixed point of a nonexpansive single-valued mapping in Hilbert spaces. The shrinking projection method is a popular method and plays an important role in studying the strong convergence for finding fixed points of nonlinear mappings. Many researchers developed the shrinking projection method for solving variational inequality problems, equilibrium problems, and fixed point problems in Hilbert spaces; see, for example, [13, 14].

Motivated and inspired by the results mentioned and related literature, we propose an iterative algorithm based on the shrinking projection method for finding a common element of the set of solutions of split generalized equilibrium problems and the set of common fixed points of a countable family of nonexpansive multivalued mappings in real Hilbert spaces. Then we prove strong convergence theorems that extend and improve the corresponding results of Kazmi and Rizvi [7], Suantai et al. [15], and others. Finally, we give some examples and numerical results to illustrate our main results.

2 Preliminaries

Let C be a nonempty closed convex subset of a real Hilbert space H. We denote the strong convergence and the weak convergence of a sequence $\{x_n\}$ to a point $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. It is also well known [16] that a Hilbert space H satisfies *Opial's condition*, that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\limsup_{n \to \infty} \|x_n - x\| < \limsup_{n \to \infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$.

The following three lemmas are useful for our main results.

Lemma 2.1 *In a real Hilbert space H, the following inequalities hold*:
(1) $\|x - y\|^2 \leq \|x\|^2 - \|y\|^2 - 2\langle x - y, y \rangle, \forall x, y \in H$;

(2) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \forall x, y \in H$;

(3) If $\{x_n\}$ is a sequence in H that converges weakly to $z \in H$, then

$$\limsup_{n\to\infty} \|x_n - y\|^2 = \limsup_{n\to\infty} \|x_n - z\|^2 + \|z - y\|^2, \quad \forall y \in H.$$

Lemma 2.2 ([17]) *Let H be a Hilbert space. Let $x_1, x_2, \ldots, x_N \in H$, and let $\alpha_1, \alpha_2, \ldots, \alpha_N$ be real numbers such that $\sum_{i=1}^{N} \alpha_i = 1$. Then*

$$\left\| \sum_{i=1}^{N} \alpha_i x_i \right\|^2 = \sum_{i=1}^{N} \alpha_i \|x_i\|^2 - \sum_{1 \leq i,j \leq N} \alpha_i \alpha_j \|x_i - x_j\|^2.$$

Lemma 2.3 ([18]) *Let H be a Hilbert space, and let $\{x_n\}$ be a sequence in H. Let $u, v \in H$ be such that $\lim_{n\to\infty} \|x_n - u\|$ and $\lim_{n\to\infty} \|x_n - v\|$ exist. If $\{x_{n_k}\}$ and $\{x_{m_k}\}$ are subsequences of $\{x_n\}$ that converge weakly to u and v, respectively, then $u = v$.*

A single-valued mapping $T : C \to H$ is called *δ-inverse strongly monotone* [19] if there exists a positive real number δ such that

$$\langle x - y, Tx - Ty\rangle \geq \delta \|Tx - Ty\|^2, \quad \forall x, y \in C.$$

For each $\gamma \in (0, 2\delta]$, we see that $I - \gamma T$ is a nonexpansive single-valued mapping, that is,

$$\|(I - \gamma T)x - (I - \gamma T)y\| \leq \|x - y\|, \quad \forall x, y \in C.$$

We denote by $CB(C)$ and $K(C)$ the collections of all nonempty closed bounded subsets and nonempty compact subsets of C, respectively. The *Hausdorff metric* $\mathcal{H}$ on $CB(C)$ is defined by

$$\mathcal{H}(A, B) := \max \left\{ \sup_{x\in A} \text{dist}(x, B), \sup_{y\in B} \text{dist}(y, A) \right\}, \quad \forall A, B \in CB(C),$$

where $\text{dist}(x, B) = \inf\{d(x, y) : y \in B\}$ is the distance from a point x to a subset B. Let $S : C \to CB(C)$ be a multivalued mapping. An element $x \in C$ is called a *fixed point* of S if $x \in Sx$. The set of all fixed points of S is denoted by $F(S)$, that is, $F(S) = \{x \in C : x \in Sx\}$. Recall that a multivalued mapping $S : C \to CB(C)$ is called

 (i) *nonexpansive* if

$$\mathcal{H}(Sx, Sy) \leq \|x - y\|, \quad \forall x, y \in C;$$

(ii) *quasi-nonexpansive* if $F(S) \neq \emptyset$ and

$$\mathcal{H}(Sx, Sp) \leq \|x - p\|, \quad \forall x \in C, \forall p \in F(S).$$

If S is a nonexpansive single-valued mapping on a closed convex subset of a Hilbert space, then $F(S)$ is closed and convex. The closedness of $F(S)$ can be easily extended to the multivalued case. However, the convexity of $F(S)$ cannot be extended (see, e.g., [20]). However, if S is a nonexpansive multivalued mapping and $Sp = \{p\}$ for each $p \in F(S)$, then $F(S)$ is always closed and convex as the following result shows.

Lemma 2.4 ([21]) *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $S : C \to CB(C)$ be a nonexpansive multivalued mapping with $F(S) \neq \emptyset$ and $Sp = \{p\}$ for each $p \in F(S)$. Then $F(S)$ is a closed and convex subset of C.*

Lemma 2.5 ([22]) *Let C be a nonempty closed convex subset of a real Hilbert space H. Given $x, y, z \in H$ and a real number α, the set $\{u \in C : \|y - u\|^2 \leq \|x - u\|^2 + \langle z, u \rangle + \alpha\}$ is closed and convex.*

Lemma 2.6 ([23, 24]) *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $P_C : H \to C$ be the metric projection. Then*

$$\|y - P_C x\|^2 + \|x - P_C x\|^2 \leq \|x - y\|^2, \quad \forall x \in H, y \in C.$$

For solving the generalized equilibrium problem, we assume that the bifunctions $F_1 : C \times C \to \mathbb{R}$ and $\varphi_1 : C \times C \to \mathbb{R}$ satisfy the following assumption.

Assumption 2.7 Let C be nonempty closed and convex subset of a Hilbert space H_1. Let $F_1 : C \times C \to \mathbb{R}$ and $\varphi_1 : C \times C \to \mathbb{R}$ be two bifunctions satisfy the following conditions:
   (A1) $F_1(x, x) = 0$ for all $x \in C$,
   (A2) F_1 is monotone, that is, $F_1(x, y) + F_1(y, x) \leq 0$ for all $x, y \in C$,
   (A3) F_1 is upper hemicontinuous, *that is*, for all $x, y, z \in C$,
        $\lim_{t \downarrow 0} F_1(tz + (1 - t)x, y) \leq F_1(x, y)$,
   (A4) for each $x \in C$, $y \mapsto F_1(x, y)$ is convex and lower semicontinuous,
   (A5) $\varphi_1(x, x) \geq 0$ for all $x \in C$,
   (A6) for each $y \in C$, $x \mapsto \varphi_1(x, y)$ is upper semicontinuous,
   (A7) for each $x \in C$, $y \mapsto \varphi_1(x, y)$ is convex and lower semicontinuous,
and assume that for fixed $r > 0$ and $z \in C$, there exists a nonempty compact convex subset K of H_1 and $x \in C \cap K$ such that

$$F_1(y, x) + \varphi_1(y, x) + \frac{1}{r} \langle y - x, x - z \rangle < 0, \quad \forall y \in C \setminus K.$$

Lemma 2.8 ([25]) *Let C be nonempty closed and convex subset of a Hilbert space H_1. Let $F_1 : C \times C \to \mathbb{R}$ and $\varphi_1 : C \times C \to \mathbb{R}$ be two bifunctions satisfy Assumption 2.7. Assume that φ_1 is monotone. For $r > 0$ and $x \in H_1$, define a mapping $T_r^{(F_1, \varphi_1)} : H_1 \to C$ as follows:*

$$T_r^{(F_1, \varphi_1)}(x) = \left\{ z \in C : F_1(z, y) + \varphi_1(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \forall y \in C \right\}$$

for all $x \in H_1$. Then:
   (1) *For each $x \in H_1$, $T_r^{(F_1, \varphi_1)} \neq \emptyset$,*
   (2) *$T_r^{(F_1, \varphi_1)}$ is single-valued,*
   (3) *$T_r^{(F_1, \varphi_1)}$ is firmly nonexpansive, that is, for any $x, y \in H_1$,*

$$\left\| T_r^{(F_1, \varphi_1)} x - T_r^{(F_1, \varphi_1)} y \right\|^2 \leq \left\langle T_r^{(F_1, \varphi_1)} x - T_r^{(F_1, \varphi_1)} y, x - y \right\rangle,$$

   (4) *$F(T_r^{(F_1, \varphi_1)}) = \mathrm{GEP}(F_1, \varphi_1)$,*
   (5) *$\mathrm{GEP}(F_1, \varphi_1)$ is compact and convex.*

Further, assume that $F_2 : Q \times Q \to \mathbb{R}$ and $\varphi_2 : Q \times Q \to \mathbb{R}$ satisfy Assumption 2.7, where Q is a nonempty closed and convex subset of a Hilbert space H_2. For all $s > 0$ and $w \in H_2$, define the mapping $T_s^{(F_2,\varphi_2)} : H_2 \to Q$ by

$$T_s^{(F_2,\varphi_2)}(v) = \left\{ w \in Q : F_2(w,d) + \varphi_2(w,d) + \frac{1}{r}\langle d - w, w - v \rangle \geq 0, \forall d \in Q \right\}.$$

Then we have:

(6) *For each $v \in H_2$, $T_s^{(F_2,\varphi_2)} \neq \emptyset$,*

(7) *$T_s^{(F_2,\varphi_2)}$ is single-valued,*

(8) *$T_s^{(F_2,\varphi_2)}$ is firmly nonexpansive,*

(9) *$F(T_s^{(F_2,\varphi_2)}) = \text{GEP}(F_2,\varphi_2)$,*

(10) *$\text{GEP}(F_2,\varphi_2)$ is closed and convex,*

where $\text{GEP}(F_2,\varphi_2)$ is the solution set of the following generalized equilibrium problem:

Find $y^* \in Q$ such that $F_2(y^*,y) + \varphi_2(y^*,y) \geq 0$ for all $y \in Q$.

Moreover, $\text{SGEP}(F_1,\varphi_1,F_2,\varphi_2)$ is a closed and convex set.

Lemma 2.9 ([11]) *Let C be nonempty closed and convex subset of a Hilbert space H_1. Let $F_1 : C \times C \to \mathbb{R}$ and $\varphi_1 : C \times C \to \mathbb{R}$ be two bifunctions satisfying Assumption 2.7, and let $T_r^{(F_1,\varphi_1)}$ be defined as in Lemma 2.8 for $r > 0$. Let $x, y \in H_1$ and $r_1, r_2 > 0$. Then*

$$\left\| T_{r_2}^{(F_1,\varphi_1)}y - T_{r_1}^{(F_1,\varphi_1)}x \right\| \leq \|y - x\| + \left| \frac{r_2 - r_1}{r_2} \right| \left\| T_{r_2}^{(F_1,\varphi_1)}y - y \right\|.$$

3 Main results

In this section, we prove strong convergence theorems for finding a common element of the set of solutions of split generalized equilibrium problems and the set of common fixed points of a countable family of nonexpansive multivalued mappings in real Hilbert spaces and give a numerical example to support our main result.

We now state and prove our main result.

Theorem 3.1 *Let C be a nonempty closed convex subset of a real Hilbert space H_1, and let Q be a nonempty closed convex subset of a real Hilbert space H_2. Let $A : H_1 \to H_2$ be a bounded linear operator, and let $\{S_i\}$ be a countable family of nonexpansive multivalued mappings of C into $CB(C)$. Let $F_1, \varphi_1 : C \times C \to \mathbb{R}$, $F_2, \varphi_2 : Q \times Q \to \mathbb{R}$ be bifunctions satisfying Assumption 2.7. Let φ_1, φ_2 be monotone, φ_1 be upper hemicontinuous, and F_2 and φ_2 be upper semicontinuous in the first argument. Assume that $\Gamma = \bigcap_{i=1}^{\infty} F(S_i) \cap \text{SGEP}(F_1,\varphi_1,F_2,\varphi_2) \neq \emptyset$ and $S_i p = \{p\}$ for each $p \in \bigcap_{i=1}^{\infty} F(S_i)$. Let $x_1 \in C$ with $C_1 = C$, and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} u_n = T_{r_n}^{(F_1,\varphi_1)}(I - \gamma A^*(I - T_{r_n}^{(F_2,\varphi_2)})A)x_n, \\ z_n = \alpha_n^{(0)}x_n + \alpha_n^{(1)}y_n^{(1)} + \cdots + \alpha_n^{(n)}y_n^{(n)}, \quad y_n^{(i)} \in S_i u_n, \\ C_{n+1} = \{p \in C_n : \|z_n - p\| \leq \|x_n - p\|\}, \\ x_{n+1} = P_{C_{n+1}}x_1, \quad n \in \mathbb{N}, \end{cases} \tag{3.1}$$

where $\{\alpha_n^{(i)}\} \subset (0,1)$ satisfy $\sum_{i=0}^{n} \alpha_n^{(i)} = 1$, $\{r_n\} \subset (0,\infty)$, and $\gamma \in (0,\frac{1}{L})$, where L is the spectral radius of A^*A, and A^* is the adjoint of A. Assume that the following conditions hold:

(C1) *The limits* $\lim_{n\to\infty}\alpha_n^{(i)} \in (0,1)$ *exist for all* $i \geq 0$,

(C2) $\liminf_{n\to\infty} r_n > 0$.

Then the sequence $\{x_n\}$ *generated by* (3.1) *converges strongly to* $P_\Gamma x_1$.

Proof We divide our proof into six steps.

Step 1. We show that $\{x_n\}$ is well-defined for every $n \in \mathbb{N}$.

By Lemmas 2.4 and 2.8 we obtain that $\mathrm{SGEP}(F_1, \varphi_1, F_2, \varphi_2)$ and $\bigcap_{i=1}^{\infty} F(S_i)$ are closed and convex subsets of C. Hence Γ is a closed and convex subset of C. It follows by Lemma 2.5 that C_{n+1} is closed and convex for each $n \in \mathbb{N}$.

Let $p \in \Gamma$. Then we have $p = T_{r_n}^{(F_1,\varphi_1)}p$ and $Ap = T_{r_n}^{(F_2,\varphi_2)}(Ap)$. It follows that $p = (I - \gamma A^*(I - T_{r_n}^{(F_2,\varphi_2)})A)p$. Since $T_{r_n}^{(F_1,\varphi_1)}$ and $T_{r_n}^{(F_2,\varphi_2)}$ both are firmly nonexpansive, for $\gamma \in (0, \frac{1}{L})$, the mapping $T_{r_n}^{(F_1,\varphi_1)}(I - \gamma A^*(I - T_{r_n}^{(F_2,\varphi_2)})A)$ is nonexpansive; see [26]. This implies that

$$\|u_n - p\| = \left\| T_{r_n}^{(F_1,\varphi_1)}\left(I - \gamma A^*\left(I - T_{r_n}^{(F_2,\varphi_2)}\right)A\right)x_n - T_{r_n}^{(F_1,\varphi_1)}\left(I - \gamma A^*\left(I - T_{r_n}^{(F_2,\varphi_2)}\right)A\right)p \right\|$$
$$\leq \|x_n - p\|. \tag{3.2}$$

Then, since $S_i p = \{p\}$ for all $p \in \bigcap_{i=1}^{\infty} F(S_i)$, we have

$$\|z_n - p\| \leq \alpha_n^{(0)}\|x_n - p\| + \alpha_n^{(1)}\left\|y_n^{(1)} - p\right\| + \cdots + \alpha_n^{(n)}\left\|y_n^{(n)} - p\right\|$$
$$= \alpha_n^{(0)}\|x_n - p\| + \alpha_n^{(1)}\mathrm{dist}\left(y_n^{(1)}, S_1 p\right) + \cdots + \alpha_n^{(n)}\mathrm{dist}\left(y_n^{(n)}, S_n p\right)$$
$$\leq \alpha_n^{(0)}\|x_n - p\| + \alpha_n^{(1)}\mathcal{H}(S_1 u_n, S_1 p) + \cdots + \alpha_n^{(n)}\mathcal{H}(S_n u_n, S_n p)$$
$$\leq \alpha_n^{(0)}\|x_n - p\| + \alpha_n^{(1)}\|u_n - p\| + \cdots + \alpha_n^{(n)}\|u_n - p\|. \tag{3.3}$$

This implies by (3.2) and $\sum_{i=0}^{n}\alpha_n^{(i)} = 1$ that

$$\|z_n - p\| \leq \|x_n - p\|. \tag{3.4}$$

This shows that $p \in C_{n+1}$ and hence $\Gamma \subset C_{n+1} \subset C_n$. Therefore, $P_{C_{n+1}}x_1$ is well-defined for every $x_1 \in C$. Hence, $\{x_n\}$ is well-defined.

Step 2. We show that $\lim_{n\to\infty} x_n = q$ for some $q \in C$.

Since Γ is a nonempty closed convex subset of H_1, there exists a unique $\omega \in \Gamma$ such that $\omega = P_\Gamma x_1$. Since $x_n = P_{C_n}x_1$ and $x_{n+1} \in C_{n+1} \subset C_n$ for all $n \in \mathbb{N}$, we have $\|x_n - x_1\| \leq \|x_{n+1} - x_1\|$ for all $n \in \mathbb{N}$. On the other hand, since $\Gamma \subset C_n$, we obtain that $\|x_n - x_1\| \leq \|\omega - x_1\|$ for all $n \in \mathbb{N}$. Hence $\{\|x_n - x_1\|\}$ is bounded; so are $\{z_n\}$ and $\{y_n^{(i)}\}$. Therefore, $\lim_{n\to\infty}\|x_n - x_1\|$ exists. By the construction of the set C_n we know that $x_m = P_{C_m}x_1 \in C_m \subset C_n$ for $m > n \geq 1$. This implies by Lemma 2.6 that

$$\|x_m - x_n\|^2 \leq \|x_m - x_1\|^2 - \|x_n - x_1\|^2 \to 0 \quad \text{as } m, n \to \infty. \tag{3.5}$$

Since $\lim_{n\to\infty}\|x_n - x_1\|$ exists, it follows that $\{x_n\}$ is a Cauchy sequence. By the completeness of H_1 and the closedness of C we get that there exists an element $q \in C$ such that $\lim_{n\to\infty} x_n = q$.

Step 3. We show that $\lim_{n\to\infty}\|y_n^{(i)} - x_n\| = 0$ for all $i \in \mathbb{N}$.

From (3.5) we have

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.6}$$

Since $x_{n+1} \in C_{n+1}$, we get that

$$\|z_n - x_n\| \le \|z_n - x_{n+1}\| + \|x_{n+1} - x_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - x_n\| \le 2\|x_{n+1} - x_n\|.$$

This implies by (3.6) that

$$\lim_{n \to \infty} \|z_n - x_n\| = 0. \tag{3.7}$$

Thus $\lim_{n \to \infty} z_n = q$.

For $p \in \Gamma$, by Lemma 2.2 we see that

$$\|z_n - p\|^2 \le \alpha_n^{(0)} \|x_n - p\|^2 + \sum_{i=1}^{n} \alpha_n^{(i)} \|y_n^{(i)} - p\|^2 - \sum_{i=1}^{n} \alpha_n^{(0)} \alpha_n^{(i)} \|y_n^{(i)} - x_n\|^2$$

$$= \alpha_n^{(0)} \|x_n - p\|^2 + \sum_{i=1}^{n} \alpha_n^{(i)} \operatorname{dist}(y_n^{(i)}, S_1 p)^2 - \sum_{i=1}^{n} \alpha_n^{(0)} \alpha_n^{(i)} \|y_n^{(i)} - x_n\|^2$$

$$= \alpha_n^{(0)} \|x_n - p\|^2 + \sum_{i=1}^{n} \alpha_n^{(i)} \mathcal{H}(S_i u_n, S_i p)^2 - \sum_{i=1}^{n} \alpha_n^{(0)} \alpha_n^{(i)} \|y_n^{(i)} - x_n\|^2$$

$$\le \alpha_n^{(0)} \|x_n - p\|^2 + \sum_{i=1}^{n} \alpha_n^{(i)} \|u_n - p\|^2 - \sum_{i=1}^{n} \alpha_n^{(0)} \alpha_n^{(i)} \|y_n^{(i)} - x_n\|^2. \tag{3.8}$$

This implies by (3.2) and $\sum_{i=0}^{n} \alpha_n^{(i)} = 1$ that

$$\|z_n - p\|^2 \le \|x_n - p\|^2 - \sum_{i=1}^{n} \alpha_n^{(0)} \alpha_n^{(i)} \|y_n^{(i)} - x_n\|^2.$$

Therefore we have

$$\alpha_n^{(0)} \alpha_n^{(i)} \|y_n^{(i)} - x_n\|^2 \le \sum_{i=1}^{n} \alpha_n^{(0)} \alpha_n^{(i)} \|y_n^{(i)} - x_n\|^2$$

$$\le \|x_n - p\|^2 - \|z_n - p\|^2$$

$$\le \|x_n - z_n\| \big(\|x_n - p\| + \|z_n - p\| \big).$$

By the given control condition on $\{\alpha_n^{(i)}\}$ and (3.7) we obtain

$$\lim_{n \to \infty} \|y_n^{(i)} - x_n\| = 0, \quad \forall i \in \mathbb{N}. \tag{3.9}$$

Step 4. We show that $\lim_{n \to \infty} \|u_n - x_n\| = 0$. For $p \in \Gamma$, we get that

$$\|u_n - p\|^2 = \big\| T_{r_n}^{(F_1, \varphi_1)} \big(I - \gamma A^* \big(I - T_{r_n}^{(F_2, \varphi_2)} \big) A \big) x_n - T_{r_n}^{(F_1, \varphi_1)} p \big\|^2$$

$$\le \big\| \big(I - \gamma A^* \big(I - T_{r_n}^{(F_2, \varphi_2)} \big) A \big) x_n - p \big\|^2$$

$$\le \|x_n - p\|^2 + \gamma^2 \big\| A^* \big(I - T_{r_n}^{(F_2, \varphi_2)} \big) A x_n \big\|^2 + 2\gamma \big\langle p - x_n, A^* \big(I - T_{r_n}^{(F_2, \varphi_2)} \big) A x_n \big\rangle$$

$$\le \|x_n - p\|^2 + \gamma^2 \big\langle A x_n - T_{r_n}^{(F_2, \varphi_2)} A x_n, A A^* \big(I - T_{r_n}^{(F_2, \varphi_2)} \big) A x_n \big\rangle$$

$$+ 2\gamma \langle A(p - x_n), Ax_n - T_{r_n}^{(F_2, \varphi_2)} Ax_n \rangle$$

$$\leq \|x_n - p\|^2 + L\gamma^2 \langle Ax_n - T_{r_n}^{(F_2, \varphi_2)} Ax_n, Ax_n - T_{r_n}^{F_2} Ax_n \rangle$$

$$+ 2\gamma \langle A(p - x_n) + \left(Ax_n - T_{r_n}^{(F_2, \varphi_2)} Ax_n \right) - \left(Ax_n - T_{r_n}^{F_2} Ax_n \right),$$

$$Ax_n - T_{r_n}^{(F_2, \varphi_2)} Ax_n \rangle$$

$$\leq \|x_n - p\|^2 + L\gamma^2 \left\| Ax_n - T_{r_n}^{(F_2, \varphi_2)} Ax_n \right\|^2$$

$$+ 2\gamma \left(\langle Ap - T_{r_n}^{(F_2, \varphi_2)} Ax_n, Ax_n - T_{r_n}^{(F_2, \varphi_2)} Ax_n \rangle - \left\| Ax_n - T_{r_n}^{(F_2, \varphi_2)} Ax_n \right\|^2 \right)$$

$$\leq \|x_n - p\|^2 + L\gamma^2 \left\| Ax_n - T_{r_n}^{(F_2, \varphi_2)} Ax_n \right\|^2$$

$$+ 2\gamma \left(\frac{1}{2} \left\| Ax_n - T_{r_n}^{(F_2, \varphi_2)} Ax_n \right\|^2 - \left\| Ax_n - T_{r_n}^{(F_2, \varphi_2)} Ax_n \right\|^2 \right)$$

$$= \|x_n - p\|^2 + \gamma (L\gamma - 1) \left\| Ax_n - T_{r_n}^{(F_2, \varphi_2)} Ax_n \right\|^2.$$

Thus by (3.8) we have

$$\|z_n - p\|^2 \leq \alpha_n^{(0)} \|x_n - p\|^2 + \sum_{i=1}^{n} \alpha_n^{(i)} \|u_n - p\|^2$$

$$\leq \alpha_n^{(0)} \|x_n - p\| + \sum_{i=1}^{n} \alpha_n^{(i)} \left(\|x_n - p\|^2 + \gamma (L\gamma - 1) \left\| Ax_n - T_{r_n}^{(F_2, \varphi_2)} Ax_n \right\|^2 \right)$$

$$= \|x_n - p\|^2 + \gamma (L\gamma - 1) \sum_{i=1}^{n} \alpha_n^{(i)} \left\| Ax_n - T_{r_n}^{(F_2, \varphi_2)} Ax_n \right\|^2$$

$$= \|x_n - p\|^2 - \gamma (1 - L\gamma) \left(1 - \alpha_n^{(0)} \right) \left\| Ax_n - T_{r_n}^{(F_2, \varphi_2)} Ax_n \right\|^2. \tag{3.10}$$

Therefore we have

$$\gamma (1 - L\gamma) \left(1 - \alpha_n^{(0)} \right) \left\| Ax_n - T_{r_n}^{(F_2, \varphi_2)} Ax_n \right\|^2 \leq \|x_n - p\|^2 - \|z_n - p\|^2$$

$$\leq \|x_n - z_n\| \left(\|x_n - p\| + \|z_n - p\| \right).$$

By the given control condition on $\{\alpha_n^{(0)}\}$, $\gamma (1 - L\gamma) > 0$, and (3.7) we obtain that

$$\lim_{n \to \infty} \left\| Ax_n - T_{r_n}^{(F_2, \varphi_2)} Ax_n \right\| = 0. \tag{3.11}$$

Since $T_{r_n}^{(F_1, \varphi_1)}$ is firmly nonexpansive and $I - \gamma A^* (I - T_{r_n}^{(F_2, \varphi_2)}) A$ is nonexpansive, we have

$$\|u_n - p\|^2 = \left\| T_{r_n}^{(F_1, \varphi_1)} \left(I - \gamma A^* \left(I - T_{r_n}^{(F_2, \varphi_2)} \right) A \right) x_n - T_{r_n}^{(F_1, \varphi_1)} p \right\|^2$$

$$\leq \langle T_{r_n}^{(F_1, \varphi_1)} \left(I - \gamma A^* \left(I - T_{r_n}^{(F_2, \varphi_2)} \right) A \right) x_n - T_{r_n}^{F_1} p,$$

$$\left(I - \gamma A^* \left(I - T_{r_n}^{(F_2, \varphi_2)} \right) A \right) x_n - p \rangle$$

$$= \langle u_n - p, \left(I - \gamma A^* \left(I - T_{r_n}^{(F_2, \varphi_2)} \right) A \right) x_n - p \rangle$$

$$= \frac{1}{2} \left(\|u_n - p\|^2 + \left\| \left(I - \gamma A^* \left(I - T_{r_n}^{(F_2, \varphi_2)} \right) A \right) x_n - p \right\|^2$$

$$- \left\| u_n - x_n - \gamma A^* \left(I - T_{r_n}^{(F_2, \varphi_2)} \right) Ax_n \right\|^2 \right)$$

$$\leq \frac{1}{2} \big(\|u_n - p\|^2 + \|x_n - p\|^2 - \big(\|u_n - x_n\|^2 + \gamma^2 \big\| A^* \big(I - T_{r_n}^{(F_2,\varphi_2)} \big) A x_n \big\|^2$$
$$- 2\gamma \big\langle u_n - x_n, A^* \big(I - T_{r_n}^{(F_2,\varphi_2)} \big) A x_n \big\rangle \big) \big),$$

which implies that

$$\|u_n - p\|^2 \leq \|x_n - p\|^2 - \|u_n - x_n\|^2 + 2\gamma \big\langle u_n - x_n, A^* \big(I - T_{r_n}^{(F_2,\varphi_2)} \big) A x_n \big\rangle$$
$$\leq \|x_n - p\|^2 - \|u_n - x_n\|^2 + 2\gamma \|u_n - x_n\| \big\| A^* \big(I - T_{r_n}^{(F_2,\varphi_2)} \big) A x_n \big\|. \qquad (3.12)$$

This implies by (3.8) that

$$\|z_n - p\|^2 \leq \alpha_n^{(0)} \|x_n - p\|^2 + \sum_{i=1}^n \alpha_n^{(i)} \|u_n - p\|^2$$
$$\leq \alpha_n^{(0)} \|x_n - p\| + \sum_{i=1}^n \alpha_n^{(i)} \big(\|x_n - p\|^2 - \|u_n - x_n\|^2$$
$$+ 2\gamma \|u_n - x_n\| \big\| A^* \big(I - T_{r_n}^{(F_2,\varphi_2)} \big) A x_n \big\| \big)$$
$$= \|x_n - p\|^2 - \big(1 - \alpha_n^{(0)} \big) \|u_n - x_n\|^2$$
$$+ 2\gamma \big(1 - \alpha_n^{(0)} \big) \|u_n - x_n\| \big\| A^* \big(I - T_{r_n}^{(F_2,\varphi_2)} \big) A x_n \big\|$$
$$= \|x_n - p\|^2 - \big(1 - \alpha_n^{(0)} \big) \|u_n - x_n\|^2$$
$$+ 2\gamma \big(1 - \alpha_n^{(0)} \big) \|u_n - x_n\| \big\| A^* \big(I - T_{r_n}^{(F_2,\varphi_2)} \big) A x_n \big\|.$$

Therefore we have

$$\big(1 - \alpha_n^{(0)} \big) \|u_n - x_n\|^2$$
$$\leq \|x_n - p\|^2 - \|z_n - p\|^2 + 2\gamma \big(1 - \alpha_n^{(0)} \big) \|u_n - x_n\| \big\| A^* \big(I - T_{r_n}^{(F_2,\varphi_2)} \big) A x_n \big\|$$
$$\leq \|x_n - p\|^2 - \|z_n - p\|^2 + 2\gamma \big(1 - \alpha_n^{(0)} \big) M \big\| A^* \big(I - T_{r_n}^{(F_2,\varphi_2)} \big) A x_n \big\|$$
$$\leq \|x_n - z_n\| \big(\|x_n - p\| + \|z_n - p\| \big) + 2\gamma \big(1 - \alpha_n^{(0)} \big) M \big\| A^* \big(I - T_{r_n}^{(F_2,\varphi_2)} \big) A x_n \big\|,$$

where $M = \sup\{\|u_n - x_n\| : n \in \mathbb{N}\}$. This implies by Condition (C1), (3.7), and (3.11) that

$$\lim_{n \to \infty} \|u_n - x_n\| = 0. \qquad (3.13)$$

Step 5. We show that $q \in \bigcap_{i=1}^\infty F(S_i)$.

By (3.9) and (3.13), for all $i \in \mathbb{N}$, we get that

$$\lim_{n \to \infty} \mathrm{dist}(u_n, S_i u_n) \leq \lim_{n \to \infty} \big\| u_n - y_n^{(i)} \big\|$$
$$\leq \lim_{n \to \infty} \|u_n - x_n\| + \lim_{n \to \infty} \big\| x_n - y_n^{(i)} \big\|$$
$$= 0. \qquad (3.14)$$

For each $i \in \mathbb{N}$, we get

$$
\begin{aligned}
\text{dist}(q, S_i q) &\leq \|q - u_n\| + \|u_n - y_n^{(i)}\| + \text{dist}(y_n^{(i)}, S_i q) \\
&\leq \|q - u_n\| + \text{dist}(u_n, S_i u_n) + \mathcal{H}(S_i u_n, S_i q) \\
&\leq 2\|q - u_n\| + \text{dist}(u_n, S_i u_n) \\
&\leq 2(\|q - z_n\| + \|z_n - x_n\|) + \text{dist}(u_n, S_i u_n).
\end{aligned}
$$

Since $\lim_{n \to \infty} z_n = q$, it follows by (3.7) and (3.14) that

$$
\text{dist}(q, S_i q) = 0 \quad \text{for all } i \in \mathbb{N}.
$$

This shows that $q \in S_i q$ for all $i \in \mathbb{N}$, and hence $q \in \bigcap_{i=1}^{\infty} F(S_i)$.

Step 6. We show that $q \in \text{SGEP}(F_1, \varphi_1, F_2, \varphi_2)$.

First, we will show that $q \in \text{GEP}(F_1, \varphi_1)$.

Since $u_n = T_{r_n}^{(F_1, \varphi_1)}(I - \gamma A^*(I - T_{r_n}^{(F_2, \varphi_2)})A)x_n$, we have

$$
F_1(u_n, y) + \varphi_1(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n - \gamma A^*(I - T_{r_n}^{(F_2, \varphi_2)})Ax_n\rangle \geq 0, \quad \forall y \in C,
$$

which implies that

$$
F_1(u_n, y) + \varphi_1(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle - \frac{1}{r_n}\langle y - u_n, \gamma A^*(I - T_{r_n}^{(F_2, \varphi_2)})Ax_n\rangle \geq 0, \quad \forall y \in C.
$$

It follows from the monotonicity of F_1 and φ_1 that

$$
\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle - \frac{1}{r_n}\langle y - u_n, \gamma A^*(I - T_{r_n}^{(F_2, \varphi_2)})Ax_n\rangle \geq F_1(y, u_n) + \varphi_1(y, u_n), \quad \forall y \in C.
$$

By (3.13) and $\lim_{n \to \infty} x_n = q$ we get that $\lim_{n \to \infty} u_n = q$. It follows by Condition (C2), (3.11), (3.13), Assumption 2.7, (A4) and (A7), that $0 \geq F_1(y, q) + \varphi_1(y, q)$ for all $y \in C$. Put $y_t = ty + (1 - t)q$ for all $t \in (0, 1]$ and $y \in C$. Consequently, we get $y_t \in C$, and hence $F_1(y_t, q) + \varphi_1(y_t, q) \leq 0$. So by Assumption 2.7, (A1)–(A7), we have

$$
\begin{aligned}
0 &\leq F_1(y_t, y_t) + \varphi_1(y_t, y_t) \\
&\leq t(F_1(y_t, y) + \varphi_1(y_t, y)) + (1 - t)(F_1(y_t, q) + \varphi_1(y_t, q)) \\
&\leq t(F_1(y_t, y) + \varphi_1(y_t, y)) + (1 - t)(F_1(q, y_t) + \varphi_1(q, y_t)) \\
&\leq F_1(y_t, y) + \varphi_1(y_t, y).
\end{aligned}
$$

Hence we have

$$
F_1(y_t, y) + \varphi_1(y_t, y) \geq 0, \quad \forall y \in C.
$$

Letting $t \to 0$, by Assumption 2.7 (A3) and the upper hemicontinuity of φ_1 we have

$$
F_1(q, y) + \varphi_1(q, y) \geq 0, \quad \forall y \in C.
$$

This implies that $q \in \text{GEP}(F_1, \varphi_1)$.

Next, we show that $Aq \in \text{GEP}(F_2, \varphi_2)$.

Since A is a bounded linear operator, we have $Ax_n \to Aq$. Then, it follows from (3.11) that

$$T_{r_n}^{(F_2,\varphi_2)} Ax_n \to Aq. \tag{3.15}$$

By the definition of $T_{r_n}^{(F_2,\varphi_2)} Ax_n$ we have

$$F_2\big(T_{r_n}^{(F_2,\varphi_2)} Ax_n, y\big) + \varphi_2\big(T_{r_n}^{(F_2,\varphi_2)} Ax_n, y\big) + \frac{1}{r_n}\big\langle y - T_{r_n}^{(F_2,\varphi_2)} Ax_n, T_{r_n}^{(F_2,\varphi_2)} Ax_n - Ax_n \big\rangle \geq 0$$

for all $y \in Q$. Since F_2 and φ_2 are upper semicontinuous in the first argument, it follows by (3.15) that

$$F_2(Aq, y) + \varphi_2(Aq, y) \geq 0, \quad \forall y \in Q.$$

This shows that $Aq \in \text{GEP}(F_2, \varphi_2)$. Therefore $q \in \text{SGEP}(F_1, \varphi_1, F_2, \varphi_2)$.

By Steps 5 and 6 we get that $q \in \Gamma$.

Step 7. Finally, we show that $q = P_\Gamma x_1$.

Since $x_n = P_{C_n} x_1$ and $\Gamma \subset C_n$, we obtain $\langle x_1 - x_n, x_n - p \rangle \geq 0$ for all $p \in \Gamma$. Thus we get $\langle x_1 - q, q - p \rangle \geq 0$ for all $p \in \Gamma$. This shows that $q = P_\Gamma x_1$.

By Steps 1–7 we can conclude that $\{x_n\}$ converges strongly to $P_\Gamma x_1$. This completes the proof.   □

If $\varphi_1 = \varphi_2 = 0$, then the split generalized equilibrium problem reduces to the split equilibrium problem. So, the following result can be immediately obtained from Theorem 3.1.

Corollary 3.2 *Let C be a nonempty closed convex subset of a real Hilbert space H_1, and let Q be a nonempty closed convex subset of a real Hilbert space H_2. Let $A : H_1 \to H_2$ be a bounded linear operator, and let $\{S_i\}$ be a countable family of nonexpansive multivalued mappings of C into $CB(C)$. Let $F_1 : C \times C \to \mathbb{R}$, $F_2 : Q \times Q \to \mathbb{R}$ be bifunctions satisfying Assumption 2.7. Let F_2 be upper semicontinuous in the first argument. Assume that $\Gamma = \bigcap_{i=1}^{\infty} F(S_i) \cap \text{SEP}(F_1, F_2) \neq \emptyset$ and $S_i p = \{p\}$ for each $p \in \bigcap_{i=1}^{\infty} F(S_i)$. Let $x_1 \in C$ with $C_1 = C$, and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} u_n = T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{F_2})A)x_n, \\ z_n = \alpha_n^{(0)} x_n + \alpha_n^{(1)} y_n^{(1)} + \cdots + \alpha_n^{(n)} y_n^{(n)}, \quad y_n^{(i)} \in S_i u_n, \\ C_{n+1} = \{p \in C_n : \|z_n - p\| \leq \|x_n - p\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad n \in \mathbb{N}, \end{cases} \tag{3.16}$$

where $\{\alpha_n^{(i)}\} \subset (0, 1)$ satisfy $\sum_{i=0}^{n} \alpha_n^{(i)} = 1$, $\{r_n\} \subset (0, \infty)$, and $\gamma \in (0, \frac{1}{L})$, where L is the spectral radius of A^*A, and A^* is the adjoint of A. Assume that the following conditions hold:

(C1) *The limits $\lim_{n\to\infty} \alpha_n^{(i)} \in (0, 1)$ exist for all $i \geq 0$,*

(C2) *$\liminf_{n\to\infty} r_n > 0$.*

Then the sequence $\{x_n\}$ generated by (3.16) converges strongly to $P_\Gamma x_1$.

If $F_1 = F_2 = F$, $H_1 = H_2 = H$, and $\varphi_1 = \varphi_2 = 0$, then the following result can be immediately obtained from Theorem 3.1.

Corollary 3.3 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : H \to H$ be a bounded linear operator, and let $\{S_i\}$ be a countable family of nonexpansive multivalued mappings of C into $CB(C)$. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying Assumption 2.7. Assume that $\Gamma = \bigcap_{i=1}^{\infty} F(S_i) \cap EP(F) \neq \emptyset$ and $S_i p = \{p\}$ for each $p \in \bigcap_{i=1}^{\infty} F(S_i)$. Let $x_1 \in C$ with $C_1 = C$, and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} u_n = T_{r_n}^F (I - \gamma A^* (I - T_{r_n}^F) A) x_n, \\ z_n = \alpha_n^{(0)} x_n + \alpha_n^{(1)} y_n^{(1)} + \cdots + \alpha_n^{(n)} y_n^{(n)}, \quad y_n^{(i)} \in S_i u_n, \\ C_{n+1} = \{p \in C_n : \|z_n - p\| \le \|x_n - p\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad n \in \mathbb{N}, \end{cases} \tag{3.17}$$

where $\{\alpha_n^{(i)}\} \subset (0,1)$ satisfy $\sum_{i=0}^{n} \alpha_n^{(i)} = 1$, $\{r_n\} \subset (0, \infty)$, and $\gamma \in (0, \frac{1}{L})$, where L is the spectral radius of A^*A, and A^* is the adjoint of A. Assume that the following conditions hold:

(C1) *The limits $\lim_{n \to \infty} \alpha_n^{(i)} \in (0,1)$ exist for all $i \ge 0$,*

(C2) $\liminf_{n \to \infty} r_n > 0$.

Then the sequence $\{x_n\}$ generated by (3.17) converges strongly to $P_\Gamma x_1$.

4 Numerical example

In this section, we present a numerical example to demonstrate the performance and convergence of our theoretical results. All codes were written in Scilab.

Example 4.1 Let $H_1 = H_2 = \mathbb{R}$ and $C = Q = [0, 10]$. Let $A : H_1 \to H_2$ be defined by $Ax = x$ for each $x \in H_1$. Then $A^* y = y$ for each $y \in H_2$. For $x \in C$, $i = 1, 2, \ldots$, we define the multivalued mappings S_i on C as follows:

$$S_i x = \left[0, \frac{x}{10i} \right] \quad \text{for all } i \in \mathbb{N}.$$

Obviously, S_i is nonexpansive for all $i \in \mathbb{N}$, $S_i(0) = \{0\}$, and $\bigcap_{i=1}^{\infty} F(S_i) = \{0\}$. Define the bifunctions $F_1, \varphi_1 : C \times C \to \mathbb{R}$ by $F_1(x, y) = y^2 + 3xy - 4x^2$ and $\varphi_1(x, y) = y^2 - x^2$ for $x, y \in C$. Define $F_2, \varphi_2 : Q \times Q \to \mathbb{R}$ by $F_2(w, v) = 2v^2 + wv - 3w^2$ and $\varphi_2(w, v) = w - v$ for $w \in Q$ and $v \in Q$. Choose $r_n = \frac{n}{n+1}$, $\gamma = \frac{1}{10}$, and the sequences $\{\lambda_n^{(i)}\}$ defined by

$$\lambda_n^{(i)} = \begin{cases} \frac{1}{b^{i+1}} \left(\frac{n}{n+1} \right), & n \ge i + 1, \\ 1 - \frac{n}{n+1} \left(\sum_{k=1}^{n} \frac{1}{b^k} \right), & n = i, \\ 0, & n < i, \end{cases}$$

where $b > 1$. It is easy to check that F_1, F_2, φ_1, φ_2, $\{\lambda_n^{(i)}\}$, and $\{r_n\}$ satisfy all the conditions in Theorem 3.1 with $\Gamma = \{0\}$.

For all $x \in C$ and $n \in \mathbb{N}$, we compute $T_r^{(F_2, \varphi_2)} Ax$. Find w such that

$$0 \leq F_2(w, v) + \varphi_2(w, v) + \frac{1}{r} \langle v - w, w - Ax \rangle$$

$$= 2v^2 + wv - 3w^2 + w - v + \frac{1}{r}(v - w)(w - x)$$

$$\Leftrightarrow$$

$$0 \leq 2rv^2 + rwv - 3rw^2 + r_n w - rv + (v - w)(w - x)$$

$$= 2rv^2 + rwv - 3rw^2 + rw - rv + wv - vx - w^2 + wx$$

$$= 2rv^2 + (rw - r + w - x)v + \left(-3rw^2 + rw - w^2 + wx\right)$$

for all $v \in Q$. Let $J_2(v) = 2rv^2 + (rw - r + w - x)v + (-3rw^2 + rw - w^2 + wx)$. $J_2(v)$ is s a quadratic function of v with coefficients $a = 2r$, $b = rw - r - x - w$, and $c = -3rw^2 + rw - w^2 + wx$. Determine the discriminant Δ of J_2:

$$\Delta = b^2 - 4ac$$

$$= (rw - r + w - x)^2 - 4(2r)\left(-3rw^2 + rw - w^2 + wx\right)$$

$$= 25r^2 w^2 - 10r^2 w + 10rw^2 - 10rwx + r^2 - 2rw + 2rx + w^2 - 2wx + x^2$$

$$= \left(25r^2 + 10r + 1\right)w^2 + \left(-10r^2 - 10rx - 2r - 2x\right)w + \left(2rx + x^2 + r^2\right)$$

$$= (5r + 1)^2 w^2 - 2w(5r + 1)(x + r) + (x + r)^2$$

$$= \left((5r + 1)w - (x + r)\right)^2.$$

We know that $J_2(v) \geq 0$ for all $v \in \mathbb{R}$. If it has at most one solution in $\mathbb{R}$, then $\Delta \leq 0$, so we have

$$w = \frac{x + r}{5r + 1}.$$

This implies that

$$T_r^{(F_2, \varphi_2)} Ax = \frac{x + r}{5r + 1}.$$

Furthermore, we get

$$\left(I - \gamma A^* \left(I - T_r^{(F_2, \varphi_2)}\right)A\right)x = x - \gamma A^*\left(Ax - T_r^{(F_2, \varphi_2)} Ax\right)$$

$$= x - \frac{1}{10}A^*\left(x - \frac{x + r}{5r + 1}\right)$$

$$= x - \frac{1}{10}\left(\frac{5rx - r}{5r + 1}\right)$$

$$= \frac{45xr + 10x + r}{10(5r + 1)}.$$

Next, we find $u \in C$ such that $F_1(u,z) + \varphi_1(u,z) + \frac{1}{r}\langle z - u, u - s\rangle \geq 0$ for all $z \in C$, where $s = (I - \gamma A^*(I - T_r^{(F_2,\varphi_2)})A)x$. Note that

$$0 \leq F_1(u,z) + \varphi_1(u,z) + \frac{1}{r}\langle z - u, u - s\rangle$$

$$= 2z^2 + 3uz - 5u^2 + \frac{1}{r}\langle v - u, u - s\rangle$$

$$\Leftrightarrow$$

$$0 \leq 2rz^2 + 3ruz - 5ru^2 + (z - u)(u - s)$$

$$= 2rz^2 + 3ruz - 5ru^2 + uz - sz - u^2 + us$$

$$= 2rz^2 + (3ru + u - s)z + \left(-5ru^2 - u^2 + us\right)$$

for all $z \in C$. Let $J_1(z) = 2rz^2 + (3ru + u - s)z + (-5ru^2 - u^2 + us)$. $J_1(z)$ be a quadratic function of z with coefficients $a = 2r$, $b = 3ru + u - s$, and $c = -5ru^2 - u^2 + us$. Determine the discriminant Δ of J_1:

$$\Delta = (3ru + u - s)^2 - 4(2r)\left(-5ru^2 - u^2 + us\right)$$

$$= 49r^2u^2 + 14ru^2 - 14rus + u^2 - 2us + s^2$$

$$= \left((7r + 1)u - s\right)^2.$$

We know that $J_1(z) \geq 0$ for all $z \in \mathbb{R}$. If it has at most one solution in $\mathbb{R}$, then $\Delta \leq 0$, so we have

$$u = \frac{s}{7r + 1}.$$

This implies that

$$u_n = T_{r_n}^{(F_1,\varphi_1)}\left(I - \gamma A^*\left(I - T_{r_n}^{(F_2,\varphi_2)}\right)A\right)x_n,$$

$$= \frac{45x_n r_n + 10x_n + r_n}{10(5r_n + 1)(7r_n + 1)}.$$

We put $y_n^{(i)} = \frac{u_n}{10i}$ for all $i \in \mathbb{N}$. Then algorithm (3.1) becomes:

$$\begin{cases} u_n = \frac{45x_n r_n + 10x_n + r_n}{10(5r_n + 1)(7r_n + 1)}, \quad r_n = \frac{n}{n+1}, \\ z_n = \lambda_n^{(0)}x_n + \frac{\lambda_n^{(1)}u_n}{10} + \frac{\lambda_n^{(2)}u_n}{20} + \cdots + \frac{\lambda_n^{(n)}u_n}{10n}, \\ C_{n+1} = \{p \in C_n : \|z_n - p\| \leq \|x_n - p\|\}, \\ x_{n+1} = P_{C_{n+1}}x_1, \quad n \in \mathbb{N}. \end{cases} \tag{4.1}$$

For arbitrary $x_1 \in C = C_1 = [0, 10]$, we get that $0 \leq z_1 \leq x_1 \leq 10$. Then $C_2 = \{p \in C_1 : |z_1 - p| \leq |x_1 - p|\} = [0, \frac{x_1 + z_1}{2}]$. Since $\frac{x_1 + z_1}{2} \leq x_1$, it follows that $x_2 = P_{C_2}x_1 = \frac{x_1 + z_1}{2}$. Continuing this process, we get $C_{n+1} = [0, \frac{x_n + z_n}{2}]$, and hence $x_{n+1} = P_{C_{n+1}}x_1 = \frac{x_n + z_n}{2}$. Now, we rewrite
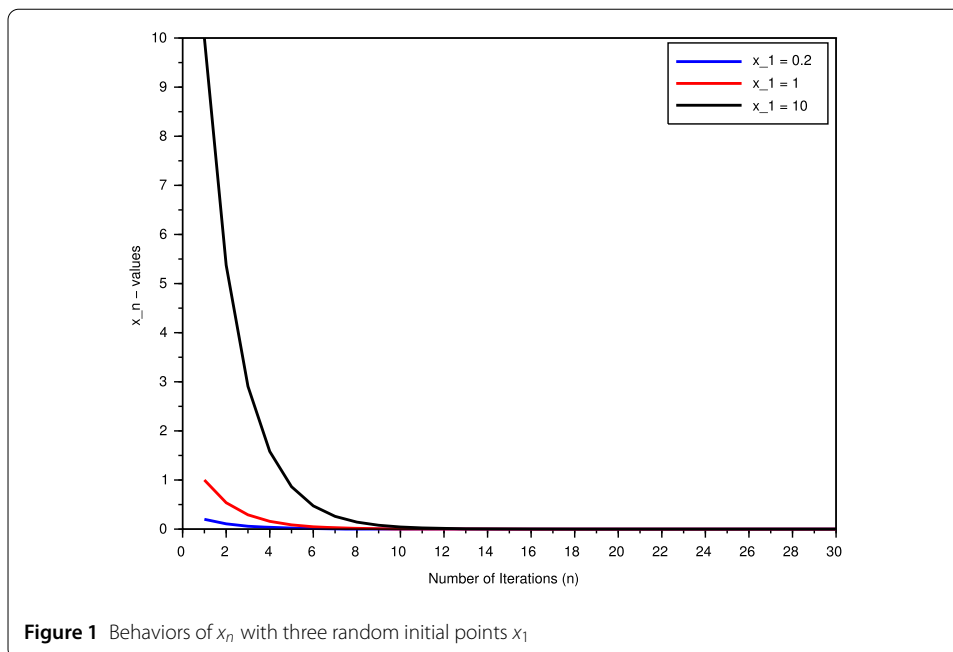
Figure 1 Behaviors of x_n with three random initial points x_1

algorithm (4.1) as follows:

$$
\begin{cases}
u_n = \frac{45x_n r_n + 10x_n + r_n}{10(5r_n+1)(7r_n+1)}, & r_n = \frac{n}{n+1}, \\
z_n = \lambda_n^{(0)} x_n + \frac{\lambda_n^{(1)} u_n}{10} + \frac{\lambda_n^{(2)} u_n}{20} + \cdots + \frac{\lambda_n^{(n)} u_n}{10n}, \\
x_{n+1} = \frac{x_n + z_n}{2}, & n \in \mathbb{N}.
\end{cases}
\tag{4.2}
$$

In this example, we set the parameter on $\{\lambda_n^{(i)}\}$ by $b = 9$. Then we obtain

$$
\left(\lambda_n^{(i)}\right) =
\begin{pmatrix}
\frac{1}{18} & \frac{17}{18} & 0 & 0 & 0 & \cdots & 0 & \cdots \\
\frac{2}{27} & \frac{2}{243} & \frac{223}{243} & 0 & 0 & \cdots & 0 & \cdots \\
\frac{1}{12} & \frac{1}{108} & \frac{1}{972} & \frac{881}{972} & 0 & \cdots & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \\
\frac{n}{9(n+1)} & \frac{n}{9^2(n+1)} & \frac{n}{9^3(n+1)} & \frac{n}{9^4(n+1)} & \frac{n}{9^5(n+1)} & \cdots & \frac{n}{9^i(n+1)} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots &
\end{pmatrix}.
$$

Figure 1 indicates the behavior of x_n for algorithm (4.2), which converges to the same solution, that is, $0 \in \Gamma$ as a solution of this example.

Now, we test the effect of the parameters in $\{\lambda_n^{(i)}\}$ on the convergence of algorithm (4.2). In this test, Figure 2 presents the behavior of x_n by choosing three different parameters in $\{\lambda_n^{(i)}\}$, that is, $b = 2$, $b = 9$, and $b = 100$.

5 Conclusions

The results presented in this paper modify, extend, and improve the corresponding results of Kazmi and Rizvi [7], Suantai et al. [15], and others. The main aim of this paper is to propose an iterative algorithm based on the shrinking projection method to find an element

Figure 2 Behaviors of x_n with three different parameters in $\{\lambda_n^{(i)}\}$
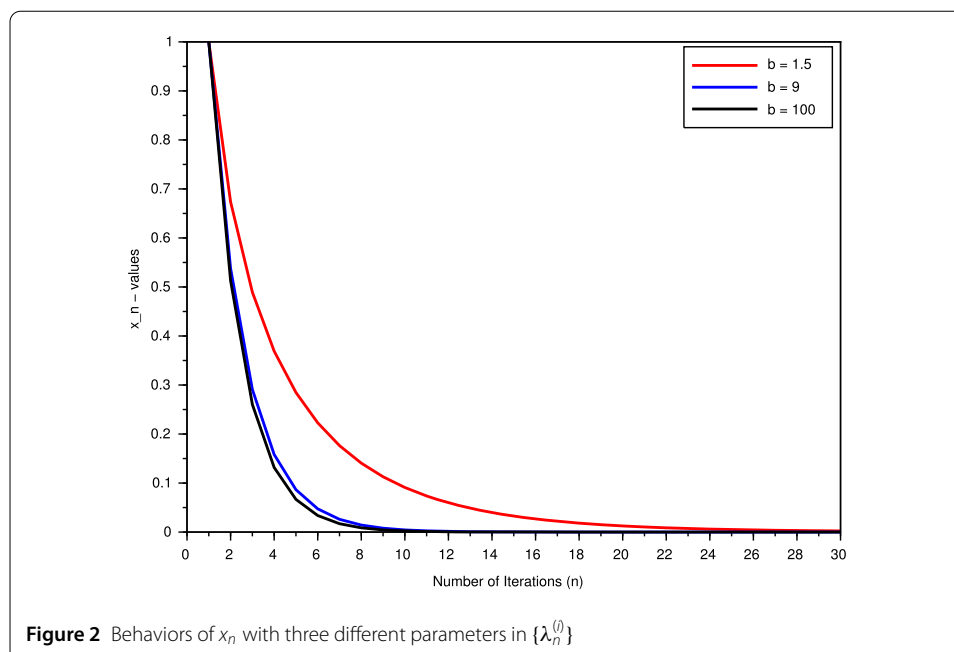
for solving a class of split generalized equilibrium problems and fixed point problems for a countable family of nonexpansive multivalued mappings in real Hilbert spaces.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed equally and significantly in writing this paper. Both authors read and approved the final manuscript.

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