Shoaib et al. *Fixed Point Theory and Applications* (2018) 2018:2
https://doi.org/10.1186/s13663-017-0626-8

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH

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Fixed point results for fuzzy mappings in a b-metric space

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Abstract
In this paper, we establish some fixed point results for fuzzy mappings in a complete dislocated b-metric space. Our results generalize and extend the results of Joseph *et al.* (SpringerPlus 5:Article ID 217, 2016). We also give examples to support our results, and applications relating the results to a fixed point for multivalued mappings and fuzzy mappings are studied.

MSC: 46S40; 47H10; 54H25

Keywords: fixed point; closed ball; complete dislocated b-metric space; fuzzy mapping; Hausdorff metric space

1 Introduction and preliminaries

Fixed point theory plays an important role in various fields of mathematics. It provides very important tools for finding the existence and uniqueness of solutions. The Banach contraction theorem has an important role in fixed point theory, and it has become very popular due to iterations which can be easily implemented on the computers. The idea of a fuzzy set was first laid down by Zadeh [2]. Later on Weiss [3] and Butnariu [4] gave the idea of a fuzzy mapping and obtained many fixed point results. Afterward, Heilpern [5] initiated the idea of fuzzy contraction mappings and proved a fixed point theorem for fuzzy contraction mappings which is a fuzzy analogue of Nadler's [6] fixed point theorem for multivalued mappings.

Recently, Beg *et al.* [7] proved the result concerning the existence of fixed points of a mapping satisfying locally contractive conditions on a closed ball (see also [8–16]). It is also possible that the mapping satisfies locally contractive conditions on a sequence contained in a closed ball in M. One can obtain fixed point results for such a mapping by using the suitable conditions.

The notion of dislocated topologies has useful applications in the context of logic programming semantics (see [17]). A dislocated metric space (metric-like space) (see [18, 19]) is a generalization of partial metric space (see [20]). Aydi *et al.* [21] established a fixed point theorem for set-valued quasi contraction in b-metric spaces. Nawab *et al.* [22] introduced the new concept of dislocated b-metric space as a generalization of metric space and established to prove some common fixed point results for four mappings satisfying the generalized weak contractive conditions in a partially ordered dislocated b-metric space.

In this paper, we obtain a fixed point and a common fixed point for fuzzy mappings for a generalized contraction on a closed ball in a complete b-metric space. An example which supports the proved results is also given. We give the following definitions and results which will be needed in the sequel.

Definition 1.1 ([22]) Let X be a nonempty set. A function $d_{lb}: X \times X \to [0,\infty)$ is called dislocated b-metric (or simply d_{lb}-metric) if, for any $x,y,z \in X$ and $b \geq 1$, the following conditions hold:

(i) If $d_{lb}(x,y) = 0$, then $x = y$;

(ii) $d_{lb}(x,y) = d_{lb}(y,x)$;

(iii) $d_{lb}(x,y) \leq b[d_{lb}(x,z) + d_{lb}(z,y)]$.

The pair (X,d_{lb}) is called a dislocated b-metric space. It should be noted that the class of d_{lb} metric spaces is effectively larger than that of d_l metric spaces, since d_{lb} is a d_l metric when $b = 1$.

It is clear that if $d_{lb}(x,y) = 0$, then from (i), $x = y$. But if $x = y$, $d_{lb}(x,y)$ may not be 0. For $x \in X$ and $\varepsilon > 0$, $\overline{B(x,\varepsilon)} = \{y \in X : d_{lb}(x,y) \leq \varepsilon\}$ is a closed ball in (X,d_{lb}).

Example 1.2 If $X = \mathbb{R}^+ \cup \{0\}$, then $d_{lb}(x,y) = (x+y)^2$ defines a dislocated b-metric d_{lb} on X.

Definition 1.3 ([22]) Let (X,d_{lb}) be a dislocated b-metric space.

(i) A sequence $\{x_n\}$ in (X,d_{lb}) is called Cauchy sequence if, given $\varepsilon > 0$, there corresponds $n_0 \in N$ such that, for all $n,m \geq n_0$, we have $d_{lb}(x_m,x_n) < \varepsilon$ or $\lim_{n,m\to\infty} d_{lb}(x_n,x_m) = 0$.

(ii) A sequence $\{x_n\}$ dislocated b-converges (for short d_{lb}-converges) to x if $\lim_{n\to\infty} d_{lb}(x_n,x) = 0$. In this case x is called a d_{lb}-limit of $\{x_n\}$.

Definition 1.4 Let K be a nonempty subset of dislocated b-metric space X, and let $x \in X$. An element $y_0 \in K$ is called a best approximation in K if

$$d_{lb}(x,K) = d_{lb}(x,y_0), \quad \text{where } d_{lb}(x,K) = \inf_{y\in K} d_{lb}(x,y).$$

If each $x \in X$ has at least one best approximation in K, then K is called a proximinal set. We denote by $P(X)$ the set of all proximinal subsets of X.

Definition 1.5 The function $H_{d_{lb}}: P(X) \times P(X) \to R^+$, defined by

$$H_{d_{lb}}(A,B) = \max\Big\{\sup_{a\in A} d_{lb}(a,B), \sup_{b\in B} d_{lb}(A,b)\Big\},$$

is called dislocated Hausdorff b-metric on $P(X)$.

A fuzzy set in X is a function with domain X and values in $[0,1]$, $F(X)$ is the collection of all fuzzy sets in X. If A is a fuzzy set and $x \in X$, then the function value $A(x)$ is called the grade of membership of x in A. The α-level set of a fuzzy set A is denoted by $[A]_\alpha$ and defined as follows:

$$[A]_\alpha = \big\{x : A(x) \geq \alpha\big\}, \quad \text{where } \alpha \in (0,1],$$

$$[A]_0 = \overline{\big\{x : A(x) > 0\big\}}.$$

Let X be any nonempty set and Y be a metric space. A mapping T is called fuzzy mapping if T is a mapping from X into $F(Y)$. A fuzzy mapping T is a fuzzy subset on $X \times Y$ with membership function $T(x)(y)$. The function $T(x)(y)$ is the grade of membership of y in $T(x)$. For convenience, we denote the α-level set of $T(x)$ by $[Tx]_\alpha$ instead of $[T(x)]_\alpha$ [23].

Definition 1.6 ([23]) A point $x \in X$ is called a fuzzy fixed point of a fuzzy mapping $T : X \to F(X)$ if there exists $\alpha \in (0,1]$ such that $x \in [Tx]_\alpha$.

Lemma 1.7 *Let A and B be nonempty proximal subsets of a dislocated b-metric space (X, d_{lb}). If $a \in A$, then*

$$d(a,B) \leq H(A,B).$$

Lemma 1.8 *Let (X, d_{lb}) be a dislocated metric space. Let $(P(X), H_{d_{lb}})$ be a dislocated Hausdorff b-metric space. Then, for all $A, B \in P(X)$ and for each $a \in A$, there exists $b_a \in B$ satisfying*

$$d_{lb}(a,B) = d_{lb}(a,b_a),$$

then

$$H_{d_{lb}}(A,B) \geq d_{lb}(a,b_a).$$

2 Main results

Theorem 2.1 *Let (X, d_{lb}) be a complete dislocated b-metric space with constant $b \geq 1$. Let $T : X \to F(X)$ be a fuzzy mapping, and let x_0 be any arbitrary point in X. Suppose there exists $\alpha(x) \in (0,1]$ for all $x \in X$ satisfying the following conditions*:

$$\begin{aligned} &H_{d_{lb}}\bigl([Tx]_{\alpha(x)}, [Ty]_{\alpha(y)}\bigr) \\ &\quad \leq a_1 d_{lb}\bigl(x, [Tx]_{\alpha(x)}\bigr) + a_2 d_{lb}\bigl(y, [Ty]_{\alpha(y)}\bigr) + a_3 d_{lb}\bigl(x, [Ty]_{\alpha(y)}\bigr) + a_4 d_{lb}\bigl(y, [Tx]_{\alpha(x)}\bigr) \\ &\qquad + a_5 d_{lb}(x,y) + a_6 \frac{d_{lb}(x, [Tx]_{\alpha(x)})(1 + d_{lb}(x, [Tx]_{\alpha(x)}))}{1 + d_{lb}(x,y)} \end{aligned} \tag{2.1}$$

and

$$d_{lb}\bigl(x_0, [Tx_0]_{\alpha(x_0)}\bigr) \leq \mu(1 - b\mu)r \tag{2.2}$$

for all $x, y \in \overline{B_{d_{lb}}(x_0, r)}$, $r > 0$ and $b\mu < 1$, where $\mu = \frac{(a_1 + ba_3 + a_5 + a_6)}{1 - (a_2 + ba_3)}$. Also, $a_i \geq 0$, where $i = 1, 2, \ldots, 6$ with $ba_1 + a_2 + b(1+b)a_3 + b(a_5 + a_6) < 1$ and $\sum_{i=1}^{6} a_i < 1$. Then there exists x^* in $\overline{B_{d_{lb}}(x_0, r)}$ such that $x^* \in [Tx^*]_{\alpha(x^*)}$.

Proof Let x_0 be any arbitrary point in X such that $x_1 \in [Tx_0]_{\alpha(x_0)}$. Consider the sequence $\{x_n\}$ of points in X such that $x_n \in [Tx_{n-1}]_{\alpha(x_{n-1})}$. First we show that $x_n \in \overline{B_{d_{lb}}(x_0, r)}$ for all $n \in \mathbb{N}$. Using (2.2), we get

$$d_{lb}(x_0, x_1) = d_{lb}\bigl(x_0, [Tx_0]_{\alpha(x_0)}\bigr) \leq \mu(1 - b\mu)r < r,$$

which implies $x_1 \in \overline{B_{d_{lb}}(x_0, r)}$. Let $x_2, x_3, \ldots, x_j \in \overline{B_{d_{lb}}(x_0, r)}$, $j \in \mathbb{N}$. Now, by using Lemma 1.8, we get

$$\begin{aligned} d_{lb}(x_j, x_{j+1}) &\le H_{d_{lb}}\big([Tx_{j-1}]_{\alpha(x_{j-1})}, [Tx_j]_{\alpha(x_j)}\big) \\ &\le a_1 d_{lb}\big(x_{j-1}, [Tx_{j-1}]_{\alpha(x_{j-1})}\big) + a_2 d_{lb}\big(x_j, [Tx_j]_{\alpha(x_j)}\big) \\ &\quad + a_3 d_{lb}\big(x_{j-1}, [Tx_j]_{\alpha(x_j)}\big) + a_4 d_{lb}\big(x_j, [Tx_{j-1}]_{\alpha(x_{j-1})}\big) \\ &\quad + a_5 d_{lb}(x_{j-1}, x_j) \\ &\quad + a_6 \frac{d_{lb}(x_{j-1}, [Tx_{j-1}]_{\alpha(x_{j-1})})(1 + d_{lb}(x_{j-1}, [Tx_{j-1}]_{\alpha(x_{j-1})}))}{1 + d_{lb}(x_{j-1}, x_j)} \\ &\le a_1 d_{lb}(x_{j-1}, x_j) + a_2 d_{lb}(x_j, x_{j+1}) + a_3 d_{lb}(x_{j-1}, x_{j+1}) \\ &\quad + a_4 d_{lb}(x_j, x_j) + a_5 d_{lb}(x_{j-1}, x_j) \\ &\quad + a_6 \frac{d_{lb}(x_{j-1}, x_j)(1 + d_{lb}(x_{j-1}, x_j))}{1 + d_{lb}(x_{j-1}, x_j)} \\ &\le a_1 d_{lb}(x_{j-1}, x_j) + a_2 d_{lb}(x_j, x_{j+1}) + ba_3\big[d_{lb}(x_{j-1}, x_j) \\ &\quad + d_{lb}(x_j, x_{j+1})\big] + a_5 d_{lb}(x_{j-1}, x_j) + a_6 d_{lb}(x_{j-1}, x_j), \\ d_{lb}(x_j, x_{j+1}) &\le \frac{a_1 + ba_3 + a_5 + a_6}{1 - (a_2 + ba_3)} d_{lb}(x_{j-1}, x_j). \end{aligned}$$

Then we have

$$d_{lb}(x_j, x_{j+1}) \le \mu d_{lb}(x_{j-1}, x_j). \tag{2.3}$$

Continuing in this way and by using (2.3), we have

$$d_{lb}(x_j, x_{j+1}) \le \mu^j d_{lb}(x_0, x_1), \quad j \in \mathbb{N}. \tag{2.4}$$

Now,

$$\begin{aligned} d_{lb}(x_0, x_{j+1}) &\le b d_{lb}(x_0, x_1) + b^2 d_{lb}(x_1, x_2) + \cdots + b^{j+1} d_{lb}(x_j, x_{j+1}) \\ &\le b d_{lb}(x_0, x_1) + b^2 \mu\big(d_{lb}(x_0, x_1)\big) + \cdots \\ &\quad + b^{j+1} \mu^{j+1}\big(d_{lb}(x_0, x_1)\big) \quad \text{by (2.4)} \\ &= \frac{b(1 - (b\mu)^{j+1})}{1 - b\mu} d_{lb}(x_0, x_1), \\ d_{lb}(x_0, x_{j+1}) &\le \frac{b(1 - (b\mu)^{j+1})}{1 - b\mu} \mu(1 - b\mu) r < r \quad \text{by (2.2)}, \end{aligned}$$

which implies $x_{j+1} \in \overline{B_{d_{lb}}(x_0, r)}$. Hence, by induction $x_n \in \overline{B_{d_{lb}}(x_0, r)}$ for all $n \in N$. Now inequality (2.4) can be written as

$$d_{lb}(x_n, x_{n+1}) \le \mu^n\big(d_{lb}(x_0, x_1)\big) \quad \text{for all } n \in N. \tag{2.5}$$

Now, for any positive integers m, n ($n > m$), we have

$$\begin{aligned} d_{lb}(x_m, x_n) &\leq b\big(d_{lb}(x_m, x_{m+1})\big) + b^2\big(d_{lb}(x_{m+1}, x_{m+2})\big) + \cdots \\ &\quad + b^{n-m}\big(d_{lb}(x_{n-1}, x_n)\big) \\ &\leq b\mu^m d_{lb}(x_0, x_1) + b^2\mu^{m+1} d_{lb}(x_0, x_1) + \cdots \\ &\quad + b^{n-m}\mu^{n-1} d_{lb}(x_0, x_1) \quad \text{by (2.5)} \\ &\leq b\mu^m\big(1 + b\mu + \cdots + b^{n-m-1}\mu^{n-m-1}\big) d_{lb}(x_0, x_1) \\ &\leq \frac{b\mu^m}{1 - b\mu} d_{lb}(x_0, x_1) \to 0 \quad \text{as } m \to \infty. \end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence in $\overline{B_{d_{lb}}(x_0, r)}$. As $\overline{B_{d_{lb}}(x_0, r)}$ is complete, there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$. Now, by Lemma 1.7 and (2.1), we get

$$\begin{aligned} d_{lb}\big(z, [Tz]_{\alpha(z)}\big) &\leq b\big[d_{lb}(z, x_{n+1}) + d_{lb}\big(x_{n+1}, [Tz]_{\alpha(z)}\big)\big] \\ &\leq b\big[d_{lb}(z, x_{n+1}) + H_{d_{lb}}\big([Tx_n]_{\alpha(x_n)}, [Tz]_{\alpha(z)}\big)\big] \\ &\leq \Bigg[d_{lb}(z, x_{n+1}) + a_1 d_{lb}\big(x_n, [Tx_n]_{\alpha(x_n)}\big) + a_2 d_{lb}\big(z, [Tz]_{\alpha(z)}\big) \\ &\quad + a_3 d_{lb}\big(x_n, [Tz]_{\alpha(z)}\big) + a_4 d_{lb}\big(z, [Tx_n]_{\alpha(x_n)}\big) + a_5 d_{lb}(x_n, z) \\ &\quad + a_6 \frac{d_{lb}(x_n, [Tx_n]_{\alpha(x_n)})(1 + d_{lb}(x_n, [Tx_n]_{\alpha(x_n)}))}{1 + d_{lb}(x_n, z)}\Bigg]. \end{aligned}$$

Taking limit $n \to \infty$, we get

$$\big(1 - b(a_2 + a_3)\big) d_{lb}\big(z, [Tz]_{\alpha(z)}\big) \leq 0.$$

So, we get

$$z \in [Tz]_{\alpha(z)}.$$

Hence, $z \in X$ is a fixed point. □

Theorem 2.2 *Let (X, d_{lb}) be a complete dislocated b-metric space with constant $b \geq 1$. Let $S, T : X \to F(X)$ be two fuzzy mappings, and let x_0 be any arbitrary point in X. Suppose there exist $\alpha_S(x), \alpha_T(x) \in (0, 1]$ for all $x \in X$ satisfying the following conditions*:

$$\begin{aligned} H_{d_{lb}}\big([Tx]_{\alpha_T(x)}, [Sy]_{\alpha_S(y)}\big) &\leq a_1 d_{lb}\big(x, [Tx]_{\alpha_T(x)}\big) + a_2 d_{lb}\big(y, [Sy]_{\alpha_S(y)}\big) \\ &\quad + a_3 d_{lb}\big(x, [Sy]_{\alpha_S(y)}\big) + a_4 d_{lb}\big(y, [Tx]_{\alpha_T(x)}\big) + a_5 d_{lb}(x, y) \end{aligned} \tag{2.6}$$

and

$$d_{lb}\big(x_0, [Tx_0]_{\alpha(x_0)}\big) \leq \mu(1 - b\mu) r \tag{2.7}$$

for all $x, y \in \overline{B_{d_{lb}}(x_0, r)}$, $r > 0$ and $b\mu < 1$, where $\mu = \frac{a_1 + a_2 + ba_3 + ba_4 + 2a_5}{2 - (a_1 + a_2 + ba_3 + ba_4)}$. Also, $a_i \geq 0$, where $i = 1, 2, \ldots, 6$ with $(a_1 + a_2)(b + 1) + b(a_3 + a_4)(b + 1) + 2ba_5 < 2$ and $\sum_{i=1}^{5} a_i < 1$. Then there exists x^* in $\overline{B_{d_{lb}}(x_0, r)}$ such that x^* is a common fixed point of S and T.

Proof Let x_0 be any arbitrary point in X such that $x_1 \in [Tx_0]_{\alpha_T(x_0)}$. Consider the sequence $\{x_n\}$ of points in X such that $x_{2i+1} \in [Tx_{2i}]_{\alpha(x_{2i})}$, $x_{2i+2} \in [Sx_{2i+1}]_{\alpha(x_{2i+1})}$ for $i = 0, 1, 2, \dots$. First we show that $x_n \in \overline{B_{d_{lb}}(x_0, r)}$ for all $n \in \mathbb{N}$. Using (2.7), we get

$$d_{lb}(x_0, x_1) = d_{lb}\big(x_0, [Tx_0]_{\alpha(x_0)}\big) \le (1 - b\mu)r < r,$$

which implies $x_1 \in \overline{B_{d_{lb}}(x_0, r)}$. Let $x_2, x_3, \dots, x_j \in \overline{B_{d_{lb}}(x_0, r)}$, $j \in \mathbb{N}$. If $j = 2i + 1$, where $i = 0, 1, 2, \dots, \frac{j-1}{2}$. Now, by using Lemma 1.8, we get

$$\begin{aligned}
d_{lb}(x_{2i+1}, x_{2i+2}) &\le H_{d_{lb}}\big([Tx_{2i}]_{\alpha(x_{2i})}, [Sx_{2i+1}]_{\alpha(x_{2i+1})}\big) \\
&\le a_1 d\big(x_{2i}, [Tx_{2i}]_{\alpha(x_{2i})}\big) + a_2 d\big(x_{2i+1}, [Sx_{2i+1}]_{\alpha(x_{2i+1})}\big) \\
&\quad + a_3 d\big(x_{2i}, [Sx_{2i+1}]_{\alpha(x_{2i+1})}\big) + a_4 d\big(x_{2i+1}, [Tx_{2i}]_{\alpha(x_{2i})}\big) \\
&\quad + a_5 d(x_{2i}, x_{2i+1}) \\
&\le a_1 d(x_{2i}, x_{2i+1}) + a_2 d(x_{2i+1}, x_{2i+2}) + a_3 d(x_{2i}, x_{2i+2}) \\
&\quad + a_4 d(x_{2i+1}, x_{2i+1}) + a_5 d(x_{2i}, x_{2i+1}) \\
&\le a_1 d(x_{2i}, x_{2i+1}) + a_2 d(x_{2i+1}, x_{2i+2}) \\
&\quad + a_3 b\big[d(x_{2i}, x_{2i+1}) + d(x_{2i+1}, x_{2i+2})\big] + a_5 d(x_{2i}, x_{2i+1}).
\end{aligned}$$

Now, we have

$$d_{lb}(x_{2i+1}, x_{2i+2}) \le \frac{a_1 + ba_3 + a_5}{1 - (a_2 + ba_3)} d(x_{2i}, x_{2i+1}). \tag{2.8}$$

Similarly, by symmetry, we have

$$\begin{aligned}
d_{lb}(x_{2i+2}, x_{2i+1}) &\le H_{d_{lb}}\big([Sx_{2i+1}]_{\alpha(x_{2i+1})}, [Tx_{2i}]_{\alpha(x_{2i})}\big) \\
&\le a_1 d\big(x_{2i+1}, [Sx_{2i+1}]_{\alpha(x_{2i+1})}\big) + a_2 d\big(x_{2i}, [Tx_{2i}]_{\alpha(x_{2i})}\big) \\
&\quad + a_3 d\big(x_{2i+1}, [Tx_{2i}]_{\alpha(x_{2i})}\big) + a_4 d\big(x_{2i}, [Sx_{2i+1}]_{\alpha(x_{2i+1})}\big) \\
&\quad + a_5 d(x_{2i+1}, x_{2i}) \\
&\le a_1 d(x_{2i+1}, x_{2i+2}) + a_2 d(x_{2i}, x_{2i+1}) + a_3 d(x_{2i+1}, x_{2i+1}) \\
&\quad + a_4 d(x_{2i}, x_{2i+2}) + a_5 d(x_{2i+1}, x_{2i}) \\
&\le a_1 d(x_{2i+1}, x_{2i+2}) + a_2 d(x_{2i}, x_{2i+1}) \\
&\quad + a_4 b\big[d(x_{2i}, x_{2i+1}) + d(x_{2i+1}, x_{2i+2})\big] + a_5 d(x_{2i+1}, x_{2i}).
\end{aligned}$$

So, we have

$$d_{lb}(x_{2i+2}, x_{2i+1}) \le \frac{a_2 + ba_4 + a_5}{1 - (a_1 + ba_4)} d(x_{2i}, x_{2i+1}). \tag{2.9}$$

Adding (2.8) and (2.9), we get

$$d_{lb}(x_{2i+1}, x_{2i+2}) \le \frac{a_1 + a_2 + ba_3 + ba_4 + 2a_5}{2 - (a_1 + a_2 + ba_3 + ba_4)} d(x_{2i}, x_{2i+1}). \tag{2.10}$$

As

$$\mu = \frac{a_1 + a_2 + ba_3 + ba_4 + 2a_5}{2 - (a_1 + a_2 + ba_3 + ba_4)} < \frac{1}{b},$$

then, by (2.10), we have

$$d_{lb}(x_{2i+1}, x_{2i+2}) \leq \mu d(x_{2i}, x_{2i+1}). \tag{2.11}$$

Similarly, if $j = 2i + 2$, where $i = 0, 1, 2, \dots, \frac{j-2}{2}$, we have

$$d_{lb}(x_{2i+2}, x_{2i+3}) \leq \mu d(x_{2i+1}, x_{2i+2}). \tag{2.12}$$

Now, by (2.11)

$$d_{lb}(x_{2i+1}, x_{2i+2}) \leq \mu^{2i+1} d_{lb}(x_0, x_1). \tag{2.13}$$

Also, by (2.12)

$$d_{lb}(x_{2i+2}, x_{2i+3}) \leq \mu^{2i+2} d_{lb}(x_0, x_1). \tag{2.14}$$

By combining (2.13) and (2.14), we get

$$d_{lb}(x_j, x_{j+1}) \leq \mu^j d_{lb}(x_0, x_1) \quad \text{for all } j \in \mathbb{N}. \tag{2.15}$$

Now,

$$\begin{aligned}
d_{lb}(x_0, x_{j+1}) &\leq bd_{lb}(x_0, x_1) + b^2 d_{lb}(x_1, x_2) + \cdots + b^{j+1} d_{lb}(x_j, x_{j+1}) \\
&\leq bd_{lb}(x_0, x_1) + b^2 \mu \bigl(d_{lb}(x_0, x_1)\bigr) + \cdots \\
&\quad + b^{j+1} \mu^j \bigl(d_{lb}(x_0, x_1)\bigr) \quad \text{by (2.15)} \\
&= \frac{b(1 - (b\mu)^{j+1})}{1 - b\mu} d_{lb}(x_0, x_1), \\
d_{lb}(x_0, x_{j+1}) &\leq \frac{b(1 - (b\mu)^{j+1})}{1 - b\mu} \mu(1 - b\mu) r < r \quad \text{by (2.7)}
\end{aligned}$$

which implies $x_{j+1} \in \overline{B_{d_{lb}}(x_0, r)}$. Hence, by induction $x_n \in \overline{B_{d_{lb}}(x_0, r)}$ for all $n \in N$. Now inequality (2.15) can be written as

$$d_{lb}(x_n, x_{n+1}) \leq \mu^n \bigl(d_{lb}(x_0, x_1)\bigr) \quad \text{for all } n \in N. \tag{2.16}$$

Now, for any positive integers m, n ($n > m$), we have

$$\begin{aligned}
d_{lb}(x_m, x_n) &\leq b\bigl(d_{lb}(x_m, x_{m+1})\bigr) + b^2\bigl(d_{lb}(x_{m+1}, x_{m+2})\bigr) + \cdots \\
&\quad + b^{n-m}\bigl(d_{lb}(x_{n-1}, x_n)\bigr) \\
&\leq b\mu^m d_{lb}(x_0, x_1) + b^2 \mu^{m+1} d_{lb}(x_0, x_1) + \cdots
\end{aligned}$$

$$
\begin{aligned}
&+ b^{n-m}\mu^{n-1}d_{lb}(x_0,x_1) \quad \text{by (2.16)}\\
&\leq b\mu^m\bigl(1+b\mu+\cdots+b^{n-m-1}\mu^{n-m-1}\bigr)d_{lb}(x_0,x_1)\\
&\leq \frac{b\mu^m}{1-b\mu}d_{lb}(x_0,x_1)\to 0.
\end{aligned}
$$

Hence, $\{x_n\}$ is a Cauchy sequence in $\overline{B_{d_{lb}}(x_0,r)}$. As $\overline{B_{d_{lb}}(x_0,r)}$ is complete, there exists $z\in X$ such that $x_n\to z$ as $n\to\infty$. Now, by Lemma 1.7 and (2.1), we prove $z\in X$ to be the common fixed point of S and T.

$$
\begin{aligned}
d_{lb}\bigl(z,[Sz]_{\alpha(z)}\bigr) &\leq b\bigl[d_{lb}(z,x_{2n+1})+d_{lb}\bigl(x_{2n+1},[Sz]_{\alpha(z)}\bigr)\bigr]\\
&\leq b\bigl[d_{lb}(z,x_{2n+1})+H_{d_{lb}}\bigl([Tx_{2n}]_{\alpha(x_{2n})},[Sz]_{\alpha(z)}\bigr)\bigr]\\
&\leq b\bigl[d_{lb}(z,x_{2n+1})+a_1d_{lb}\bigl(x_{2n},[Tx_{2n}]_{\alpha(x_{2n})}\bigr)+a_2d_{lb}\bigl(z,[Sz]_{\alpha(z)}\bigr)\\
&\quad + a_3d_{lb}\bigl(x_{2n},[Sz]_{\alpha(z)}\bigr)+a_4d_{lb}\bigl(z,[Tx_{2n}]_{\alpha(x_{2n})}\bigr)+a_5d_{lb}(x_{2n},z)\bigr].
\end{aligned}
$$

Taking limit $n\to\infty$, we get

$$
\bigl(1-b(a_2+a_3)\bigr)d_{lb}\bigl(z,[Sz]_{\alpha(z)}\bigr)\leq 0.
$$

So, we get

$$
z\in[Sz]_{\alpha(z)}.
$$

This implies that $z\in X$ is a fixed point of S. Similarly, we can prove that z is a fixed point of T. Hence, z is a common fixed point of S and T. □

Example 2.3 Let $X=\mathbb{Q}^+\cup\{0\}$ and $d_{lb}(x,y)=(x+y)^2$, whenever $x,y\in X$, then (X,d_{lb}) is a complete dislocated b-metric space with $b>1$. Define a fuzzy mapping $T:X\to F(X)$ by

$$
T(x)(t)=\left\{\begin{array}{ll}
1, & 0\leq t\leq x/4\\
1/2, & x/4<t\leq x/3\\
1/4, & x/3<t\leq x/2\\
0, & x/2<t\leq 1
\end{array}\right|.
$$

For all $x\in X$, there exists $\alpha(x)=1$ such that

$$
[Tx]_{\alpha(x)}=\left[0,\frac{x}{4}\right].
$$

Consider $x_0=1$ and $r=4$, then $\overline{B_{d_{lb}}(x_0,r)}=[0,1]$. Let $a_1=\frac{1}{10}$, $a_2=\frac{1}{20}$, $a_3=\frac{1}{30}$, $a_4=\frac{1}{40}$, $a_5=\frac{1}{50}$, $a_6=\frac{1}{60}$.Then

$$
\begin{aligned}
H_{d_{lb}}\bigl([Tx]_{\alpha(x)},[Ty]_{\alpha(y)}\bigr) &\leq \frac{1}{10}\left(x+\frac{x}{4}\right)^2+\frac{1}{20}\left(y+\frac{y}{4}\right)^2+\frac{1}{30}\left(x+\frac{y}{4}\right)^2\\
&\quad +\frac{1}{40}\left(y+\frac{x}{4}\right)^2+\frac{1}{50}(x-y)^2+\frac{1}{60}\left(\frac{(x+\frac{x}{4})^2(1+(x+\frac{x}{4})^2)}{1+(x+y)^2}\right)
\end{aligned}
$$

and

$$d_{lb}\big(x_0,[Tx_0]_{\alpha(x_0)}\big)\leq \mu(1-b\mu)r,$$

where

$$\mu=\frac{(a_1+ba_3+a_5+a_6)}{1-(a_2+ba_3)}<\frac{1}{b}.$$

Since all the conditions of Theorem 2.1 are satisfied, there exists $0\in\overline{B_{d_{lb}}(x_0,r)}$ which is the fixed point of T.

Example 2.4 Consider $X=\{0,1,2\}$. Let $d_{lb}:X\times X\to[0,\infty)$ be the mapping defined by

$$d_{lb}(x,y)=\begin{cases}0, & x=y \text{ and } x,y\in\{0,1\},\\ \frac{2}{5}, & x=y \text{ and } x,y\in\{2\},\\ \frac{1}{2}, & x\neq y \text{ and } x,y\in\{0,2\},\\ 1, & x\neq y \text{ and } x,y\in\{0,1\},\\ \frac{1}{4}, & x\neq y \text{ and } x,y\in\{1,2\}.\end{cases}$$

It is clear that d is a complete dislocated b-metric space with the constant $b=\frac{4}{3}$. Note that $d(2,2)\neq 0$, so d is not a b-metric and also d is not a metric. Consider $x_0=1$ and $r=1$, then $\overline{B_{d_{lb}}(x_0,r)}=0$. Define the fuzzy mapping $S,T:X\to F(X)$ by

$$(T0)(t)=\begin{cases}\frac{3}{4}, & t=0,\\ 0, & t=1,2,\end{cases}$$

$$(T1)(t)=\begin{cases}0, & t=0,1,\\ \frac{3}{4}, & t=2,\end{cases}$$

$$(T2)(t)=\begin{cases}0, & t=0,2,\\ \frac{3}{4}, & t=1,\end{cases}$$

and

$$(S0)(t)=(S1)(t)=(S2)(t)=\begin{cases}\frac{3}{4}, & t=0,\\ 0, & t=1,2.\end{cases}$$

Define $\alpha_S(x)=\alpha_T(x)=\alpha$, where $\alpha\in(0,\frac{3}{4}]$. Now we have

$$[Tx]_{\alpha_T(x)}=\begin{cases}\{0\}, & x=0,\\ \{2\}, & x=1,\\ \{1\}, & x=2,\end{cases}$$

and

$$[Sx]_{\alpha_S(x)}=\{0\}\quad \text{for all } x\in X.$$

For $x, y \in X$, we get

$$H\big([Sx]_{\alpha_S(x)}, [Ty]_{\alpha_T(y)}\big) = \begin{cases} H(\{0\},\{0\}) = 0, & y = 0, \\ H(\{0\},\{2\}) = \frac{1}{2}, & y = 1, \\ H(\{0\},\{1\}) = 1, & y = 2. \end{cases}$$

Let $a_1 = a_2 = \frac{1}{4}$, $a_3 = 0$, $a_4 = \frac{1}{4}$, $a_5 = 0$, we can see that $b\mu = \frac{20}{21} < 1$, where $\mu = \frac{a_1+a_2+ba_3+ba_4+2a_5}{2-(a_1+a_2+ba_3+ba_4)} = \frac{5}{7}$. Also, $a_i \geq 0$, where $i = 1, 2, \ldots, 6$ with $(a_1 + a_2)(b + 1) + b(a_3 + a_4) \times (b + 1) + 2ba_5 = \frac{35}{18} < 2$ and $\sum_{i=1}^{5} a_i < 1$. It easy to prove that condition (2.6) in Theorem 2.2 holds. Then there exists $0 \in [Sx]_{\alpha_S(x)} \cap [Ty]_{\alpha_T(y)}$.

3 Application

In this section, we indicate that Theorem 2.1 and Theorem 2.2 can be utilized to derive a common fixed point for a multivalued mapping in a dislocated b-metric space.

Theorem 3.1 *Let (X, d_{lb}) be a complete dislocated b-metric space with constant $b \geq 1$. Suppose that $R : X \to P(X)$ are two multivalued mappings satisfying the following conditions*:

$$\begin{aligned} H_{d_{lb}}(Rx, Ry) &\leq a_1 d_{lb}(x, Rx) + a_2 d_{lb}(y, Ry) + a_3 d_{lb}(x, Ry) \\ &\quad + a_4 d_{lb}(y, Rx) + a_5 d_{lb}(x, y) + a_6 \frac{d_{lb}(x, Rx)(1 + d_{lb}(x, Rx))}{1 + d_{lb}(x, y)} \end{aligned} \tag{3.1}$$

and

$$d_{lb}(x_0, Rx_0) \leq \mu(1 - b\mu)r \tag{3.2}$$

for all $x, y \in \overline{B_{d_{lb}}(x_0, r)}$, $r > 0$ and $b\mu < 1$, where $\mu = \frac{(a_1+ba_3+a_5+a_6)}{1-(a_2+ba_3)}$. Also, $a_i \geq 0$, where $i = 1, 2, \ldots, 6$ with $ba_1 + a_2 + b(1 + b)a_3 + b(a_5 + a_6) < 1$ and $\sum_{i=1}^{6} a_i < 1$. Then there exists x^* in $\overline{B_{d_{lb}}(x_0, r)}$ such that $x^* \in Rx^*$.

Proof Let $\alpha : X \to (0, 1]$ be an arbitrary mapping. Consider two fuzzy mappings $T : X \to F(X)$ defined by

$$(Tx)(t) = \begin{cases} \alpha(x), & t \in Rx, \\ 0, & t \notin Rx. \end{cases}$$

We obtain that

$$[Tx]_{\alpha(x)} = \big\{t : Tx(t) \geq \alpha(x)\big\} = Rx.$$

Hence, condition (3.1) becomes condition (2.1) in Theorem 2.1. This implies that there exists $z \in X$ such that $z \in [Tz]_{\alpha(z)} = Rz$. □

Theorem 3.2 *Let (X, d_{lb}) be a complete dislocated b-metric space with constant $b \geq 1$. Suppose that $R, G : X \to P(X)$ are two multivalued mappings satisfying the following con-*

ditions:

$$H_{d_{lb}}(Rx,Gy) \leq a_1 d_{lb}(x,Rx) + a_2 d_{lb}(y,Gy) + a_3 d_{lb}(x,Gy) + a_4 d_{lb}(y,Rx) + a_5 d_{lb}(x,y) \tag{3.3}$$

and

$$d_{lb}(x_0,Rx_0) \leq \mu(1-b\mu)r \tag{2.7}$$

for all $x,y \in \overline{B_{d_{lb}}(x_0,r)}$, $r > 0$ *and* $b\mu < 1$, *where* $\mu = \frac{a_1+a_2+ba_3+ba_4+2a_5}{2-(a_1+a_2+ba_3+ba_4)}$. *Also*, $a_i \geq 0$, *where* $i = 1,2,\ldots,6$ *with* $(a_1+a_2)(b+1)+b(a_3+a_4)(b+1)+2ba_5 < 2$ *and* $\sum_{i=1}^{5} a_i < 1$. *Then there exists* x^* *in* $\overline{B_{d_{lb}}(x_0,r)}$ *such that* x^* *is a common fixed point of* R *and* G.

Proof Let $\alpha : X \to (0,1]$ be an arbitrary mapping. Consider two fuzzy mappings $S,T : X \to F(X)$ defined by

$$(Sx)(t) = \begin{cases} \alpha(x), & t \in Rx, \\ 0, & t \notin Rx, \end{cases}$$

$$(Tx)(t) = \begin{cases} \alpha(x), & t \in Gx, \\ 0, & t \notin Gx. \end{cases}$$

We obtain that

$$[Sx]_{\alpha(x)} = \{t : Sx(t) \geq \alpha(x)\} = Rx$$

and

$$[Tx]_{\alpha(x)} = \{t : Tx(t) \geq \alpha(x)\} = Gx.$$

Hence, condition (3.3) becomes condition (2.6) of Theorem 2.2. This implies that there exists $z \in [Sz]_{\alpha(z)} \cap [Tz]_{\alpha(z)} = Rz \cap Gz$. □

4 Conclusion

In the present work we have shown the new concept of fuzzy mappings in a complete dislocated b-metric space. We have also obtained fixed point and common fixed point results for fuzzy mappings in a complete dislocated b-metric space. Our results generalize and extend the concept of Joseph *et al.* [1] and references therein. We have also given examples to support our results, showing that d is a complete dislocated b-metric space but is not b-metric and metric space. Finally, we related the results to a fixed point for multivalued mappings and fuzzy mappings.

Acknowledgements

This project was supported by the Theoretical and Computational Science (TaCS) Center under Computational and Applied Science for Smart Innovation Cluster (CLASSIC), Faculty of Science, KMUTT. PK acknowledges the financial support provided by King Mongkut's University of Technology Thonburi through the "KMUTT 55th Anniversary Commemorative Fund". SP supported by Rajamangala University of Technology Rattanakosin Research and Development Institute, Rajamangala University of Technology Rattanakosin (RMUTR).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 7 April 2017 Accepted: 15 December 2017 Published online: 05 January 2018

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