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A generalization of Hegedüs-Szilágyi's fixed point theorem in complete metric spaces

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Abstract

In 1980, Hegedüs and Szilágyi proved some fixed point theorem in complete metric spaces. Introducing a new contractive condition, we generalize Hegedüs-Szilágyi's fixed point theorem. We discuss the relationship between the new contractive condition and other contractive conditions. We also show that we cannot extend Hegedüs-Szilágyi's fixed point theorem to Meir-Keeler type.

MSC: 54H25

Keywords: Hegedüs-Szilágyi's fixed point theorem; complete metric space

1 Introduction and preliminaries

Throughout this paper we denote by $\mathbb{N}$ the set of all positive integers and by $\mathbb{R}$ the set of all real numbers.

Let T be a mapping on a metric space (X,d). Throughout this paper, we define $D_T(x)$ and $D_T(x,y)$ by

$$D_T(x) = \sup\bigl\{d(u,v) : u,v \in \bigl\{x, Tx, T^2x, \ldots\bigr\}\bigr\},$$
$$D_T(x,y) = \sup\bigl\{d(u,v) : u,v \in \bigl\{x, Tx, T^2x, \ldots, y, Ty, T^2y, \ldots\bigr\}\bigr\}$$

for any $x, y \in X$. That is, $D_T(x)$ is the diameter of the orbit $\{x, Tx, T^2x, \ldots\}$ of x.

Hegedüs and Szilágyi in [1] proved the following fixed point theorem. The author thinks that the proof in [1] is splendid.

Theorem 1 (Theorem 5 in [1]) *Let (X,d) be a complete metric space, and let T be a mapping on X. Assume $D_T(x) < \infty$ for all $x \in X$. Assume also that there exists a function φ from $[0,\infty)$ into itself satisfying the following:*

(i) *$\varphi(t) < t$ holds for all $t \in (0,\infty)$;*
(ii) *φ is upper semicontinuous from the right;*
(iii) *$d(Tx,Ty) \le \varphi \circ D_T(x,y)$ holds for all $x, y \in X$.*

Then T has a unique fixed point z. Moreover, $\{T^n x\}$ converges to z for any $x \in X$.

Remark 1 See also [2–4]. Note that in the proof of Theorem 1 in [3], we need an additional assumption such as the nondecreasingness of φ.

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We state Boyd-Wong's [5], Meir-Keeler's [6] and Matkowski's [7] fixed point theorems.

Theorem 2 (Theorem 1 in [5]) *Let (X,d) be a complete metric space, and let T be a mapping on X. Assume that there exists a function φ from $[0,\infty)$ into itself satisfying* (i) *and* (ii) *of Theorem* 1 *and the following*:

(iii) $d(Tx,Ty) \le \varphi \circ d(x,y)$ *holds for all* $x,y \in X$.

Then T has a unique fixed point z. Moreover, $\{T^n x\}$ converges to z for any $x \in X$.

Theorem 3 ([6]) *Let (X,d) be a complete metric space, and let T be a mapping on X. Assume that, for any $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$d(x,y) < \varepsilon + \delta \quad \textit{implies} \quad d(Tx,Ty) < \varepsilon$$

for all $x,y \in X$. Then T has a unique fixed point z. Moreover, $\{T^n x\}$ converges to z for any $x \in X$.

Theorem 4 (Theorem 1.2 in [7]) *Let (X,d) be a complete metric space, and let T be a mapping on X. Assume that there exists a function φ from $[0,\infty)$ into itself satisfying the following*:

(i) φ *is nondecreasing*;

(ii) $\lim_n \varphi^n(t) = 0$ *holds for all* $t \in (0,\infty)$;

(iii) $d(Tx,Ty) \le \varphi \circ d(x,y)$ *holds for all* $x,y \in X$.

Then T has a unique fixed point z. Moreover, $\{T^n x\}$ converges to z for any $x \in X$.

From the above, we can tell that Theorem 1 is of Boyd-Wong [5] type (see Definition 8). So it is a very natural question of whether we can extend Theorem 1 to Meir-Keeler [6] type. It is also a natural question of whether we can prove a Matkowski [7] type fixed point theorem.

In this paper, we answer the above two questions; one is negative and the other is affirmative. Indeed, we generalize Theorem 1. The assumption of the new theorem (Theorem 5) is weaker than a Matkowski type condition (see Corollary 7). We also give a counterexample for a Meir-Keeler type condition (Example 16). We further discuss the relationship between the assumption of Theorem 5 and other contractive conditions.

2 Main results

In this section, we generalize Theorem 1.

Theorem 5 *Let (X,d) be a complete metric space, and let T be a mapping on X. Assume $D_T(x) < \infty$ for all $x \in X$. Assume also that there exists a function φ from $[0,\infty)$ into itself satisfying the following*:

(i) $\varphi(t) < t$ *holds for all* $t \in (0,\infty)$;

(ii) *For any $\varepsilon > 0$, there exists $\delta > 0$ such that, for any $t \in (0,\infty)$,*

$$\varepsilon < t < \varepsilon + \delta \quad \textit{implies} \quad \varphi(t) \le \varepsilon.$$

(iii) *For any* $x,y \in X$,

$$d(Tx,Ty) \le \varphi \circ D_T(x,y)$$

holds.

Then T has a unique fixed point z. Moreover, $\{T^n x\}$ converges to z for any $x \in X$.

Remark 2

- $D_T(x, y) < \infty$ obviously holds for any $x, y \in X$.
- Since $D_T(x, y) = 0$ implies $d(Tx, Ty) = 0$, without loss of generality, we may assume $\varphi(0) = 0$.
- We do not assume that φ is nondecreasing. So, in general, $D_T(Tx, Ty) \le \varphi \circ D_T(x, y)$ does not hold.

Before proving Theorem 5, we need one lemma.

Lemma 6 *Let $x, y \in X$. Assume that either of the following holds*:

(a) $x = y$;

(b) $\lim_n D_T(T^n x) = \lim_n D_T(T^n y) = 0$.

Then $\lim_n D_T(T^n x, T^n y) = 0$ *holds.*

Proof Since

$$\{T^n x, T^{n+1} x, \ldots, T^n y, T^{n+1} y, \ldots\} \supset \{T^{n+1} x, T^{n+2} x, \ldots, T^{n+1} y, T^{n+2} y, \ldots\}$$

for $n \in \mathbb{N}$, $\{D_T(T^n x, T^n y)\}$ is nonincreasing. So $\{D_T(T^n x, T^n y)\}$ converges to some $\varepsilon \in [0, \infty)$. Arguing by contradiction, we assume $\varepsilon > 0$. We consider the following two cases:

- $\varepsilon < D_T(T^n x, T^n y)$ holds for any $n \in \mathbb{N}$;
- $\varepsilon = D_T(T^n x, T^n y)$ holds for some $n \in \mathbb{N}$.

In the first case, we choose $\delta \in (0, \infty)$ such that

$$\varepsilon < t < \varepsilon + \delta \quad \text{implies} \quad \varphi(t) \le \varepsilon.$$

We choose $\nu \in \mathbb{N}$ satisfying

$$D_T(T^\nu x, T^\nu y) < \varepsilon + \delta.$$

In the case of (b), without loss of generality, we may assume

$$D_T(T^\nu x) \le \varepsilon/2 \quad \text{and} \quad D_T(T^\nu y) \le \varepsilon/2. \tag{1}$$

Fix $m \ge \nu$ and $n \ge \nu$. Then since

$$\begin{aligned}\varepsilon &< D_T(T^{\max\{m,n\}} x, T^{\max\{m,n\}} y)\\ &\le D_T(T^m x, T^n y)\\ &\le D_T(T^{\min\{m,n\}} x, T^{\min\{m,n\}} y) \le D_T(T^\nu x, T^\nu y) < \varepsilon + \delta,\end{aligned}$$

we have

$$d(T^{m+1} x, T^{n+1} y) \le \varphi \circ D_T(T^m x, T^n y) \le \varepsilon.$$

Since m, n are arbitrary, considering (1), we obtain

$$\varepsilon < D_T\bigl(T^{\nu+1}x, T^{\nu+1}y\bigr) \le \varepsilon,$$

which implies a contradiction. In the second case, we choose $\nu \in \mathbb{N}$ satisfying

$$D_T\bigl(T^{\nu}x, T^{\nu}y\bigr) = \varepsilon.$$

In the case of (b), without loss of generality, we may assume

$$D_T\bigl(T^{\nu}x\bigr) \le \varphi(\varepsilon) \quad \text{and} \quad D_T\bigl(T^{\nu}y\bigr) \le \varphi(\varepsilon). \tag{2}$$

Fix $m \ge \nu$ and $n \ge \nu$. Then since

$$\begin{aligned}\varepsilon &\le D_T\bigl(T^{\max\{m,n\}}x, T^{\max\{m,n\}}y\bigr)\\ &\le D_T\bigl(T^{m}x, T^{n}y\bigr)\\ &\le D_T\bigl(T^{\min\{m,n\}}x, T^{\min\{m,n\}}y\bigr) \le D_T\bigl(T^{\nu}x, T^{\nu}y\bigr) = \varepsilon,\end{aligned}$$

we have

$$d\bigl(T^{m+1}x, T^{n+1}y\bigr) \le \varphi \circ D_T\bigl(T^{m}x, T^{n}y\bigr) = \varphi(\varepsilon).$$

Since m, n are arbitrary, considering (2), we obtain

$$\varepsilon \le D_T\bigl(T^{\nu+1}x, T^{\nu+1}y\bigr) \le \varphi(\varepsilon) < \varepsilon,$$

which implies a contradiction. Therefore we have shown $\lim_n D_T(T^n x, T^n y) = 0$. □

Proof of Theorem 1 Fix $x \in X$. By Lemma 6(a), $\{D_T(T^n x)\}$ converges to 0. Thus $\{T^n x\}$ is a Cauchy sequence in X. Since X is complete, $\{T^n x\}$ converges to some $z \in X$. By Lemma 6(a) again, $\{D_T(T^n z)\}$ also converges to 0. So, by Lemma 6(b), we obtain

$$\lim_{n\to\infty} D_T\bigl(T^{n}x, T^{n}z\bigr) = 0. \tag{3}$$

So $\{T^n z\}$ also converges to z. Hence

$$D_T(z) = D_T(Tz) \tag{4}$$

holds. Arguing by contradiction, we assume $\varepsilon := D_T(z) > 0$. Since $\lim_n D_T(T^n z) = 0$ holds, there exists $\nu \in \mathbb{N}$ satisfying

$$\varepsilon = D_T(z) = \cdots = D_T\bigl(T^{\nu-1}z\bigr) = D_T\bigl(T^{\nu}z\bigr) > D_T\bigl(T^{\nu+1}z\bigr),$$

where $T^0 z = z$. This implies

$$\varepsilon = D_T\bigl(T^{\nu}z\bigr) = \sup\bigl\{d\bigl(T^{\nu}z, T^{n}z\bigr) : n > \nu\bigr\}.$$

For $n > \nu$, we have

$$d(T^\nu z, T^n z) \le \varphi \circ D_T(T^{\nu-1}z, T^{n-1}z) = \varphi \circ D_T(T^{\nu-1}z) = \varphi(\varepsilon).$$

Since n is arbitrary, we obtain

$$\varepsilon = \sup\{d(T^\nu z, T^n z) : n > \nu\} \le \varphi(\varepsilon) < \varepsilon,$$

which implies a contradiction. Therefore we have shown $D_T(z) = 0$. Hence z is a fixed point of T. Since (3) holds for any $x \in X$, we obtain the uniqueness of the fixed point. □

By Theorem 5, we obtain a Matkowski type fixed point theorem.

Corollary 7 *Let (X, d) be a complete metric space, and let T be a mapping on X. Assume $D_T(x) < \infty$ for all $x \in X$. Assume also that there exists a function φ from $[0, \infty)$ into itself satisfying the following*:
(i) *φ is nondecreasing*;
(ii) *$\lim_n \varphi^n(t) = 0$ holds for all $t \in (0, \infty)$*;
(iii) *$d(Tx, Ty) \le \varphi \circ D_T(x, y)$ holds for all $x, y \in X$.*
Then T has a unique fixed point z. Moreover, $\{T^n x\}$ converges to z for any $x \in X$.

3 Comparison

In this section, using subsets of $(0, \infty)^2$, we discuss the relationship between the new contractive condition in Theorem 5 and other contractive conditions. See [1, 8–11] and the references therein.

Definition 8 Let Q be a subset of $(0, \infty)^2$.
(1) Q is said to be *contractive* (*Cont* for short) [12, 13] if there exists $r \in (0, 1)$ such that $u \le rt$ holds for any $(t, u) \in Q$.
(2) Q is said to be *Browder* (*Bro* for short) [14] if there exists a function φ from $(0, \infty)$ into itself satisfying the following:
  (2-i) φ is nondecreasing and right-continuous;
  (2-ii) $\varphi(t) < t$ holds for any $t \in (0, \infty)$;
  (2-iii) $u \le \varphi(t)$ holds for any $(t, u) \in Q$.
(3) Q is said to be *Boyd-Wong* (*BW* for short) [5] if there exists a function φ from $(0, \infty)$ into itself satisfying the following:
  (3-i) φ is upper semicontinuous from the right;
  (3-ii) $\varphi(t) < t$ holds for any $t \in (0, \infty)$;
  (3-iii) $u \le \varphi(t)$ holds for any $(t, u) \in Q$.
(4) Q is said to be *Meir-Keeler* (*MK* for short) [6] if, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $u < \varepsilon$ holds for any $(t, u) \in Q$ with $t < \varepsilon + \delta$.
(5) Q is said to be *Matkowski* (*Mat* for short) [7] if there exists a function φ from $(0, \infty)$ into itself satisfying the following:
  (5-i) φ is nondecreasing;
  (5-ii) $\lim_n \varphi^n(t) = 0$ for any $t \in (0, \infty)$;
  (5-iii) $u \le \varphi(t)$ holds for any $(t, u) \in Q$.

(6) Q is said to be of *New-type* (*NT* for short) if there exists a function φ from $(0, \infty)$ into itself satisfying the following:
  (6-i) $\varphi(t) < t$ for any $t \in (0, \infty)$;
  (6-ii) For any $\varepsilon > 0$, there exists $\delta > 0$ such that $\varepsilon < t < \varepsilon + \delta$ implies $\varphi(t) \leq \varepsilon$;
  (6-iii) $u \leq \varphi(t)$ holds for any $(t, u) \in Q$.

(7) Q is said to be *CJM* [15–18] if the following hold:
  (7-i) For any $\varepsilon > 0$, there exists $\delta > 0$ satisfying $u \leq \varepsilon$ holds for any $(t, u) \in Q$ with $t < \varepsilon + \delta$;
  (7-ii) $u < t$ holds for any $(t, u) \in Q$.

It is obvious that the following implications hold:

$$\begin{tikzcd}
\text{Cont} \arrow[r] & \text{Bro} \arrow[r] \arrow[dr] & \text{BW} \arrow[r] \arrow[dr] & \text{MK} \arrow[r] & \text{CJM} \\
& & \text{Mat} \arrow[r] & \text{NT} \arrow[ur] &
\end{tikzcd}$$

It is well known that the converse implication of (Cont $\rightarrow$ Bro) does not hold. The following three examples tell us that for each implication except (Cont $\rightarrow$ Bro), there exists a counterexample for its converse implication. In particular, MK and NT are independent.

Example 9 Let $u \in (0, \infty)$ and define Q by

$$Q = \{(t, u) : u < t\}.$$

Then Q is Mat. However, Q is not MK.

Remark 3 We note that the converse implication of (BW $\rightarrow$ NT) does not hold.

Example 10 Let $t, u \in (0, \infty)$ with $t < u$. Define Q by

$$Q = \{((1 - \lambda)t + \lambda u, \lambda u) : \lambda \in (0, 1)\}.$$

Then Q is BW. However, Q is not Mat.

Remark 4 We note that the converse implication of (Mat $\rightarrow$ NT) does not hold.

Example 11 Let $t \in (0, \infty)$ and define Q by

$$Q = \{(t, u) : 0 < u < t\}.$$

Then Q is MK. However, Q is not NT.

Remark 5 We note that the converse implication of (NT $\rightarrow$ CJM) does not hold.

In the remainder of this section, we let (X, d) be a complete metric space, and let T be a mapping on X satisfying $D_T(x) < \infty$ for all $x \in X$. Define subsets P_T and Q_T of $(0, \infty)^2$

by

$$P_T = \{(d(x,y), d(Tx,Ty)) : x, y \in X\} \cap (0,\infty)^2,$$
$$Q_T = \{(D_T(x,y), d(Tx,Ty)) : x, y \in X\} \cap (0,\infty)^2.$$

We will give three mappings such that Q_T for each mapping matches one of Examples 9-11, respectively.

Lemma 12 *Let X be a nonempty set. Let f be a function from X into $[0,\infty)$ such that $\{x \in X : f(x) = 0\}$ consists of at most one element. Define a function d from $X \times X$ into $[0,\infty)$ by*

$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ \max\{f(x), f(y)\} & \text{if } x \neq y. \end{cases} \tag{5}$$

Let T be a mapping on X satisfying the following:
- $f(x) > 0$ *implies* $Tx \neq x$ *and* $f(Tx) \leq f(x)$;
- $f(x) = 0$ *implies* $Tx = x$.

Then the following hold:
(i) (X,d) *is a metric space*;
(ii) *if either* $\{x \in X : f(x) = 0\} \neq \varnothing$ *or* $\inf f(X) > 0$ *holds, then X is complete*;
(iii) $P_T = Q_T$.

Proof We have essentially proved (i) and (ii); see Lemma 7 in [19]. Let us prove (iii). Fix $x, y \in X$ with $x \neq y$ and $f(x) \leq f(y)$. Then we have $f(y) > 0$ and hence $Ty \neq y$. We have

$$\cdots \leq f(T^n y) \leq \cdots \leq f(T^2 y) \leq f(Ty) \leq f(y),$$
$$\cdots \leq f(T^n x) \leq \cdots \leq f(T^2 x) \leq f(Tx) \leq f(x) \leq f(y).$$

Hence

$$D_T(x,y) = f(y) = d(x,y)$$

holds. Therefore $P_T = Q_T$ holds. □

Example 13 Let $X = [0,\infty)$ and define a function d from $X \times X$ into $[0,\infty)$ by (5), where $f(x) = x$. That is,

$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ \max\{x, y\} & \text{if } x \neq y \end{cases} \tag{6}$$

holds. Define a mapping T on X by

$$Tx = \begin{cases} 0 & \text{if } x \leq 1, \\ 1 & \text{if } x > 1. \end{cases}$$

Then the following hold:

(i) (X, d) is a complete metric space;
(ii) $f(x) > 0$ implies $f(Tx) < f(x)$;
(iii) $f(x) = 0$ implies $Tx = x$;
(iv) $P_T = Q_T = \{(t, 1) : 1 < t\}$;
(v) P_T and Q_T are Mat;
(vi) neither P_T nor Q_T are MK.

Proof We can prove (i)-(iii) easily. Using Lemma 12, we can prove (iv). (v) and (vi) follow from Example 9. □

Example 14 Put $X = [0, 2)$ and define f and d as in Example 13. Define a mapping T on X by

$$Tx = \begin{cases} 0 & \text{if } x \leq 1, \\ 2x - 2 & \text{if } x \geq 1. \end{cases}$$

Then (i)-(iii) of Example 13 and the following hold:
(iv) $P_T = Q_T = \{(1 + \lambda, 2\lambda) : \lambda \in (0, 1)\}$;
(v) P_T and Q_T are BW;
(vi) neither P_T nor Q_T are Mat.

Proof We can prove (i)-(iii) easily. Using Lemma 12, we can prove (iv). (v) and (vi) follow from Example 10. □

Example 15 Let $X = [0, 1) \cup (1, \infty)$ and define a function d from $X \times X$ into $[0, \infty)$ by (5), where $f(x) = \min\{x, 1\}$. Define a mapping T on X by

$$Tx = \begin{cases} 0 & \text{if } x < 1, \\ 1/x & \text{if } x > 1. \end{cases}$$

Then (i)-(iii) of Example 13 and the following hold:
(iv) $P_T = Q_T = \{(1, u) : 0 < u < 1\}$;
(v) P_T and Q_T are MK;
(vi) neither P_T nor Q_T are NT.

Proof We can prove (i)-(iii) easily. Using Lemma 12, we can prove (iv). (v) and (vi) follow from Example 11. □

We finally give the following example, which tells us that we cannot extend Theorem 1 to a Meir-Keeler type contractive condition.

Example 16 Let $X = [0, 1)$ and define a function d from $X \times X$ into $[0, \infty)$ by (6). Define a mapping T on X by

$$Tx = \begin{cases} 1/2 & \text{if } x = 0, \\ \sqrt{x} & \text{if } x \neq 0. \end{cases}$$

Then the following hold:

(i) (X, d) is a complete metric space;

(ii) $d(x, y) < 1$ holds for any $x, y \in X$;

(iii) for any $x \in X$, $\{T^n x\}$ converges to 1 in the Euclidean space $\mathbb{R}^1$;

(iv) $D_T(x) = 1$ holds for any $x \in X$;

(v) $TX = (0, 1)$;

(vi) $Q_T = \{(1, u) : 0 < u < 1\}$;

(vii) Q_T is MK;

(viii) Q_T is not NT;

(ix) T does not have a fixed point.

Proof We can easily prove (i)-(vi) and (ix). (vii) and (viii) follow from Example 11. □

4 Conclusions

In this paper, introducing a new contractive condition (see Definition 8(6)), we generalize Hegedüs-Szilágyi's fixed point theorem (Theorem 1) in complete metric spaces proved in 1980. In Section 3, we discuss the relationship between the new contractive condition and other contractive conditions. We also show that we cannot extend Theorem 1 to Meir-Keeler type (see Example 16).

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Competing interests
The author declares that he has no competing interests.

Authors' contributions
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