Suzuki *Fixed Point Theory and Applications* (2017) 2017:30
https://doi.org/10.1186/s13663-017-0623-y

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH

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Characterizations of contractive conditions by using convergent sequences

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Abstract

We give characterizations of the contractive conditions, by using convergent sequences. Since we use a unified method, we can compare the contractive conditions very easily. We also discuss the contractive conditions of integral type by a unified method.

MSC: Primary 54E99; secondary 54H25

Keywords: contractive condition; contraction of integral type

1 Introduction and preliminaries

Throughout this paper we denote by $\mathbb{N}$ the set of all positive integers.

The fixed point theorem for contractions is referred to as the *Banach contraction principle*.

Theorem 1 ([1, 2]) *Let (X,d) be a complete metric space and let T be a contraction on X, that is, there exists $r \in [0,1)$ such that $d(Tx,Ty) \le r d(x,y)$ for all $x,y \in X$. Then T has a unique fixed point z. Moreover, $\{T^n x\}$ converges to z for any $x \in X$.*

Theorem 1 has many generalizations. In other words, we have studied many contractive conditions. Using the subsets Q of $[0,\infty)^2$ defined by

$$Q = \{(d(x,y), d(Tx,Ty))\},$$

Hegedüs and Szilágyi in [3] studied some contractive conditions. See Jachymski [4] and the references therein. Using the subsets Q of $(0,\infty)^2$ defined by

$$Q = \{(d(x,y), d(Tx,Ty))\} \cap (0,\infty)^2,$$

we studied contractive conditions in [5]. Using subsets of $(0,\infty)^2$, we list some contractive conditions, which are weaker than or equivalent to the Browder contractive condition.

Definition 2 Let Q be a subset of $(0,\infty)^2$.

(1) Q is said to be *CJM* [6–9] if the following hold:

(1-i) For any $\varepsilon > 0$, there exists $\delta > 0$ such that $u \le \varepsilon$ holds for any $(t,u) \in Q$ with $t < \varepsilon + \delta$.

(1-ii) $u < t$ holds for any $(t, u) \in Q$.

(2) Q is said to be *Meir-Keeler* (*MK*, for short) [10] if for any $\varepsilon > 0$, there exists $\delta > 0$ such that $u < \varepsilon$ holds for any $(t, u) \in Q$ with $t < \varepsilon + \delta$.

(3) Q is said to be *Boyd-Wong* (*BW*, for short) [11] if there exists a function φ from $(0, \infty)$ into itself satisfying the following:
  - (3-i) φ is upper semicontinuous from the right.
  - (3-ii) $\varphi(t) < t$ holds for any $t \in (0, \infty)$.
  - (3-iii) $u \le \varphi(t)$ holds for any $(t, u) \in Q$.

(4) Q is said to be of *New Type* (*NT*, for short) [12] if there exists a function φ from $(0, \infty)$ into itself satisfying the following:
  - (4-i) $\varphi(t) < t$ for any $t \in (0, \infty)$.
  - (4-ii) For any $\varepsilon > 0$, there exists $\delta > 0$ such that $\varepsilon < t < \varepsilon + \delta$ implies $\varphi(t) \le \varepsilon$.
  - (4-iii) $u \le \varphi(t)$ holds for any $(t, u) \in Q$.

(5) Q is said to be *Matkowski* (*Mat*, for short) [13] if there exists a function φ from $(0, \infty)$ into itself satisfying the following:
  - (5-i) φ is nondecreasing.
  - (5-ii) $\lim_n \varphi^n(t) = 0$ for any $t \in (0, \infty)$.
  - (5-iii) $u \le \varphi(t)$ holds for any $(t, u) \in Q$.

(6) Q is said to be a *Browder* (*Bro*, for short) [14] if there exists a function φ from $(0, \infty)$ into itself satisfying the following:
  - (6-i) φ is nondecreasing and right continuous.
  - (6-ii) $\varphi(t) < t$ holds for any $t \in (0, \infty)$.
  - (6-iii) $u \le \varphi(t)$ holds for any $(t, u) \in Q$.

We know the following implications:

$$
\begin{array}{ccccccc}
\text{Bro} & \rightarrow & \text{BW} & \rightarrow & \text{MK} & \rightarrow & \text{CJM} \\
 & \searrow & & \searrow & & \nearrow & \\
 & & \text{Mat} & \rightarrow & \text{NT} & &
\end{array}
$$

We note that for each implication, there exists a counterexample for its converse implication.

There is a problem in the above list. The expressions as regards the conditions vary. So we cannot understand easily the relationship between the contractive conditions. Motivated by this, in this paper, we give characterizations (Theorems 5-10) of the contractive conditions, by using convergent sequences. Since we use a unified method, we can compare the contractive conditions very easily. For example, we can prove quite easily that if Q is BW and Matkowski, then Q is Browder (see Theorem 12). We also discuss the contractive conditions of integral type by a unified method.

2 New definitions

We introduce the following definitions in order to treat the contractive conditions appearing in Section 1 by a unified method.

Definition 3 Let Q be a subset of $(0, \infty)^2$.

(1) A sequence $\{(t_n, u_n)\}$ is said to satisfy *Condition* Δ if $\{(t_n, u_n)\}$ does not converge to (t, t) for any $t \in (0, \infty)$.
(2) Q is said to satisfy *Condition* $\mathrm{C}(0,0,0)$ if the following hold:
 (2-i) $u < t$ holds for any $(t, u) \in Q$.
 (2-ii) Every sequence $\{(t_n, u_n)\}$ in Q satisfies Condition Δ provided $\{t_n\}$ and $\{u_n\}$ are strictly decreasing.
(3) Q is said to satisfy *Condition* $\mathrm{C}(0,0,1)$ if the following hold:
 (3-i) Q satisfies Condition $\mathrm{C}(0,0,0)$.
 (3-ii) Every sequence $\{(t_n, u_n)\}$ in Q satisfies Condition Δ provided $\{t_n\}$ is strictly decreasing and $\{u_n\}$ is constant.
(4) Q is said to satisfy *Condition* $\mathrm{C}(0,0,2)$ if the following hold:
 (4-i) Q satisfies Condition $\mathrm{C}(0,0,0)$.
 (4-ii) Every sequence $\{(t_n, u_n)\}$ in Q satisfies Condition Δ provided $\{t_n\}$ is strictly decreasing and $\{u_n\}$ is nondecreasing.
(5) Q is said to satisfy *Condition* $\mathrm{C}(0,1,0)$ if the following hold:
 (5-i) Q satisfies Condition $\mathrm{C}(0,0,0)$.
 (5-ii) Every sequence $\{(t_n, u_n)\}$ in Q satisfies Condition Δ provided $\{t_n\}$ is constant and $\{u_n\}$ is strictly increasing.
(6) Q is said to satisfy *Condition* $\mathrm{C}(1,0,0)$ if the following hold:
 (6-i) Q satisfies Condition $\mathrm{C}(0,0,0)$.
 (6-ii) Every sequence $\{(t_n, u_n)\}$ in Q satisfies Condition Δ provided $\{t_n\}$ and $\{u_n\}$ are strictly increasing.
(7) Let $(p, q, r) \in \{0,1\}^2 \times \{0,1,2\}$. Then Q is said to satisfy *Condition* $\mathrm{C}(p,q,r)$ if Q satisfies Conditions $\mathrm{C}(p,0,0)$, $\mathrm{C}(0,q,0)$ and $\mathrm{C}(0,0,r)$.

Proposition 4 *Let* $p_1, p_2, q_1, q_2 \in \{0,1\}$ *and let* $r_1, r_2 \in \{0,1,2\}$. *Let* Q *be a subset of* $(0,\infty)^2$. *Then the following are equivalent*:
 (i) *Q satisfies Conditions* $\mathrm{C}(p_1, q_1, r_1)$ *and* $\mathrm{C}(p_2, q_2, r_2)$.
 (ii) *Q satisfies Condition* $\mathrm{C}(\max\{p_1,p_2\}, \max\{q_1,q_2\}, \max\{r_1,r_2\})$.

Proof Obvious. □

3 Characterizations

In this section, we give characterizations of the contractive conditions appearing in Section 1 by a unified method.

Theorem 5 *Let* Q *be a subset of* $(0,\infty)^2$. *Then the following are equivalent*:
 (i) *Q is CJM.*
 (ii) *Q satisfies Condition* $\mathrm{C}(0,0,0)$.

Theorem 6 *Let* Q *be a subset of* $(0,\infty)^2$. *Then the following are equivalent*:
 (i) *Q is MK.*
 (ii) *Q satisfies Condition* $\mathrm{C}(0,0,1)$, *that is, the following hold*:
  (a) $u < t$ *holds for any* $(t, u) \in Q$.
  (b) *Every sequence* $\{(t_n, u_n)\}$ *in* Q *satisfies Condition* Δ *provided* $\{t_n\}$ *is strictly decreasing and* $\{u_n\}$ *is nonincreasing.*

Theorem 7 *Let Q be a subset of $(0,\infty)^2$. Then the following are equivalent*:

(i) *Q is BW.*

(ii) *Q satisfies Condition* $\mathrm{C}(0,1,2)$, *that is, the following hold*:

  (a) *$u < t$ holds for any $(t,u) \in Q$.*

  (b) *Every sequence $\{(t_n,u_n)\}$ in Q satisfies Condition Δ provided $\{t_n\}$ is nonincreasing.*

Remark In order to prove (ii) $\Rightarrow$ (i), we define a function φ from $(0,\infty)$ into itself by

$$\varphi(t) = \limsup\bigl[\psi(u) : u \to t, t \le u\bigr],$$

where ψ is a function from $(0,\infty)$ into itself defined by

$$\psi(t) = \max\bigl\{t/2, \sup\bigl\{u : (t,u) \in Q\bigr\}\bigr\}. \tag{1}$$

See Lemma 4 in [5].

Theorem 8 *Let Q be a subset of $(0,\infty)^2$. Then the following are equivalent*:

(i) *Q is NT.*

(ii) *Q satisfies Condition* $\mathrm{C}(0,1,0)$, *that is, the following hold*:

  (a) *$u < t$ holds for any $(t,u) \in Q$.*

  (b) *Every sequence $\{(t_n,u_n)\}$ in Q satisfies Condition Δ provided $\{t_n\}$ and $\{u_n\}$ are strictly decreasing.*

  (c) *Every sequence $\{(t_n,u_n)\}$ in Q satisfies Condition Δ provided $\{t_n\}$ is constant and $\{u_n\}$ is strictly increasing.*

Remark In order to prove (ii) $\Rightarrow$ (i), we define a function φ from $(0,\infty)$ into itself by

$$\varphi(t) = \max\bigl\{t/2, \sup\bigl\{u : (t,u) \in Q\bigr\}\bigr\}.$$

Theorem 9 *Let Q be a subset of $(0,\infty)^2$. Then the following are equivalent*:

(i) *Q is Matkowski.*

(ii) *Q satisfies Condition* $\mathrm{C}(1,1,0)$, *that is, the following hold*:

  (a) *$u < t$ holds for any $(t,u) \in Q$.*

  (b) *Every sequence $\{(t_n,u_n)\}$ in Q satisfies Condition Δ provided $\{t_n\}$ and $\{u_n\}$ are strictly decreasing.*

  (c) *Every sequence $\{(t_n,u_n)\}$ in Q satisfies Condition Δ provided $\{t_n\}$ is nondecreasing and $\{u_n\}$ are strictly increasing.*

Theorem 10 *Let Q be a subset of $(0,\infty)^2$. Then the following are equivalent*:

(i) *Q is Browder.*

(ii) *Q satisfies Condition* $\mathrm{C}(1,1,2)$, *that is, the following hold*:

  (a) *$u < t$ holds for any $(t,u) \in Q$.*

  (b) *Every sequence $\{(t_n,u_n)\}$ in Q satisfies Condition Δ.*

Remark In order to prove (ii) $\Rightarrow$ (i), we define a function φ from $(0,\infty)$ into itself by

$$\varphi(t) = \frac{t}{2} + \frac{1}{2}\max\bigl\{\sup\bigl\{\psi(u) : u \in (0,\infty), u \le t\bigr\},$$

$$\sup\big\{\psi(u)+2(t-u):u\in(0,\infty),t\leq u\big\}\big\},$$

where ψ is a function from $(0,\infty)$ into itself defined by (1). See Lemma 6 in [15]. See also the proof of Proposition 1 in [16].

We only give a proof of Theorem 9. The reason of this is that we can prove the other theorems easily by using the method in the proof of Theorem 9.

Lemma 11 *Let φ be a nondecreasing function from $(0,\infty)$ into itself. Then the following are equivalent*:

(i) $\lim_n \varphi^n(t) = 0$ *holds for any* $t\in(0,\infty)$.

(ii) $\varphi(t) < t$ *holds for any* $t\in(0,\infty)$. *For any* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that*

$$\varepsilon < t < \varepsilon+\delta \quad \textit{implies} \quad \varphi(t)\leq\varepsilon.$$

Proof We first show (i) $\Rightarrow$ (ii). We assume (i). Arguing by contradiction, there exists $\tau\in(0,\infty)$ satisfying $\varphi(\tau)\geq\tau$. Since φ is nondecreasing, we have $\varphi^2(\tau)\geq\varphi(\tau)$. Continuing this argument, we can show that $\{\varphi^n(\tau)\}$ is nondecreasing. Since $\varphi(\tau)\geq\tau>0$ holds, $\{\varphi^n(\tau)\}$ cannot converge to 0. This is a contradiction. So we have shown $\varphi(t)<t$ for any $t\in(0,\infty)$. Also, arguing by contradiction, we assume that there exists $\varepsilon>0$ such that for any $\delta>0$, there exists t satisfying

$$\varepsilon < t < \varepsilon+\delta \quad \text{and} \quad \varphi(t)>\varepsilon.$$

Since φ is nondecreasing, we have

$$\varphi(t)>\varepsilon \quad \text{provided} \quad t>\varepsilon.$$

Fix $\tau>\varepsilon$. Then we have $\varphi(\tau)>\varepsilon$. Hence $\varphi^2(\tau)>\varepsilon$ holds. Continuing this argument, we have $\varphi^n(\tau)>\varepsilon$ for any $n\in\mathbb{N}$. Therefore $\{\varphi^n(\tau)\}$ cannot converge to 0, which implies a contradiction. Therefore we have shown (ii).

Let us prove (ii) $\Rightarrow$ (i): We assume (ii). Arguing by contradiction, we assume that there exists $t\in(0,\infty)$ such that $\{\varphi^n(t)\}$ does not converge to 0. Since $\{\varphi^n(t)\}$ is strictly decreasing, $\{\varphi^n(t)\}$ converges to some $\varepsilon\in(0,\infty)$. We can choose $\delta>0$ satisfying

$$\varepsilon < t < \varepsilon+\delta \quad \text{implies} \quad \varphi(t)\leq\varepsilon.$$

Choose $\nu\in\mathbb{N}$ satisfying $\varphi^\nu(t)<\varepsilon+\delta$. Then we have $\varphi^{\nu+1}(t)\leq\varepsilon$, which implies a contradiction. Therefore we have shown (i). □

Proof of Theorem 9 We first prove (i) $\Rightarrow$ (ii). We assume (i). Then there exists a function φ from $(0,\infty)$ into itself satisfying the following:

(1) φ is nondecreasing.

(2) $\lim_n \varphi^n(t) = 0$ for any $t\in(0,\infty)$.

(3) $u\leq\varphi(t)$ holds for any $(t,u)\in Q$.

For any $(t,u) \in Q$, we have by Lemma 11

$$u \leq \varphi(t) < t.$$

Thus (a) of Theorem 9 holds. Also, arguing by contradiction, we assume that (b) of Theorem 9 does not hold. Then there exists a sequence $\{(t_n,u_n)\}$ in Q such that $\{(t_n,u_n)\}$ converges to (τ,τ) for some $\tau \in (0,\infty)$ and $\{t_n\}$ and $\{u_n\}$ are strictly decreasing. By Lemma 11 again, there exists $\delta > 0$ such that

$$\tau < t < \tau + \delta \quad \text{implies} \quad \varphi(t) \leq \tau.$$

For sufficiently large $n \in \mathbb{N}$, since $\tau < t_n < \tau + \delta$ holds, we have

$$u_n \leq \varphi(t_n) \leq \tau.$$

Since $\{u_n\}$ is strictly decreasing, $\{u_n\}$ does not converge to τ, which implies a contradiction. We have shown (b) of Theorem 9. Also, arguing by contradiction, we assume that (c) of Theorem 9 does not hold. Then there exists a sequence $\{(t_n,u_n)\}$ in Q such that $\{(t_n,u_n)\}$ converges to (τ,τ) for some $\tau \in (0,\infty)$, $\{t_n\}$ is nondecreasing and $\{u_n\}$ is strictly increasing. By (1) and (3), we have

$$u_n \leq \varphi(t_n) \leq \varphi(\tau).$$

As n tends to ∞, we obtain $\tau \leq \varphi(\tau)$. By Lemma 11, this is a contradiction. We have shown (a)-(c) of Theorem 9.

Let us prove (ii) $\Rightarrow$ (i). We assume (a)-(c) of Theorem 9. Define a function φ from $(0,\infty)$ into itself by

$$\varphi(s) = \max\bigl\{s/2, \sup\bigl\{u : (t,u) \in Q, t \leq s\bigr\}\bigr\},$$

where $\sup \varnothing = -\infty$. It is obvious that φ is nondecreasing. It is also obvious that $u \leq \varphi(t)$ holds for any $(t,u) \in Q$. Since Q satisfies Condition $\mathrm{C}(0,0,0)$, we have

$$\varphi(s) \leq \max\bigl\{s/2, \sup\bigl\{t : (t,u) \in Q, t \leq s\bigr\}\bigr\} \leq s.$$

Arguing by contradiction, we assume $\varphi(\tau) = \tau$ for some $\tau \in (0,\infty)$. Then there exists a sequence $\{(t_n,u_n)\}$ in Q satisfying $t_n \leq \tau$ for $n \in \mathbb{N}$ and $\lim_n u_n = \tau$. Since $u_n < t_n \leq \tau$ holds for $n \in \mathbb{N}$, we can choose a subsequence $\{f(n)\}$ of the sequence $\{n\}$ in $\mathbb{N}$ such that $\{t_{f(n)}\}$ is nondecreasing and $\{u_{f(n)}\}$ is strictly increasing. Since $\{t_{f(n)}\}$ converges to τ, we obtain a contradiction. Therefore we have shown

$$\varphi(s) < s$$

for any $s \in (0,\infty)$. We will show the following:

- For $\varepsilon \in (0,\infty)$, there exists $\delta > 0$ such that

$$\varepsilon < t < \varepsilon + \delta \quad \text{implies} \quad \varphi(t) \leq \varepsilon.$$

Arguing by contradiction, we assume that there exists $\varepsilon > 0$ such that for any $\delta \in (0, \varepsilon)$, there exists t satisfying

$$\varepsilon < t < \varepsilon + \delta \quad \text{and} \quad \varphi(t) > \varepsilon.$$

So we can choose a sequence $\{t_n\}$ in $(\varepsilon, 2\varepsilon)$ such that $\{t_n\}$ is strictly decreasing, $\{t_n\}$ converges to ε and $\varepsilon < \varphi(t_n)$ for $n \in \mathbb{N}$. Noting $t_n/2 < \varepsilon$, we have

$$\varphi(t_n) = \sup\bigl\{u : (t, u) \in Q, t \leq t_n\bigr\}.$$

So there exists $(t'_n, u'_n) \in Q$ satisfying $u'_n > \varepsilon$ and $t'_n \leq t_n$. Since $(t'_n, u'_n) \in Q$ holds, we have

$$\varepsilon < u'_n < t'_n \leq t_n.$$

Hence $\{t'_n\}$ and $\{u'_n\}$ converge to ε. So we can choose a subsequence $\{f(n)\}$ of $\{n\}$ such that $\{t'_{f(n)}\}$ and $\{u'_{f(n)}\}$ are strictly decreasing. This contradicts that Q satisfies Condition C(0,0,0). By Lemma 11, we obtain (2). □

By Proposition 4, we can prove the following.

Theorem 12 *Let Q be a subset of* $(0, \infty)^2$. *Then the following are equivalent*:
(i) *Q is Browder.*
(ii) *Q is BW and Matkowski.*

Proof By Proposition 4 and Theorems 7, 9 and 10, both are equivalent to Condition C(1,1,2). □

4 Integral type

There is another merit in our approach.

Branciari in [17] introduced contractions of integral type as follows: A mapping T on a metric space (X, d) is a *Branciari contraction* if there exist $r \in [0, 1)$ and a locally integrable function f from $[0, \infty)$ into itself such that

$$\int_0^s f(t)\,dt > 0 \quad \text{and} \quad \int_0^{d(Tx,Ty)} f(t)\,dt \leq r \int_0^{d(x,y)} f(t)\,dt$$

holds for all $s > 0$ and $x, y \in X$. Jachymski in [4] proved that the concepts of the Branciari contraction and the Browder contraction are equivalent.

In [5, 15, 18, 19], we also studied contractions of integral type for several contractive conditions stated in Section 1. We note that we have used various methods to prove theorems there. Motivated by this fact, in this paper, we study contractions of integral type by a unified method.

Throughout this section, we let Q be a subset of $(0, \infty)^2$. Let θ be a nondecreasing function from $(0, \infty)$ into itself and define a subset R of $(0, \infty)^2$ by

$$R = \bigl\{\bigl(\theta(t), \theta(u)\bigr) : (t, u) \in Q\bigr\}.$$

Lemma 13 *If R satisfies Condition* C(0,0,0), *then Q also satisfies Condition* C(0,0,0).

Proof We assume that R satisfies Condition C(0,0,0). Fix $(t,u) \in Q$. Arguing by contradiction, we assume $u \geq t$. Then since θ is nondecreasing, we have $\theta(u) \geq \theta(t)$, which implies a contradiction. Therefore we have shown

$$u < t \quad \text{for any } (t,u) \in Q.$$

Also, arguing by contradiction, we assume the following:

- There exists a sequence $\{(t_n, u_n)\}$ in Q such that $\{t_n\}$ and $\{u_n\}$ are strictly decreasing and $\lim_n t_n = \lim_n u_n = \tau$ holds for some $\tau \in (0,\infty)$.

We consider the following two cases:

- $\lim[\theta(t) : t \to \tau + 0] = \theta(s)$ holds for some $s \in (\tau,\infty)$.
- $\lim[\theta(t) : t \to \tau + 0] < \theta(s)$ holds for any $s \in (\tau,\infty)$.

In the first case, we have $\theta(t_n) = \theta(u_n)$ for sufficiently large $n \in \mathbb{N}$, which implies a contradiction. In the second case, taking subsequences, without loss of generality, we may assume that $\{\theta(t_n)\}$ and $\{\theta(u_n)\}$ are strictly decreasing. We have

$$\lim_{n\to\infty} \theta(t_n) = \lim_{t\to\tau+0} \theta(t) = \lim_{n\to\infty} \theta(u_n),$$

which also implies a contradiction. Therefore Q satisfies Condition C(0,0,0). □

Remark Compare this proof with the proof of Theorem 2.7 in [18]. In our new proof, the reason why we do not need any continuity of θ is quite clear.

Lemma 14 *If R satisfies Condition* C(0,0,1) *and θ is right continuous, then Q also satisfies Condition* C(0,0,1).

Proof We assume that R satisfies Condition C(0,0,1). Then by Lemma 13, Q satisfies Condition C(0,0,0). Arguing by contradiction, we assume that there exists a sequence $\{(t_n,\tau)\}$ in Q such that $\{t_n\}$ is strictly decreasing and $\lim_n t_n = \tau$ holds. Since θ is right continuous, we have

$$\lim_{n\to\infty} \theta(t_n) = \lim_{t\to\tau+0} \theta(t) = \theta(\tau).$$

Since $\theta(\tau) < \theta(t_n)$ holds for $n \in \mathbb{N}$, taking a subsequence, without loss of generality, we may assume that $\{\theta(t_n)\}$ is strictly decreasing. Hence R does not satisfy Condition C(0,0,1), which implies a contradiction. □

Remark Compare this proof with the proof of Theorem 2.1 in [18]. In our new proof, the reason why we need the right continuity of θ is quite clear.

Lemma 15 *If R satisfies Condition* C(0,1,0) *and θ is left continuous, then Q also satisfies Condition* C(0,1,0).

Proof We assume that R satisfies Condition C(0,1,0). Then by Lemma 13, Q satisfies Condition C(0,0,0). Arguing by contradiction, we assume that there exists a sequence $\{(\tau,u_n)\}$

in Q such that $\{u_n\}$ is strictly increasing and $\lim_n u_n = \tau$ holds. Since θ is left continuous, we have

$$\lim_{n\to\infty} \theta(u_n) = \lim_{t\to\tau-0} \theta(t) = \theta(\tau).$$

Since $\theta(u_n) < \theta(\tau)$ holds for $n \in \mathbb{N}$, taking a subsequence, without loss of generality, we may assume that $\{\theta(u_n)\}$ is strictly increasing. Hence R does not satisfy Condition $\mathrm{C}(0,1,0)$, which implies a contradiction. □

Remark Compare this proof with the proof of Proposition 2.1 in [19]. In our new proof, the reason why we need the left continuity of θ is quite clear.

Lemma 16 *If R satisfies Condition* $\mathrm{C}(0,1,2)$ *and θ is continuous, then Q also satisfies Condition* $\mathrm{C}(0,1,2)$.

Proof We assume that R satisfies Condition $\mathrm{C}(0,1,2)$. Then by Lemma 13, Q satisfies Condition $\mathrm{C}(0,0,0)$. Arguing by contradiction, we assume that there exists a sequence $\{(t_n,u_n)\}$ in Q such that $\{t_n\}$ is nonincreasing, $\{u_n\}$ is nondecreasing and $\lim_n t_n = \lim_n u_n = \tau$ holds. Since θ is nondecreasing, we note that $\{\theta(t_n)\}$ is nonincreasing and $\{\theta(u_n)\}$ is nondecreasing. Since θ is continuous, we have

$$\begin{aligned}\lim_{n\to\infty} \theta(t_n) &= \lim_{t\to\tau+0} \theta(t) = \theta(\tau)\\ &= \lim_{t\to\tau-0} \theta(t) = \lim_{n\to\infty} \theta(u_n).\end{aligned}$$

Hence R does not satisfy Condition $\mathrm{C}(0,1,2)$, which implies a contradiction. □

Remark Compare this proof with the proofs of Proposition 8 in [15] and Proposition 9 in [5]. In our new proof, the reason why we need the continuity of θ is quite clear.

Lemma 17 *If R satisfies Condition* $\mathrm{C}(1,0,0)$, *then Q also satisfies Condition* $\mathrm{C}(1,0,0)$.

Proof We assume that R satisfies Condition $\mathrm{C}(1,0,0)$. Then by Lemma 13, Q satisfies Condition $\mathrm{C}(0,0,0)$. Arguing by contradiction, we assume that there exists a sequence $\{(t_n,u_n)\}$ in Q such that $\{t_n\}$ and $\{u_n\}$ are strictly increasing and $\lim_n t_n = \lim_n u_n = \tau$ holds for some $\tau \in (0,\infty)$. We consider the following two cases:

- $\theta(s) = \lim[\theta(t) : t \to \tau - 0]$ holds for some $s \in (0,\tau)$.
- $\theta(s) < \lim[\theta(t) : t \to \tau - 0]$ holds for any $s \in (0,\tau)$.

In the first case, we have $\theta(t_n) = \theta(u_n)$ for sufficiently large $n \in \mathbb{N}$, which implies a contradiction. In the second case, taking subsequences, without loss of generality, we may assume that $\{\theta(t_n)\}$ and $\{\theta(u_n)\}$ are strictly increasing. Since θ is nondecreasing, we have

$$\lim_{n\to\infty} \theta(t_n) = \lim_{t\to\tau-0} \theta(t) = \lim_{n\to\infty} \theta(u_n).$$

Hence R does not satisfy Condition $\mathrm{C}(1,0,0)$, which implies a contradiction. □

Theorem 18 *If R is CJM, then Q is also CJM.*

Proof By Theorem 5, R satisfies Condition C$(0,0,0)$. So by Lemma 13, Q satisfies Condition C$(0,0,0)$. By Theorem 5 again, Q is CJM. □

Theorem 19 *If R is MK and θ is right continuous, then Q is also MK.*

Proof By Theorem 6, R satisfies Condition C$(0,0,1)$. So by Lemma 14, Q satisfies Condition C$(0,0,1)$. By Theorem 6 again, Q is MK. □

Theorem 20 *If R is BW and θ is continuous, then Q is also BW.*

Proof By Theorem 7, R satisfies Condition C$(0,1,2)$. So by Lemma 16, Q satisfies Condition C$(0,1,2)$. By Theorem 7 again, Q is BW. □

Theorem 21 *If R is NT and θ is left continuous, then Q is also NT.*

Proof By Theorem 8, R satisfies Condition C$(0,1,0)$. So by Lemma 15, Q satisfies Condition C$(0,1,0)$. By Theorem 8 again, Q is NT. □

Theorem 22 *If R is Matkowski and θ is left continuous, then Q is also Matkowski.*

Proof By Theorem 9, R satisfies Condition C$(1,1,0)$. So by Lemmas 15 and 17, Q satisfies Condition C$(1,1,0)$. By Theorem 9 again, Q is Matkowski. □

Theorem 23 *If R is Browder and θ is continuous, then Q is also Browder.*

Proof By Theorem 10, R satisfies Condition C$(1,1,2)$. So by Lemmas 16 and 17, Q satisfies Condition C$(1,1,2)$. By Theorem 10 again, Q is Browder. □

5 Conclusions

In this paper, we give characterizations of the contractive conditions, by using convergent sequences (see Theorems 5-10). Since we use a unified method, we can compare the contractive conditions very easily (see Theorem 12). We also discuss the contractive conditions of integral type by a unified method (see Theorems 18-23).

Acknowledgements
The author is supported in part by JSPS KAKENHI Grant Number 16K05207 from Japan Society for the Promotion of Science.

Competing interests
The author declares that he has no competing interests.

Authors' contributions
The author read and approved the final manuscript.

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Received: 15 July 2017 Accepted: 27 November 2017 Published online: 21 December 2017

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