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On approximate solutions for a class of semilinear fractional-order differential equations in Banach spaces

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Abstract

We apply the topological degree theory for condensing maps to study approximation of solutions to a fractional-order semilinear differential equation in a Banach space. We assume that the linear part of the equation is a closed unbounded generator of a C_0-semigroup. We also suppose that the nonlinearity satisfies a regularity condition expressed in terms of the Hausdorff measure of noncompactness. We justify the scheme of semidiscretization of the Cauchy problem for a differential equation of a given type and evaluate the topological index of the solution set. This makes it possible to obtain a result on the approximation of solutions to the problem.

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1 Introduction

The problem of approximation of solutions to semilinear differential equations in Banach spaces has attracted the attention of many researchers (see, *e.g.*, [1–10], and the references therein). In these works, it was supposed that the linear part of the equation is a compact analytic semigroup or condensing C_0-semigroup.

Recently this approach was extended to the case of a fractional-order semilinear equation in a Banach space. In particular, Liu, Li, and Piskarev [11], using the general approximation scheme of Vainikko [12], considered approximation of the Cauchy problem for the case where the linear part generates an analytic and compact resolution operator and the corresponding nonlinearity is sufficiently smooth.

Our main goal in this paper is to apply the topological degree theory for condensing maps to study the problem of approximation of solutions to a Caputo fractional-order semilinear differential equation in a Banach space under the assumption that the linear part of the equation is a closed unbounded generator of a C_0-semigroup. It is also supposed that the nonlinearity is a continuous map satisfying a regularity condition expressed in terms of the Hausdorff measure of noncompactness.

It is worth noting that the interest to the theory of differential equations of fractional order essentially strengthened in the recent years due to interesting applications in physics, enginery, biology, economics, and other branches of natural sciences (see, *e.g.*, monographs [13–21], and the references therein). Among a large amount of papers dedicated to fractional-order equations, let us mention works [22–30], where existence results of various types were obtained. Notice that results on the existence of solutions to the Cauchy and the periodic problems for semilinear differential inclusions in a Banach space under conditions similar to the above mentioned were obtained in the authors' papers [31, 32].

In this paper, we justify the scheme of semidiscretization of the Cauchy problem for a differential equation of a given type and evaluate the topological index of the solution set. This makes it possible to present the main result of the paper (Theorem 3) on the approximation of solutions to the above problem.

2 Differential equations of fractional order

In this section, we recall some notions and definitions (details can be found, *e.g.*, in [17, 19, 20]). Let E be a real Banach space.

Definition 1 The Riemann-Liouville fractional derivative of order $q \in (0,1)$ of a continuous function $g : [0,a] \to E$ is the function $D^q g$ of the following form:

$$D^q g(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t (t-s)^{-q} g(s)\,ds,$$

provided that the right-hand side of this equality is well defined.

Here Γ is the Euler gamma function

$$\Gamma(r) = \int_0^\infty s^{r-1} e^{-s}\,ds.$$

Definition 2 The Caputo fractional derivative of order $q \in (0,1)$ of a continuous function $g : [0,a] \to E$ is the function ${}^C D^q g$ defined in the following way:

$${}^C D^q g(t) = \bigl(D^q \bigl(g(\cdot) - g(0)\bigr)\bigr)(t),$$

provided that the right-hand side of this equality is well defined.

Consider the Cauchy problem for a Caputo fractional-order semilinear differential equation of the form

$${}^C D^q x(t) = Ax(t) + f\bigl(t, x(t)\bigr), \quad t \in [0,a], \tag{2.1}$$

with the initial condition

$$x(0) = x_0, \tag{2.2}$$

where $0 < q < 1$, and a linear operator A satisfies the following condition:

(A) $A : D(A) \subseteq E \to E$ is a linear closed (not necessarily bounded) operator generating a C_0-semigroup $\{U(t)\}_{t \geq 0}$ of bounded linear operators in E, and $f : [0,a] \times E \to E$ is a continuous map obeying the following properties:

(f1) f maps bounded sets into bounded ones;
(f2) there exists $\mu \in \mathbb{R}_+$ such that, for every bounded set $\Delta \subset E$, we have:

$$\chi\big(f(t,\Delta)\big) \le \mu\chi(\Delta)$$

for $t \in [0,a]$, where χ is the Hausdorff measure of noncompactness in E:

$$\chi(\Delta) = \inf\{\varepsilon > 0, \text{for which } \Delta \text{ has a finite } \varepsilon\text{-net in } E\}.$$

Notice that we may assume, without loss of generality, that the semigroup $\{\mathcal{U}(t)\}_{t\ge 0}$ is bounded:

$$\sup_{t\ge 0}\big\|\mathcal{U}(t)\big\| = M < \infty.$$

Otherwise, we may replace the operator A with the operator $\widetilde{A}x = Ax - \omega x$ and, respectively, the nonlinearity f with $\widetilde{f}(t,x) = f(t,x) + \omega x$, where $\omega > 0$ is a constant greater than the order of growth of the semigroup.

Definition 3 (cf. [22, 29, 30]) A mild solution of problem (2.1)-(2.2) is a function $x \in C([0,a];E)$ that can be represented as

$$x(t) = \mathcal{G}(t)x_0 + \int_0^t (t-s)^{q-1}\mathcal{T}(t-s)f\big(s,x(s)\big)\,ds, \quad t \in [0,a],$$

where

$$\mathcal{G}(t) = \int_0^\infty \xi_q(\theta)U\big(t^q\theta\big)\,d\theta, \qquad \mathcal{T}(t) = q\int_0^\infty \theta\xi_q(\theta)U\big(t^q\theta\big)\,d\theta,$$
$$\xi_q(\theta) = \frac{1}{q}\theta^{-1-\frac{1}{q}}\Psi_q\big(\theta^{-1/q}\big),$$
$$\Psi_q(\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-qn-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q), \quad \theta \in \mathbb{R}_+.$$

Remark 1 (See, *e.g.*, [29]) $\xi_q(\theta) \ge 0$, $\int_0^\infty \xi_q(\theta)\,d\theta = 1$, $\int_0^\infty \theta\xi_q(\theta)\,d\theta = \frac{1}{\Gamma(q+1)}$.

Lemma 1 (See [29], Lemma 3.4) *The operator functions $\mathcal{G}$ and $\mathcal{T}$ possess the following properties*:

(1) *For each $t \in [0,a]$, $\mathcal{G}(t)$ and $\mathcal{T}(t)$ are linear bounded operators*; *more precisely, for each $x \in E$, we have*

$$\big\|\mathcal{G}(t)x\big\|_E \le M\|x\|_E, \tag{2.3}$$

$$\big\|\mathcal{T}(t)x\big\|_E \le \frac{qM}{\Gamma(1+q)}\|x\|_E. \tag{2.4}$$

(2) *The operator functions $\mathcal{G}$ and $\mathcal{T}$ are strongly continuous*, *that is*, *the functions $t \in [0,a] \to \mathcal{G}(t)x$ and $t \in [0,a] \to \mathcal{T}(t)x$ are continuous for each $x \in E$.*

2.1 Measures of noncompactness and condensing maps

Let $\mathcal{E}$ be a Banach space. Introduce the following notation:

- $\mathrm{Pb}(\mathcal{E}) = \{A \subseteq \mathcal{E} : A \neq \varnothing \text{ is bounded}\}$;
- $\mathrm{Pv}(\mathcal{E}) = \{A \in \mathrm{Pb}(\mathcal{E}) : A \text{ is convex}\}$;
- $K(\mathcal{E}) = \{A \in \mathrm{Pb}(\mathcal{E}) : A \text{ is compact}\}$;
- $\mathrm{Kv}(\mathcal{E}) = \mathrm{Pv}(\mathcal{E}) \cap K(\mathcal{E})$.

Definition 4 (See, *e.g.*, [33, 34]) Let $(\mathcal{A}, \geq)$ be a partially ordered set. A function $\beta : \mathrm{Pb}(\mathcal{E}) \to \mathcal{A}$ is called a measure of noncompactness (MNC) in $\mathcal{E}$ if for each $\Omega \in \mathrm{Pb}(\mathcal{E})$, we have:

$$\beta(\overline{\mathrm{co}}\Omega) = \beta(\Omega),$$

where $\overline{\mathrm{co}}\Omega$ denotes the closure of the convex hull of Ω.

A measure of noncompactness β is called:

(1) *monotone* if for each $\Omega_0, \Omega_1 \in \mathrm{Pb}(\mathcal{E})$, $\Omega_0 \subseteq \Omega_1$ implies $\beta(\Omega_0) \leq \beta(\Omega_1)$;

(2) *nonsingular* if for each $a \in \mathcal{E}$ and each $\Omega \in \mathrm{Pb}(\mathcal{E})$, we have $\beta(\{a\} \cup \Omega) = \beta(\Omega)$.

If $\mathcal{A}$ is a cone in a Banach space, then the MNC β is called:

(4) *regular* if $\beta(\Omega) = 0$ is equivalent to the relative compactness of $\Omega \in \mathrm{Pb}(\mathcal{E})$;

(5) *real* if $\mathcal{A}$ is the set of all real numbers $\mathbb{R}$ with natural ordering;

(6) *algebraically semiadditive* if $\beta(\Omega_0 + \Omega_1) \leq \beta(\Omega_0) + \beta(\Omega_1)$ for all $\Omega_0, \Omega_1 \in \mathrm{Pb}(\mathcal{E})$.

It should be mentioned that the Hausdorff MNC obeys all above properties. Other examples can be presented by the following measures of noncompactness defined on $\mathrm{Pb}(C([0,a];E))$, where $C([0,a];E)$ is the space of continuous functions with values in a Banach space E:

(i) *the modulus of fiber noncompactness*

$$\varphi(\Omega) = \sup_{t \in [0,a]} \chi_E\bigl(\Omega(t)\bigr),$$

where χ_E is the Hausdorff MNC in E, and $\Omega(t) = \{y(t) : y \in \Omega\}$;

(ii) *the modulus of equicontinuity* defined as

$$\mathrm{mod}_C(\Omega) = \lim_{\delta \to 0} \sup_{y \in \Omega} \max_{|t_1 - t_2| \leq \delta} \bigl\| y(t_1) - y(t_2) \bigr\|.$$

Notice that these MNCs satisfy all above-mentioned properties except regularity. Nevertheless, notice that due to the Arzelà-Ascoli theorem, the relation

$$\varphi(\Omega) = \mathrm{mod}_C(\Omega) = 0$$

provides the relative compactness of the set Ω.

Definition 5 A continuous map $\mathcal{F} : X \subseteq \mathcal{E} \to \mathcal{E}$ is called condensing with respect to an MNC β (or β-condensing) if for every bounded set $\Omega \subseteq X$ that is not relatively compact, we have

$$\beta\bigl(\mathcal{F}(\Omega)\bigr) \ngeq \beta(\Omega).$$

More generally, given a metric space Λ of parameters, we say that a continuous map $\Gamma : \Lambda \times X \to \mathcal{E}$ is *a condensing family with respect to an MNC β (or β-condensing family)* if for every bounded set $\Omega \subseteq X$ that is not relatively compact, we have

$$\beta\bigl(\Gamma(\Lambda \times \Omega)\bigr) \ngeq \beta(\Omega).$$

Let $V \subset \mathcal{E}$ be a bounded open set, let $\mathcal{K} \subseteq \mathcal{E}$ be a closed convex subset such that $V_{\mathcal{K}} := V \cap \mathcal{K} \neq \emptyset$, let β be a monotone nonsingular MNC in $\mathcal{E}$, and let $\mathcal{F} : \overline{V}_{\mathcal{K}} \to \mathcal{K}$ be a β-condensing map such that $x \neq \mathcal{F}(x)$ for all $x \in \partial V_{\mathcal{K}}$, where $\overline{V}_{\mathcal{K}}$ and $\partial V_{\mathcal{K}}$ denote the closure and boundary of the set $V_{\mathcal{K}}$ in the relative topology of $\mathcal{K}$.

In such a setting, *the (relative) topological degree*

$$\deg_{\mathcal{K}}(i - \mathcal{F}, \overline{V}_{\mathcal{K}})$$

of the corresponding vector field $i - \mathcal{F}$ satisfying the standard properties is defined (see, *e.g.*, [34, 35]), where i is the identity map on $\mathcal{E}$. In particular, the condition

$$\deg_{\mathcal{K}}(i - \mathcal{F}, \overline{V}_{\mathcal{K}}) \neq 0$$

implies that *the fixed point set* $\operatorname{Fix} \mathcal{F} = \{x : x = \mathcal{F}(x)\}$ is a nonempty subset of $V_{\mathcal{K}}$.

To describe the next property, let us introduce the following notion.

Definition 6 Suppose that β-condensing maps $\mathcal{F}_0, \mathcal{F}_1 : \overline{V}_{\mathcal{K}} \to \mathcal{K}$ have no fixed points on the boundary $\partial V_{\mathcal{K}}$ and there exists a β-condensing family $\Gamma : [0,1] \times \overline{V}_{\mathcal{K}} \to \mathcal{K}$ such that:

(i) $x \neq \Gamma(\lambda, x)$ for all $(\lambda, x) \in [0,1] \times \partial V_{\mathcal{K}}$;

(ii) $\Gamma(0, \cdot) = \mathcal{F}_0$; $\Gamma(1, \cdot) = \mathcal{F}_1$.

Then the vector fields $\Phi_0 = i - \mathcal{F}_0$ and $\Phi_1 = i - \mathcal{F}_1$ are called homotopic:

$$\Phi_0 \sim \Phi_1.$$

The homotopy invariance property of the topological degree asserts that if $\Phi_0 \sim \Phi_1$, then $\deg_{\mathcal{K}}(i - \mathcal{F}_0, \overline{V}_{\mathcal{K}}) = \deg_{\mathcal{K}}(i - \mathcal{F}_1, \overline{V}_{\mathcal{K}})$.

Let us also mention the following properties of the topological degree.

The normalization property. If $\mathcal{F} \equiv x_0 \in \mathcal{K}$, then

$$\deg_{\mathcal{K}}(i - \mathcal{F}, \overline{V}_{\mathcal{K}}) = \begin{cases} 1 & \text{if } x_0 \in V_{\mathcal{K}}, \\ 0 & \text{if } x_0 \notin \overline{V}_{\mathcal{K}}. \end{cases}$$

Additive dependence property. Suppose that $\{V_{\mathcal{K}j}\}_{j=1}^m$ are disjoint open subsets of $V_{\mathcal{K}}$ such that $\mathcal{F}$ has no fixed points on $\overline{V}_{\mathcal{K}} \setminus \bigcup_{j=1}^m V_{\mathcal{K}j}$. Then

$$\deg_{\mathcal{K}}(i - \mathcal{F}, \overline{V}_{\mathcal{K}}) = \sum_{j=1}^{m} \deg_{\mathcal{K}}(i - \mathcal{F}, \overline{V}_{\mathcal{K}j}).$$

The map restriction property. Let $\mathcal{F} : \overline{V}_{\mathcal{K}} \to \mathcal{K}$ be a β-condensing map without fixed points on the boundary $\partial V_{\mathcal{K}}$, and let $\mathcal{L} \subset \mathcal{K}$ be a closed convex set such that $\mathcal{F}(\overline{V}_{\mathcal{K}}) \subseteq \mathcal{L}$.

Then

$$\deg_{\mathcal{K}}(i - \mathcal{F}, \overline{V}_{\mathcal{K}}) = \deg_{\mathcal{L}}(i - \mathcal{F}, \overline{V}_{\mathcal{L}}).$$

Now, let N be an isolated component of the fixed point set $\operatorname{Fix}\mathcal{F}$, that is, there exists $\varepsilon > 0$ such that

$$W_\varepsilon(N) \cap \operatorname{Fix}\mathcal{F} = N,$$

where $W_\varepsilon(N)$ denotes the ε-neighborhood of a set N in $\mathcal{K}$. From the additive dependence property of the topological degree it follows that the degree

$$\deg_{\mathcal{K}}\bigl(i - \mathcal{F}, \overline{W_\delta(N)}\bigr)$$

does not depend on δ, $0 < \delta < \varepsilon$. This generic value of the topological degree is called *the index of the set N with respect to* $\mathcal{F}$ and is denoted as

$$\operatorname{ind}_{\mathcal{F}}(N).$$

In particular, if $x_\star$ is an isolated fixed point of $\mathcal{F}$, then $\operatorname{ind}_{\mathcal{F}}(x_\star)$ is called its index.

We further need the following stability property of a nonzero index fixed point, which may be verified by using the property of continuous dependence of the fixed point set (see, *e.g.*, [34], Proposition 3.5.2).

Proposition 1 *Let* $\{h_n\}$ *be a sequence of positive numbers converging to zero, let* $H = \overline{\{h_n\}}$, *and let* $\Gamma : H \times \overline{V} \to \mathcal{E}$ *be a* β*-condensing family. Denoting* $\mathcal{F}_n = \Gamma(h_n, \cdot)$, *suppose that the map* $\mathcal{F}_0 = \Gamma(0, \cdot)$ *has a unique fixed point* x_0 *such that* $\operatorname{ind}_{\mathcal{F}_0}(x_0) \neq 0$. *Then* $\operatorname{Fix}\mathcal{F}_n \neq \emptyset$ *for all sufficiently large n, and, moreover,* $x_n \to x_0$ *for each sequence* $\{x_n\}$ *such that* $x_n \in \operatorname{Fix}\mathcal{F}_n$, $n \geq 1$.

We further also need the following statement.

Proposition 2 (Corollary 4.2.2 of [34]) *Let E be a Banach space, and let* $\{\mathfrak{f}_i\} \subset L^1([0,a];E)$ *be a semicompact sequence, i.e., it is integrably bounded and the set* $\{\mathfrak{f}_i(t)\}$ *is relatively compact in E for a.e.* $t \in [0,a]$. *Then, for every* $\delta > 0$, *there exist a measurable set* m_δ *and a compact set* $K_\delta \subset E$ *such that* $\operatorname{meas} m_\delta < \delta$ *and* $\operatorname{dist}(\mathfrak{f}_i(t), K_\delta) < \delta$ *for all i and* $t \in [0,a] \setminus m_\delta$.

3 Scheme of semidiscretization

Along with equation (2.1), for a given sequence of positive numbers $\{h_n\}$ converging to zero, consider the equation

$$D^q x_h(t) = A_h x_h(t) + f_h\bigl(t, x_h(t)\bigr), \quad t \in [0,a], \tag{3.1}$$

where $h \in H = \overline{\{h_n\}}$ is the semidiscretization parameter, $A_h : D(A_h) \subset E_h \to E_h$ are closed linear operators in Banach spaces E_h generating C_0-semigroups $\{U_h(t)\}_{t\geq 0}$. We assume that $E_0 = E$, $A_0 = A$, $f_0 = f$, and continuous maps $f_h : [0,a] \times E_h \to E_h$ satisfying conditions $(f1)$-$(f2)$ for each $h \in H$.

We suppose that there exist linear operators $Q_h : E_h \to E$, $h \in H$, $Q_0 = I$ and projection operators $P_h : E \to E_h$, $P_0 = I$ such that

$$P_h Q_h = I_h, \tag{3.2}$$

where I_h is the identity on E_h, and

$$Q_h P_h x \to x \tag{3.3}$$

as $h \to 0$ for each $x \in E$. We suppose that the operators P_h and Q_h are uniformly bounded in the sense that there exists a constant M_1 such that

$$\|P_h\| \le M_1, \qquad \|Q_h\| \le M_1 \tag{3.4}$$

for all $h \in H$.

An initial condition for equation (3.1) is given by the equality

$$x_h(0) = P_h x_0. \tag{3.5}$$

Notice that a mild solution $x_h \in C([0,a];E_h)$ to problem (3.1), (3.5) is defined by the equality

$$x_h(t) = \mathcal{G}_h(t)P_h x_0 + \int_0^t (t-s)^{q-1}\mathcal{T}_h(t-s)f_h\bigl(s, x_h(s)\bigr)\,ds, \quad t \in [0,a],$$

where the operator functions $\mathcal{G}_h$ and $\mathcal{T}_h$ are defined analogously to Definition 3:

$$\mathcal{G}_h(t) = \int_0^\infty \xi_q(\theta)U_h\bigl(t^q\theta\bigr)\,d\theta, \qquad \mathcal{T}_h(t) = q\int_0^\infty \theta\xi_q(\theta)U_h\bigl(t^q\theta\bigr)\,d\theta.$$

We assume that

$(H1)$ for each $x \in E$,

$$Q_h U_h(t) P_h x \to U(t)x$$

as $h \to 0$ uniformly in $t \in [0,a]$.

Notice that conditions of validity of hypothesis $(H1)$ in terms of the strong convergence of the resolvents

$$Q_h(A_h + \lambda I)^{-1}P_h x \to (A + \lambda I)^{-1}x$$

being an analog of the Trotter-Kato theorem are given, for example, in [36], Chapter IX, Theorem 2.16; see also [11], Theorem 2.6.

We also consider the map $g : H \times [0,a] \times E \to E$,

$$g(h,t,x) = Q_h f_h(t, P_h x),$$

for which we suppose the following:

($g1$) g is continuous and bounded on bounded sets;
($g2$) there exists $k > 0$ such that

$$\chi_E\big(g(H,t,\Omega)\big) \le k\chi(\Omega)$$

for each $t \in [0,a]$ and bounded $\Omega \subset E$.

Consider the operator $F : H \times C([0,a];E) \to C([0,a];E)$ defined by the equality

$$F(h,x)(t) = Q_h\mathcal{G}_h(t)P_hx_0 + \int_0^t (t-s)^{q-1}Q_h\mathcal{T}_h(t-s)f_h\big(s,P_hx(s)\big)\,ds. \tag{3.6}$$

Taking into account (3.2), we conclude that the operator F may be written in the following form:

$$F(h,x)(t) = Q_h\mathcal{G}_h(t)P_hx_0 + \int_0^t (t-s)^{q-1}Q_h\mathcal{T}_h(t-s)P_hg\big(h,s,P_hx(s)\big)\,ds. \tag{3.7}$$

Notice that solutions x_h of problem (3.1), (3.5) and fixed points of $F(h,\cdot)$ are connected in the following way: if x_h is a solution of (3.1), (3.5), then $x^h = Q_hx_h$ is a fixed point of $F(h,\cdot)$, and, conversely, if x_h is a fixed point of $F(h,\cdot)$, then P_hx^h is a solution of problem (3.1), (3.5).

The continuity of the operator F follows from property ($g1$) and the next assertion.

Lemma 2 *For each $x \in E$, we have the relations*

$$Q_h\mathcal{G}_h(t)P_hx \to \mathcal{G}(t)x \tag{3.8}$$

and

$$Q_h\mathcal{T}_h(t)P_hx \to \mathcal{T}(t)x \tag{3.9}$$

as $h \to 0$ uniformly in $t \in [0,a]$.

Proof Let us prove (3.8). Since the operator Q_h is bounded, by Remark 1 and condition ($H1$), for each $x \in E$, we have:

$$\begin{aligned}\big\|Q_h\mathcal{G}_h(t)P_hx - \mathcal{G}(t)x\big\|_E &= \left\|\int_0^\infty \xi_q(\theta)Q_hU_h\big(t^q\theta\big)P_hx\,d\theta - \int_0^\infty \xi_q(\theta)U\big(t^q\theta\big)x\,d\theta\right\|_E \\ &= \left\|\int_0^\infty \xi_q(\theta)\big[Q_hU_h\big(t^q\theta\big)P_hx - U\big(t^q\theta\big)x\big]\,d\theta\right\|_E \\ &\le \int_0^\infty \xi_q(\theta)\big\|Q_hU_h\big(t^q\theta\big)P_hx - U\big(t^q\theta\big)x\big\|_E\,d\theta \to 0.\end{aligned}$$

In a similar way, we can get relation (3.9):

$$\begin{aligned}\|Q_h\mathcal{T}_h(t)P_hx-\mathcal{T}(t)x\|_E &= \left\|\int_0^\infty \theta\xi_q(\theta)Q_hU_h(t^q\theta)P_hx\,d\theta-\int_0^\infty\theta\xi_q(\theta)U(t^q\theta)x\,d\theta\right\|_E\\ &= \left\|\int_0^\infty\theta\xi_q(\theta)\bigl[Q_hU_h(t^q\theta)P_hx-U(t^q\theta)x\bigr]\,d\theta\right\|_E\\ &\le\int_0^\infty\theta\xi_q(\theta)\|Q_hU_h(t^q\theta)P_hx-U(t^q\theta)x\|_E\,d\theta\to 0.\end{aligned}$$ □

Introduce in $C([0,a];E)$ the measure of noncompactness $\nu : P(C([0,a];E))\to\mathbb{R}^2_+$ with values in the cone $\mathbb{R}^2_+$ endowed with the natural order:

$$\nu(\Omega)=\max_{\mathcal{D}\in\Delta(\Omega)}\bigl(\psi(\mathcal{D}),\operatorname{mod}_C(\mathcal{D})\bigr), \tag{3.10}$$

where $\Delta(\Omega)$ denotes the collection of all denumerable subsets of Ω,

$$\psi(\mathcal{D})=\sup_{t\in[0,a]}e^{-pt}\chi_E\bigl(\mathcal{D}(t)\bigr),$$

and a constant $p>0$ is chosen in the following way. Fix a constant $d>0$ satisfying the relation

$$\frac{qMM_1^2k}{\Gamma(1+q)}\frac{d^q}{q}<\frac{1}{4}. \tag{3.11}$$

Then p is taken so that the following estimate holds:

$$\frac{qMM_1^2k}{\Gamma(1+q)}\frac{1}{pd^{1-q}}<\frac{1}{4}. \tag{3.12}$$

The second component of the MNC ν is the modulus of equicontinuity defined in Section 2.1. It is clear the the MNC ν is regular.

Theorem 1 *The operator F is a ν-condensing family.*

Proof Let $\Omega\subset C([0,a];E)$ be a nonempty bounded subset such that

$$\nu\bigl(F(H\times\Omega)\bigr)\ge\nu(\Omega). \tag{3.13}$$

We will show that the set Ω is relatively compact.

Let the maximum in the left-hand side of relation (3.13) be achieved on a denumerable set $\mathcal{D}=\{y_m\}$. Then there exist sequences $\{x_m\}_{m=1}^\infty\subset\Omega$ and $\{h_{n_m}\}_{m=1}^\infty\subset H$ (which, for simplicity, will be denoted $\{h_m\}$) such that

$$y_m(t)=Q_{h_m}\mathcal{G}_{h_m}(t)P_{h_m}x_0+\int_0^t(t-s)^{q-1}Q_{h_m}\mathcal{T}_{h_m}(t-s)P_{h_m}g\bigl(h_m,s,P_{h_m}x_m(s)\bigr)\,ds$$

for each $m\ge 1$.

From Lemma 2 and the properties of algebraic semiadditivity and regularity of the MNC ν it follows that we can substitute the sequence $\{y_m\}$ with the sequence $\{\widetilde{y}_m\}$, where

$$\widetilde{y}_m(t) = \int_0^t (t-s)^{q-1} Q_{h_m} \mathcal{T}_{h_m}(t-s) P_{h_m} g\big(h_m, s, P_h x_m(s)\big)\, ds.$$

From inequality (3.13) it follows that

$$\psi\big(\{\widetilde{y}_m\}\big) \geq \psi\big(\{x_m\}\big) \tag{3.14}$$

and

$$\mathrm{mod}_C\big(\{\widetilde{y}_m\}\big) \geq \mathrm{mod}_C\big(\{x_m\}\big). \tag{3.15}$$

By using property $(g2)$ we get the following estimates:

$$\begin{aligned} \chi_E\big(g\big(H, s, \{x_m(s)\}\big)\big) &\leq k\chi_E\big(\{x_m(s)\}\big) \\ &= e^{ps} k e^{-ps} \chi_E\big(\{x_m(s)\}\big) \\ &\leq e^{ps} k \sup_{\xi\in[0,a]} e^{-p\xi} \chi_E\big(\{x_m(\xi)\}\big) = e^{ps} k \psi\big(\{x_m\}\big). \end{aligned}$$

The last inequality yields:

$$\begin{aligned} e^{-pt} \chi_E\big(\{\widetilde{y}_m(t)\}\big) &\leq e^{-pt} \frac{qMM_1^2 k}{\Gamma(1+q)} \int_0^t (t-s)^{q-1} e^{ps} \psi\big(\{x_m\}\big)\, ds \\ &\leq \frac{qMM_1^2 k}{\Gamma(1+q)} \psi\big(\{x_m\}\big) \\ &\quad \times \left(e^{-pt} \int_0^{t-d} (t-s)^{q-1} e^{ps}\, ds + e^{-pt} \int_{t-d}^t (t-s)^{q-1} e^{ps}\, ds \right) \\ &\leq \frac{qMM_1^2 k}{\Gamma(1+q)} \psi\big(\{x_m\}\big) \left(e^{-pt} \frac{1}{d^{1-q}} \frac{e^{p(t-d)} - 1}{p} + \frac{d^q}{q} \right) \\ &\leq \frac{qMM_1^2 k}{\Gamma(1+q)} \psi\big(\{x_m\}\big) \left(\frac{1}{d^{1-q}} \frac{e^{-pd}}{p} + \frac{d^q}{q} \right) \\ &\leq \frac{qMM_1^2 k}{\Gamma(1+q)} \psi\big(\{x_m\}\big) \left(\frac{1}{pd^{1-q}} + \frac{d^q}{q} \right). \end{aligned}$$

Now, by using inequalities (3.11) and (3.12) we have

$$\begin{aligned} &\sup_{t\in[0,a]} e^{-pt} \chi_E\big(\{\widetilde{y}_m(t)\}\big) \leq \frac{1}{2} \psi\big(\{x_m\}\big), \\ &\psi\big(\{\widetilde{y}_m(t)\}\big) \leq \frac{1}{2} \psi\big(\{x_m\}\big). \end{aligned}$$

Taking into account inequality (3.14) together with the last one, we get:

$$\psi\big(\{x_m\}\big) \leq \frac{1}{2} \psi\big(\{x_m\}\big),$$

and hence

$$\psi\big(\{x_m\}\big) = 0,$$

implying

$$\chi_E\big(\big\{x_m(t)\big\}\big) \equiv 0 \quad \text{on } [0,a]. \tag{3.16}$$

From (3.4) and conditions $(H1)$ and $(g1)$ it follows that F maps bounded sets into bounded ones, and so the set $\bigcup_{t\in[0,a]} g(H,t,\{x_m(t)\})$ is bounded. Applying condition $(g2)$ and (3.16), we get that the set $\{g(h_m,t,x_m(t))\}$ is relatively compact for all $t \in [0,a]$. Then the sequence of functions $g(h_m,\cdot,x_m(\cdot))$ is semicompact, and from Proposition 2 it follows that, for each $\delta > 0$, there exist a compact set $K_\delta \subset E$, a set $m_\delta \subset [0,a]$ of Lebesgue measure meas $m_\delta < \delta$, and a sequence of functions $\{\widetilde{g}_m\} \subset L^1([0,a];E)$ such that

$$\big\{\widetilde{g}_m(t)\big\} \subset K_\delta \quad \text{for } t \in [0,a] \setminus m_\delta,$$
$$\big\{\widetilde{g}_m(t)\big\} = 0 \quad \text{for } t \in m_\delta,$$

and

$$\big\|g\big(h_m,t,x_m(t)\big) - \widetilde{g}_m(t)\big\| < \delta \quad \text{for } t \in [0,a] \setminus m_\delta.$$

The set of functions

$$v_m(t) = Q_{h_m}\mathcal{G}_{h_m}(t)P_{h_m}x_0 + \int_{[t,0]\setminus m_\delta} (t-s)^{q-1}Q_{h_m}\mathcal{T}_{h_m}(t-s)\widetilde{g}_m(s)\,ds, \quad m \geq 1,$$

is relatively compact in $C([0,a];E)$ and forms a γ_δ-net of the set $\{y_m\}$, where $\gamma_\delta \to 0$ as $\delta \to 0$, and we have the following estimate:

$$\mathrm{mod}_C(\Omega) \leq \mathrm{mod}_C\big(\{y_m\}\big) = 0.$$

Therefore

$$\nu(\Omega) = (0,0),$$

which concludes the proof. □

4 Index of the solution set

The operator $G : C([0,a];E) \to C([0,a];E)$,

$$Gx(t) = \mathcal{G}(t)x_0 + \int_0^t (t-s)^{q-1}\mathcal{T}(t-s)f\big(s,x(s)\big)\,ds,$$

is *the solution operator* of problem (2.1), (2.2) in the sense that the set of its fixed points Fix G coincides with the set Σ of all mild solutions of this problem. From Theorem 1 it follows that $G = F(0,\cdot)$ is ν-condensing. It opens the possibility to introduce the following notion. Suppose that the set Fix G is bounded in $C([0,a];E)$. Then its index $\mathrm{ind}_G(\mathrm{Fix}\, G)$ is well defined.

Definition 7 The value $\mathrm{ind}_G(\mathrm{Fix}\, G)$ is called the index $\mathrm{ind}(\Sigma)$ of a bounded solution set Σ of problem (2.1), (2.2).

We further assume that the nonlinearity f satisfies the following condition:

$(f2')$ $\chi_E(f([0,a] \times \Delta)) \leq \mu \chi_E(\Delta)$

for each bounded $\Delta \subset E$, where $\mu \geq 0$.

Notice that this condition is fulfilled if, for example, f satisfies condition $(f2)$ and for each bounded set $\Omega \subset E$, the function $f(\cdot, x) : [0,a] \to E$ is uniformly continuous w.r.t. $x \in \Delta$.

Theorem 2 *If Σ is a bounded solution set of* (2.1), (2.2) *then*

$$\mathrm{ind}(\Sigma) = 1.$$

Proof Take an arbitrary open bounded set $V \subset C([0,a];E)$ containing the solution set Σ and let

$$\mathcal{L} = \{x \in C([0,a];E) : x(0) = x_0\}.$$

From the map restriction property of the topological degree it follows that

$$\mathrm{ind}(\Sigma) = \deg_{\mathcal{L}}(i - G, \overline{V}_{\mathcal{L}}).$$

Take an arbitrary $x^* \in \Sigma$ and for $n = 1, 2, \ldots$, consider the operators $G_n : \mathcal{L} \to \mathcal{L}$ defined as

$$\begin{aligned} G_n x(t) = {} & \int_0^t (t + 1/n - s)^{q-1} \mathcal{T}(t + 1/n - s) f\bigl(s, \overline{x}(s - 1/n)\bigr)\, ds + x^*(t) \\ & - \int_0^t (t + 1/n - s)^{q-1} \mathcal{T}(t + 1/n - s) f\bigl(s, \overline{x^*}(s - 1/n)\bigr)\, ds, \end{aligned}$$

where $\overline{x} \in C([-1,a];E)$ denotes the extension of a function $x \in \mathcal{L}$ defined as x_0 on $[-1,0]$.

Let us show that, for a sufficiently large n, the operators G_n are ν-condensing (the choice of the coefficient p is described later).

Let $\Omega \subset \mathcal{L}$ be a nonempty bounded set, and let

$$\nu\bigl(G_n(\Omega)\bigr) \geq \nu(\Omega). \tag{4.1}$$

Let us show that the set Ω is relatively compact.

Let the maximum in the left-hand part of inequality (4.1) is achieved on a denumerable set $\mathcal{D} = \{y_k\}$. Then there exists a sequence $\{x_k\} \subset \Omega$ such that

$$\begin{aligned} y_k(t) = {} & \int_0^t (t + 1/n - s)^{q-1} \mathcal{T}(t + 1/n - s) f\bigl(s, \overline{x_k}(s - 1/n)\bigr)\, ds + x^*(t) \\ & - \int_0^t (t + 1/n - s)^{q-1} \mathcal{T}(t + 1/n - s) f\bigl(s, \overline{x^*}(s - 1/n)\bigr)\, ds. \end{aligned}$$

It is clear that it is sufficient to consider the sequence $\{\widetilde{y}_k\}$ defined as

$$\widetilde{y}_k(t) = \int_0^t (t + 1/n - s)^{q-1} \mathcal{T}(t + 1/n - s) f\big(s, \overline{x_k}(s - 1/n)\big)\, ds.$$

From (4.1) it follows that

$$\psi\big(\{\widetilde{y}_k\}\big) \geq \psi\big(\{x_k\}\big), \tag{4.2}$$

$$\operatorname{mod}_C\big(\{\widetilde{y}_k\}_{n=1}^{\infty}\big) \geq \operatorname{mod}_C\big(\{x_k\}\big). \tag{4.3}$$

The following estimates hold:

$$\begin{aligned} \chi_E\big(\big\{f\big(s, \overline{x_k}(s - 1/n)\big)\big\}\big) &\leq \mu \chi_E\big(\big\{\overline{x_k}(s - 1/n)\big\}\big) \\ &\leq e^{p(s-1/n)} \mu \sup_{\xi \in [0,a]} e^{-p(\xi - 1/n)} \chi_E\big(\big\{\overline{x}_k(\xi - 1/n)\big\}\big) \\ &\leq e^{p(s-1/n)} \mu \psi\big(\{x_k\}\big). \end{aligned} \tag{4.4}$$

Now, take d such that, for all $n \geq 1$,

$$\frac{M\mu}{\Gamma(1+q)} \big((d + 1/n)^q - (1/n)^q\big) < \frac{1}{4} \tag{4.5}$$

and then choose $p > 0$ such that

$$\frac{qM\mu}{\Gamma(1+q)} \cdot \frac{1}{pd^{1-q}} < \frac{1}{4}. \tag{4.6}$$

Now, using estimate (4.4), we get that, for each $t \in [0, a]$,

$$\begin{aligned} e^{-pt} \chi_E\big(\big\{\widetilde{y}_k(t)\big\}\big) &\leq e^{-pt} \frac{qM\mu}{\Gamma(1+q)} \int_0^t (t + 1/n - s)^{q-1} e^{p(s-1/n)} \psi\big(\{\overline{x}_k\}\big)\, ds \\ &\leq \frac{qM\mu}{\Gamma(1+q)} \psi\big(\{\overline{x}_k\}\big) e^{-p/n} \\ &\quad \times \left(e^{-pt} \int_0^{t-d} (t + 1/n - s)^{q-1} e^{ps}\, ds + e^{-pt} \int_{t-d}^t (t + 1/n - s)^{q-1} e^{ps}\, ds \right) \\ &\leq \frac{qM\mu}{\Gamma(1+q)} \psi\big(\{\overline{x}_k\}\big) \left(e^{-pt} \frac{1}{(d + 1/n)^{1-q}} \frac{e^{p(t-d)} - 1}{p} + \frac{(d + 1/n)^q - (1/n)^q}{q} \right) \\ &\leq \frac{qM\mu}{\Gamma(1+q)} \psi\big(\{\overline{x}_k\}\big) \left(\frac{1}{d^{1-q}} \frac{e^{-pd}}{p} + \frac{(d + 1/n)^q - (1/n)^q}{q} \right) \\ &\leq \frac{qM\mu}{\Gamma(1+q)} \psi\big(\{\overline{x}_k\}\big) \left(\frac{1}{pd^{1-q}} + \frac{(d + 1/n)^q - (1/n)^q}{q} \right). \end{aligned}$$

Now, using inequalities (4.5) and (4.6), from the last estimate we have:

$$\sup_{t \in [-1,a]} e^{-pt} \chi_E\big(\big\{\widetilde{y}_k(t)\big\}\big) \leq \frac{1}{2} \psi\big(\{\overline{x}_k\}\big),$$

$$\psi\big(\big\{\widetilde{y}_k(t)\big\}\big) \leq \frac{1}{2} \psi\big(\{\overline{x}_k\}\big).$$

Combining the last inequality with (4.2), we get:

$$\psi\big(\{\overline{x}_k\}\big) \le \frac{1}{2}\psi\big(\{\overline{x}_k\}\big),$$

implying

$$\psi\big(\{\overline{x}_k\}\big) = 0. \tag{4.7}$$

Since the operator G_n maps bounded sets into bounded ones, the sequence of functions $\{\mathfrak{f}_k\}$ defined by $\mathfrak{f}_k(t) = f(t, \overline{x}_k(t-1/n))$ is bounded. From relation (4.7) and condition $(f2)$ it follows that the set $\{\mathfrak{f}_k(t)\}$ is relatively compact for each $t \in [0,a]$, and hence the sequence $\{\mathfrak{f}_k\}$ is weakly compact in $L_1([0,a];E)$ (see, *e.g.*, Proposition 4.2.1 in [34]). From condition (4.7) it follows that, for each $\delta > 0$, there exist a compact set $K_\delta \subset E$ and a set $m_\delta \subset [-1,a]$ with the Lebesgue measure $\operatorname{meas}(m_\delta) < \delta$ such that $\{\mathfrak{f}_k(t)\} \subset K_\delta$ for $t \in [0,a] \setminus m_\delta$. Therefore the sequence of functions

$$v_k(t) = \int_{[0,t]\setminus m_\delta} (t+1/n-s)^{q-1}\mathcal{T}(t+1/n-s)\mathfrak{f}_k(s)\,ds$$

is relatively compact in $C([0,a];E)$ and forms a γ_δ-net of the set $\{y_k\}$, where $\gamma_\delta \to 0$ as $\delta \to 0$, and hence the sequence $\{\widetilde{y}_k\}$ is relatively compact. By (4.1) we get

$$\operatorname{mod}_C(\Omega) \le \operatorname{mod}_C\big(\{y_k\}\big) = 0.$$

Therefore $\nu(\Omega) = (0,0)$.

Let us prove that, for all sufficiently large n, we have the equality

$$\deg_{\mathcal{L}}(i - G_n, \overline{V}_{\mathcal{L}}) = \deg_{\mathcal{L}}(i - G, \overline{V}_{\mathcal{L}}). \tag{4.8}$$

We will show that equality (4.8) follows from the homotopy of the above fields, which is realized for sufficiently large n by the linear transfer $\Gamma : [0,1] \times \overline{V}_{\mathcal{L}} \to C([0,a];E)$ defined as

$$\Gamma(\lambda, x) = \lambda G_n x + (1-\lambda)Gx.$$

From the properties of the MNC ν it easily follows that the family Γ is ν-condensing. Let us demonstrate that

$$x \neq \Gamma(\lambda, x)$$

for all $(\lambda, x) \in [0.1] \times \partial V_{\mathcal{L}}$. Supposing the contrary, we will have the sequences $\{x_m\}$, $\{n_m\}$, and $\{\lambda_m\}$ such that $x_m \in \partial V_{\mathcal{L}}$, $n_m \to \infty$, $\lambda_m \in [0,1]$, $\lambda_m \to \lambda_0$, and

$$x_m = \lambda_m G_{n_m} x_m + (1-\lambda_m)Gx_m. \tag{4.9}$$

Let us show that the sequence $\{x_m\}$ is relatively compact. From relations (4.9) it follows that

$$\begin{aligned} x_m(t) &= \lambda_m \int_0^t (t+1/n_m-s)^{q-1}\mathcal{T}(t+1/n_m-s)f\bigl(s,\overline{x}_m(s-1/n_m)\bigr)\,ds + \lambda_m x^*(t) \\ &\quad - \lambda_m \int_0^t (t+1/n_m-s)^{q-1}\mathcal{T}(t+1/n_m-s)f\bigl(s,\overline{x^*}(s-1/n_m)\bigr)\,ds \\ &\quad + (1-\lambda_m)Gx_m(t), \quad t\in[0,a]. \end{aligned}$$

Choose a constant $p>0$ such that, for $d>0$ satisfying the inequality

$$\frac{M\mu d^q}{\Gamma(1+q)} < \frac{1}{4}, \tag{4.10}$$

we have

$$\frac{qM\mu}{\Gamma(1+q)}\frac{1}{pd^{1-q}} < \frac{1}{4}. \tag{4.11}$$

Notice that

$$\begin{aligned} \overline{x}_m\left(t-\frac{1}{n_m}\right) &= \lambda_m \int_0^{t-1/n_m} (t-s)^{q-1}\mathcal{T}(t-s)f\bigl(s,\overline{x}_m(s-1/n_m)\bigr)\,ds + \lambda_m x^*\left(t-\frac{1}{n_m}\right) \\ &\quad - \lambda_m \int_0^{t-1/n_m} (t-s)^{q-1}\mathcal{T}(t-s)f\bigl(s,\overline{x}^*(s-1/n_m)\bigr)\,ds \\ &\quad + (1-\lambda_m)\left(\mathcal{G}\left(t-\frac{1}{n_m}\right)x_0\right. \\ &\quad \left. + \int_0^{t-1/n_m}\left(t-\frac{1}{n_m}-s\right)^{q-1}\mathcal{T}\left(t-\frac{1}{n_m}-s\right)f\bigl(s,x_m(s)\bigr)\,ds\right), \\ &\quad t\in\left[\frac{1}{n_m},a\right]. \end{aligned}$$

Let us make the substitution of variables $s+1/n_m=\xi$ in the last integral and denote ξ by s again. Then we get

$$\begin{aligned} \overline{x}_m\left(t-\frac{1}{n_m}\right) &= \lambda_m \int_0^{t-1/n_m} (t-s)^{q-1}\mathcal{T}(t-s)f\bigl(s,\overline{x}_m(s-1/n_m)\bigr)\,ds + \lambda_m x^*\left(t-\frac{1}{n_m}\right) \\ &\quad - \lambda_m \int_0^{t-1/n_m} (t-s)^{q-1}\mathcal{T}(t-s)f\bigl(s,\overline{x}^*(s-1/n_m)\bigr)\,ds \\ &\quad + (1-\lambda_m)\left(\mathcal{G}\left(t-\frac{1}{n_m}\right)x_0\right. \\ &\quad \left. + \int_{1/n_m}^{t} (t-s)^{q-1}\mathcal{T}(t-s)f\bigl(s-1/n_m,\overline{x}_m(s-1/n_m)\bigr)\,ds\right), \\ &\quad t\in\left[\frac{1}{n_m},a\right]. \end{aligned}$$

Set

$$\overline{f}(s,x) = \begin{cases} f(s,x) & \text{for } s \geq 0, \\ f(0,x) & \text{for } s < 0. \end{cases}$$

Notice that the sequences of functions

$$\left\{\lambda_m \int_{t-1/n_m}^{t} (t-s)^{q-1}\mathcal{T}(t-s) f\big(s, \overline{x}_m(s-1/n_m)\big)\, ds\right\},$$
$$\left\{\lambda_m \int_{0}^{t-1/n_m} (t-s)^{q-1}\mathcal{T}(t-s) f\big(s, x_m^*(s-1/n_m)\big)\, ds\right\},$$
$$\left\{\lambda_m \overline{x}^*(t-1/n_m)\right\},$$
$$\left\{(1-\lambda_m)\mathcal{G}(t-1/n_m)x_0\right\},$$

and

$$\left\{\int_0^{1/n_m} (t-s)^{q-1}\mathcal{T}(t-s)\overline{f}\big(s-1/n_m, \overline{x}(s-1/n_m)\big)\, ds\right\}$$

converge uniformly for t belonging to each interval $[\gamma, a]$ with $\gamma \in (0,a)$. Let us estimate for $t > 0$ the following value:

$$\begin{aligned} & e^{-pt}\chi_E\big(\{\overline{x}_m(t-1/n_m)\}\big) \\ &\quad = e^{-pt}\chi_E\left(\left\{\int_0^t \big((t-s)^{q-1}\mathcal{T}(t-s)\right.\right. \\ &\qquad \left.\left. \times \left[\lambda_0 f\big(s, \overline{x}_m(s-1/n_m)\big) + (1-\lambda_0) f\big(s-1/n_m, \overline{x}_m(s-1/n_m)\big)\right]\big)\, ds\right\}\right). \end{aligned}$$

Since by $(f2')$

$$\chi_E\big(\{\lambda_0 f\big(s, x_m(s-1/n_m)\big) + (1-\lambda_0)\overline{f}\big(s-1/n_m, x_m(s-1/n_m)\big)\}\big) \leq K\chi_E\big(\{x_m(s-1/n_m)\}\big),$$

we get

$$\begin{aligned} e^{-pt}\chi_E\big(\{\overline{x}_m(t-1/n_m)\}\big) &\leq \frac{qM\mu}{\Gamma(1+q)} \sup_{s\in[0,a]} e^{-ps}\chi_E\big(\{\overline{x}_m(s-1/n_m)\}\big) \\ &\quad \times \left(e^{-pt}\int_0^{t-d} (t-s)^{q-1}e^{ps}\, ds + e^{-pt}\int_{t-d}^{t} (t-s)^{q-1}e^{ps}\, ds\right) \\ &\leq \frac{qM\mu}{\Gamma(1+q)} \sup_{s\in[0,a]} e^{-ps}\chi_E\big(\{\overline{x}_m(s-1/n_m)\}\big) \\ &\quad \times \left(e^{-pt}\frac{1}{(d)^{1-q}}\frac{e^{p(t-d)}-1}{p} + \frac{d^q}{q}\right) \\ &\leq \frac{qM\mu}{\Gamma(1+q)} \sup_{s\in[0,a]} e^{-ps}\chi_E\big(\{\overline{x}_m(s-1/n_m)\}\big)\left(\frac{1}{d^{1-q}}\frac{e^{-pd}}{p} + \frac{d^q}{q}\right) \\ &\leq \frac{qM\mu}{\Gamma(1+q)} \sup_{s\in[0,a]} e^{-ps}\chi_E\big(\{\overline{x_m}(s-1/n_m)\}\big)\left(\frac{1}{pd^{1-q}} + \frac{d^q}{q}\right). \end{aligned}$$

Now, using (4.10) and (4.11), we have

$$\sup_{t\in[0,a]} e^{-pt}\chi_E\big(\{\overline{x}_m(t-1/n_m)\}\big) \le \frac{1}{2}\sup_{s\in[0,a]} e^{-ps}\chi_E\big(\{\overline{x}_m(s-1/n_m)\}\big),$$

implying

$$\sup_{t\in[0,a]} e^{-pt}\chi_E\big(\{\overline{x}_m(t-1/n_m)\}\big) = 0,$$

and hence

$$\chi_E\big(\{\overline{x}_m(t-1/n_m)\}\big) = 0, \quad t\in[0,a].$$

The proof of the equicontinuity of the sequence $\{x_m\}$ is similar to the previous proof of the equicontinuity of the sequence $\{\widetilde{y}_k\}$.

So the sequence $\{x_m\}$ is relatively compact, and we can assume, without loss of generality, that $x_m \to x^0$. Then $x^0 \in \partial V_{\mathcal{L}}$, that is, $x^0 \neq x^*$. Passing to the limit in (4.9) as $m\to\infty$, we get the contradiction

$$Gx^0 = x^0.$$

Now, let us show that the map $S:[0,1]\times\overline{V}_{\mathcal{L}}\to\mathcal{L}$ given by the formula

$$S(\lambda,x) = \lambda G_n x + x^* - \lambda G_n x^*$$

is the homotopy connecting maps $G_0(x)\equiv x^*$ and G_n. In fact, for each $\lambda\in[0,1]$, the equation

$$S(\lambda,x) = x \tag{4.12}$$

has a unique solution $x = x^*$. For $\lambda = 0$, this is evident, and for $\lambda \neq 0$, if equation (4.12) has a solution y, then $y(0) = x_0$, and hence $y(t-1/n) = x^*(t-1/n)$, $t\in[0,1/n]$, so $y(t) = x^*(t)$, $t\in[0,1/n]$. Then $y(t-1/n) = x^*(t-1/n)$, $t\in[1/n,2/n]$. Continuing further, we get the equality $y(t) = x^*(t)$, $t\in[0,a]$. Then, using the map restriction and normalization properties of the topological degree, we have:

$$\deg_{\mathcal{L}}(i-G_n,\overline{V}_{\mathcal{L}}) = \deg_{\mathcal{L}}(i-G_0,\overline{V}_{\mathcal{L}}) = 1,$$

yielding

$$\deg_{\mathcal{L}}(i-G,\overline{V}_{\mathcal{L}}) = 1,$$

which concludes the proof of the theorem. □

Remark 2 On the set $\mathcal{L}$, let us consider the operators $\widetilde{G}_n:\mathcal{L}\to C([0,a];E)$ given by the formula

$$\widetilde{G}_n x(t) = \mathcal{G}(t)x_0 + \int_0^t (t+1/n-s)^{q-1}\mathcal{T}(t+1/n-s)f\big(s,\overline{x}(s-1/n)\big)\,ds.$$

Fixed points x_n of the operators $\widetilde{G}_n$ may be found by the method of steps successively on the intervals $[\frac{k}{n}, \frac{k+1}{n}]$, $k = 0, 1, \dots, [an]$. Finding these fixed points is analogous to the Tonelli procedure (see, *e.g.*, [37] or [38, 39], Chapter II, Theorem 2.1) of solving problem (2.1)-(2.2). If it is known that the sequence $\{x_n\}$ is bounded on the interval $[0, a]$, then, repeating the reasonings for $\{x_m\}$ given by formula (4.9) with $\lambda_m = 1$, we get the compactness of the sequence $\{x_n\}$. If $\{x_{n_k}\}$ is a convergent subsequence, then passing to the limit in the equalities

$$x_{n_k} = G_{n_k} x_{n_k},$$

we obtain that the limit of the subsequence $\{x_{n_k}\}$ may be only a fixed point of the operator G, that is, a solution of problem (2.1)-(2.2).

Let us mention also that, instead of condition $(f2')$, in this case, it is sufficient to assume condition $(f2)$.

5 The main result

Now we are in position to present the main result of this paper.

Theorem 3 *Under conditions* (A), $(f1)$, $(f2')$, (3.2), (3.3), (3.4), $(g1)$, $(g2)$, *suppose that problem* (2.1)-(2.2) *has a unique solution* $x^\star$ *on the interval* $[0, a]$. *Then, for a sufficiently small* $h > 0$, *problems* (3.1), (3.5) *have solutions* x_h *on the interval* $[0, a]$, *and*

$$Q_h x_h \to x^\star$$

as $h \to 0$.

Proof It is sufficient to apply Proposition 1 to the operator F given by formula (3.6). □

In conclusion, let us present two examples of the construction of A_h and f_h.

Example 1 Let $h_n = 1/n$, and let $A_{h_n} = A_n$ be the Yosida approximations (see, *e.g.*, [40]), $E_{h_n} = E$, $P_{h_n} = Q_{h_n} = I$, and $f_{h_n} = f$. Then condition $(H1)$ is fulfilled for the semigroups generated by the operators A_n. So, the transfer from a unbounded operator A in equation (2.1) to a bounded operator A_h in equation (3.1) can be justified.

Example 2 Let f satisfy $(f2)$. If we set

$$f_h(t, x_h) = P_h f(t, Q_h x_h),$$

then, for the operator g, condition $(g2)$ is fulfilled with the constant $k = \mu$.

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