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Hybrid steepest iterative algorithm for a hierarchical fixed point problem

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Abstract

The purpose of this work is to introduce and study an iterative method to approximate solutions of a hierarchical fixed point problem and a variational inequality problem involving a finite family of nonexpansive mappings on a real Hilbert space. Further, we prove that the sequence generated by the proposed iterative method converges to a solution of the hierarchical fixed point problem for a finite family of nonexpansive mappings which is the unique solution of the variational inequality problem. The results presented in this paper are the extension and generalization of some previously known results in this area. An example which satisfies all the conditions of the iterative method and the convergence result is given.

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1 Introduction

Throughout this paper, we always assume that $\mathcal{V}$ is a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$, respectively. Let a nonlinear mapping $S:\mathcal{V}\to\mathcal{V}$ be a nonexpansive operator if

$$\|Su-Sv\|\le\|u-v\|,\quad \forall u,v\in\mathcal{V}.$$

A point $u\in\mathcal{V}$ is said to be a fixed point of S provided $Su=u$. In this paper, we use $\overline{F}(S)$ to denote the fixed point set which is closed and convex, see [1].

Let $S:W\to\mathcal{V}$ be a nonexpansive mapping, where W is a nonempty closed convex subset of $\mathcal{V}$. The hierarchical fixed point problem (in short, HFPP) is to find $u\in\overline{F}(S)$ such that

$$\langle u-Su,v-u\rangle\ge 0,\quad \forall v\in\overline{F}(S). \tag{1.1}$$

Many authors solve (1.1) by various methods, see [2–9] and the references therein.

Yao *et al.* [2] proposed the following iterative algorithm to solve HFPP (1.1):

$$\begin{cases} v_n = b_n S u_n + (1 - b_n) u_n, \\ u_{n+1} = P_W [a_n g(u_n) + (1 - a_n) S v_n], \quad \forall n \geq 0, \end{cases} \tag{1.2}$$

where $\{a_n\}$ and $\{b_n\}$ are sequences in $(0,1)$ and $g : W \to \mathcal{V}$ is a contraction mapping, and the sequence $\{u_n\}$ generated by (1.2) converges strongly to $z \in \overline{F}(S)$, which is also a unique solution of the variational inequality problem (VIP), *i.e.*, to find $z \in \overline{F}(S)$ such that

$$\langle (I - g)z, v - z \rangle \geq 0, \quad \forall v \in \overline{F}(S). \tag{1.3}$$

After that, Ceng *et al.* [6] introduced the following algorithm:

$$u_{n+1} = P_W \big[a_n \rho g(u_n) + (I - a_n \mu F) S(u_n) \big], \quad \forall n \geq 0, \tag{1.4}$$

where F is a Lipschitz continuous and strongly monotone mapping, g is a Lipschitz continuous mapping. Compute an iterative sequence $\{u_n\}$ generated by (1.4) converging strongly to $z \in \overline{F}(S)$, which is also a unique solution of the following variational inequality problem (VIP), *i.e.*, to find $z \in \overline{F}(S)$ such that

$$\langle \rho g(z) - \mu F(z), v - z \rangle \geq 0, \quad \forall v \in \overline{F}(S). \tag{1.5}$$

By using a T_n-mapping [10], Yao [11] proposed the following iterative method:

$$u_{n+1} = a_n c g(u_n) + b u_n + \big[(1 - b) I - a_n A \big] T_n u_n, \quad \forall n \geq 0, \tag{1.6}$$

where $c > 0$, A is a strongly positive bounded linear operator and $g : W \to \mathcal{V}$ is a contraction mapping.

Further, Ceng *et al.* [12] proposed explicit and implicit iterative schemes for finding a common solution for the set of fixed points of a nonexpansive mapping. Buong and Duong [13] studied the explicit iterative algorithm for finding the approximate solution of a VIP defined over the set of common fixed points of a finite number of nonexpansive mappings:

$$u_{k+1} = \big(1 - b_k^0\big) u_k + b_k^0 S_0^k S_p^k \cdots S_1^k u_k, \tag{1.7}$$

where $S_i^k = (1 - b_k^i) u_k + b_k^i S^i$ for $1 \leq i \leq p$, $\{S_i\}_{i=1}^p$ are p-nonexpansive mappings on a real Hilbert space $\mathcal{V}$, $S_0^k = I - \lambda_k \mu F$, and F is an η-strongly monotone and L-Lipschitz continuous mapping.

Very recently, Zhang and Yang [14] studied the more general explicit iterative algorithm

$$u_{k+1} = a_k c g(u_k) + (I - \mu a_k F) S_p^k S_{p-1}^k \cdots S_1^k u_k, \tag{1.8}$$

where g is an α-Lipschitzian, F is an η-strongly monotone and L-Lipschitz continuous mapping and $S_i^k = (1 - b_k^i) u_k + b_k^i S^i$ for $1 \leq i \leq p$. Under some assumptions, compute an iterative sequence $\{u_k\}$ proposed by the iterative algorithm (1.8) that strongly converges to the solution of the VIP, *i.e.*, to find $z \in \bigcap_{i=1}^{p} \overline{F}(S_i)$ such that

$$\bigl\langle (\mu F - \gamma g)z, v - z \bigr\rangle \geq 0, \quad \forall v \in \bigcap_{i=1}^{p} \overline{F}(S_i). \tag{1.9}$$

Inspired and motivated by the recent research, we develop an iterative algorithm for a hierarchical fixed point problem of a finite family of nonexpansive mappings on the real Hilbert space. We generate a strong convergence theorem for the sequence considered by the generalized method. Numerical examples are also given for the theoretical verification of the algorithm. The algorithm and results presented in this paper improve and extend some recent corresponding algorithms and results; see [15, 16] and the references therein.

2 Preliminaries

We recall some concepts and results which are needed in the sequel.

Definition 2.1 Let $S : W \to \mathcal{V}$ be a mapping which is said to be

(i) monotone if

$$\langle Su - Sv, u - v \rangle \geq 0, \quad \forall u, v \in W;$$

(ii) strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Su - Sv, u - v \rangle \geq \alpha \|u - v\|^2, \quad \forall u, v \in W;$$

(iii) Lipschitz continuous if there exists a constant $k > 0$ such that

$$\|Su - Sv\| \leq k\|u - v\|, \quad \forall u, v \in W.$$

If $k = 1$, then S is called nonexpansive.

Definition 2.2 A mapping $g : W \to \mathcal{V}$ is said to be σ-contraction if there exists a constant $\sigma \in (0, 1)$ such that

$$\|gu - gv\| \leq \sigma \|u - v\|, \quad \forall u, v \in W.$$

Lemma 2.1 ([6]) *Let $F : W \to \mathcal{V}$ be an η-strongly monotone and k-Lipschitz continuous mapping and $g : W \to \mathcal{V}$ be a τ-Lipschitz continuous mapping. Then the mapping $\mu F - \rho g$ is $(\mu\eta - \rho\tau)$-strongly monotone with condition $\mu\eta > \rho\tau \geq 0$, i.e.,*

$$\bigl\langle (\mu F - \rho g)u - (\mu F - \rho g)v, u - v \bigr\rangle \geq (\mu\eta - \rho\tau)\|u - v\|^2, \quad \forall u, v \in W.$$

Definition 2.3 A mapping $T : \mathcal{V} \to \mathcal{V}$ is said to be an averaged mapping if it can be written as the average of the identity I and a nonexpansive mapping, *i.e.*,

$$T \equiv (1 - \alpha)I + \alpha S,$$

where $\alpha \in (0, 1)$ and $S : \mathcal{V} \to \mathcal{V}$ is nonexpansive.

Lemma 2.2 ([17, 18]) *If the mappings $\{S_i\}_{i=1}^p$ are averaged and have a common fixed point, then*

$$\bigcap_{i=1}^p \overline{F}(S_i) = \overline{F}(S_1 S_2 \cdots S_p).$$

In particular, if $p = 2$, we have $\overline{F}(S_1) \cap \overline{F}(S_2) = \overline{F}(S_1 S_2) = \overline{F}(S_2 S_1)$.

Lemma 2.3 ([19]) *Let $\{\alpha_n\}$ be a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \le (1 - w_n)\alpha_n + t_n,$$

where $\{w_n\} \in (0,1)$ and $\{t_n\}$ is a sequence such that
(i) $\sum_{n=1}^{\infty} w_n = \infty$;
(ii) $\limsup_{n\to\infty} \frac{t_n}{w_n} \le 0$ *or* $\sum_{n=1}^{\infty} |t_n| < \infty$.
Then $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 2.4 ([1]) *Let $S : W \to W$ be a nonexpansive mapping with $\overline{F}(S) \neq \emptyset$. Then the mapping $I - S$ is demiclosed at 0, that is, if $\{u_n\}$ is a sequence converging weakly to u and $\{(I - S)u_n\}$ converges strongly to 0, then $(I - S)u = 0$.*

Lemma 2.5 ([20]) *Let $F : W \to \mathcal{V}$ be an η-strongly monotone and k-Lipschitzian mapping. Let $\frac{2\eta}{k^2} > \mu > 0$, $Q = I - \lambda\mu F$. Then Q is a $(1 - \lambda\tau)$-contraction mapping with* $\min\{1, \frac{1}{\tau}\} > \lambda > 0$, *that is,*

$$\|Qu - Qv\| \le (1 - \lambda\tau)\|u - v\|, \quad \forall u, v \in W,$$

where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu k^2)} \in (0,1]$.

Lemma 2.6 *Let $\mathcal{V}$ be a real Hilbert space. The following inequality holds*:

$$\|u + v\|^2 \le \|u\|^2 + 2\langle v, u + v\rangle, \quad \forall u, v \in \mathcal{V}.$$

3 Main results

In this section, we establish an iterative method for finding the solution of hierarchical fixed point problem (1.1).

Let W be a nonempty closed convex subset of a real Hilbert space $\mathcal{V}$, and let $\{S_i\}_{i=1}^p$ be p nonexpansive mappings on W such that $\Xi = \bigcap_{i=1}^p \overline{F}(S_i) \neq \emptyset$. Let $F : W \to W$ be an η-strongly monotone and k-Lipschitzian mapping and $g : W \to W$ be a τ-contraction mapping.

We consider the following hierarchical fixed point problem (in short, HFPP): find $u \in \Xi$ such that

$$\bigl\langle \rho g(u) - \mu F(u), v - u\bigr\rangle \le 0, \quad \forall v \in \Xi = \bigcap_{i=1}^p \overline{F}(S_i). \tag{3.1}$$

Now we define the following algorithm for finding a solution of HFPP (3.1).

Algorithm 3.1 Given arbitrarily $u_0 \in W$, compute sequences $\{u_n\}$ and $\{v_n\}$ by the iterative schemes

$$\begin{cases} v_n = b_n u_n + (1-b_n) S_p^n S_{p-1}^n \cdots S_1^n u_n, \\ u_{n+1} = a_n \rho g(v_n) + c_n v_n + [(1-c_n)I - a_n \mu F] S_p^n S_{p-1}^n \cdots S_1^n v_n, \quad \forall n \geq 0, \end{cases} \tag{3.2}$$

where $S_i^n = (1-d_n^i)I + d_n^i S_i$ and $d_n^i \in (0,1)$ for $i = 1,2,\dots,p$, let the parameters satisfy $\frac{2\eta}{k^2} > \mu > 0$ and $\frac{\nu}{\tau} > \rho > 0$, where $\nu = \mu(\eta - \frac{\mu k^2}{2})$ and $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences in $(0,1)$ satisfying the following conditions:

(i) $\lim_{n\to\infty} a_n = 0$ and $\sum_{n=1}^{\infty} a_n = \infty$ and $\sum_{n=1}^{\infty} |a_{n-1} - a_n| < \infty$.
(ii) $\{b_n\} \subset [\sigma, 1)$ and $\lim_{n\to\infty} b_n = b < 1$.
(iii) $a_n + c_n < 1$ and $\lim_{n\to\infty} c_n = 0$.
(iv) $\sum_{n=1}^{\infty} |c_{n-1} - c_n| < \infty$ and $\sum_{n=1}^{\infty} |d_{n-1}^i - d_n^i| < \infty$ for $i = 1,2,\dots,p$.

Lemma 3.1 *Let $u^* \in \Xi$. Then the sequences $\{u_n\}$ and $\{v_n\}$ defined in Algorithm* 3.1 *are bounded.*

Proof Let $u^* \in \Xi$. So, we have

$$\begin{aligned} \|v_n - u^*\| &= \|b_n u_n + (1-b_n) S_p^n S_{p-1}^n \cdots S_1^n u_n - u^*\| \\ &= \|(1-b_n)(S_p^n S_{p-1}^n \cdots S_1^n u_n - u^*) + b_n(u_n - u^*)\| \\ &\leq (1-b_n)\|u_n - u^*\| + b_n\|u_n - u^*\| \\ &= \|u_n - u^*\|. \end{aligned} \tag{3.3}$$

From (3.2) and (3.3), we have

$$\begin{aligned} \|u_{n+1} - u^*\| &= \|a_n \rho g(v_n) + c_n v_n + [(1-c_n)I - a_n \mu F] S_p^n S_{p-1}^n \cdots S_1^n v_n - u^*\| \\ &= \|a_n(\rho g(v_n) - \mu F(u^*)) + c_n(v_n - u^*) \\ &\quad + [(1-c_n)I - a_n \mu F] S_p^n S_{p-1}^n \cdots S_1^n v_n \\ &\quad - [(1-c_n)I - a_n \mu F] S_p^n S_{p-1}^n \cdots S_1^n u^*\| \\ &\leq a_n\|\rho g(v_n) - \mu F(u^*)\| + c_n\|v_n - u^*\| \\ &\quad + \|[(1-c_n)I - a_n \mu F] S_p^n S_{p-1}^n \cdots S_1^n v_n \\ &\quad - [(1-c_n)I - a_n \mu F] S_p^n S_{p-1}^n \cdots S_1^n u^*\| \\ &= a_n\|\rho g(v_n) - \mu F(u^*)\| + c_n\|v_n - u^*\| \\ &\quad + (1-c_n)\left\|\left(I - \frac{a_n \mu F}{1-c_n}\right) S_p^n S_{p-1}^n \cdots S_1^n v_n - \left(I - \frac{a_n \mu F}{1-c_n}\right) S_p^n S_{p-1}^n \cdots S_1^n u^*\right\| \\ &\leq (1-c_n)\left(1 - \frac{a_n \nu}{1-c_n}\right)\|v_n - u^*\| + c_n\|v_n - u^*\| + a_n\|\rho g(v_n) - \mu F(u^*)\| \\ &\leq (1-a_n\nu)\|u_n - u^*\| + a_n\rho\|g(v_n) - g(u^*)\| + a_n\|\rho g(u^*) - \mu F(u^*)\| \\ &\leq (1-a_n\nu)\|u_n - u^*\| + a_n\rho\tau\|v_n - u^*\| + a_n\|\rho g(u^*) - \mu F(u^*)\| \\ &\leq (1 - a_n(\nu - \rho\tau))\|u_n - u^*\| + a_n\|\rho g(u^*) - \mu F(u^*)\| \end{aligned}$$

$$\begin{aligned}
&\leq \bigl(1-a_n(\nu-\rho\tau)\bigr)\bigl\|u_n-u^*\bigr\| + a_n(\nu-\rho\tau)\frac{\|\rho g(u^*)-\mu F(u^*)\|}{(\nu-\rho\tau)} \\
&\leq \max\biggl\{\bigl\|u_n-u^*\bigr\|, \frac{\|\rho g(u^*)-\mu F(u^*)\|}{\nu-\rho\tau}\biggr\}, \qquad (3.4)
\end{aligned}$$

where the third and fifth inequalities follow from (3.3) and the second inequality follows from Lemma 2.5.

By induction on n and (3.4), we have

$$\bigl\|u_n-u^*\bigr\| \leq \max\biggl\{\bigl\|u_n-u^*\bigr\|, \frac{1}{\nu-\tau\rho}\bigl\|(\rho g-\mu F)u^*\bigr\|\biggr\} \quad \text{for } n=1,2,\dots \text{ and } u_o\in K.$$

Hence, $\{u_n\}$ is bounded; and consequently, we get $\{v_n\}$, $\{Sv_n\}$, $\{S_1u_{n+1}\}$, $\|S_1^n u_{n+1}\|$, $\|S_2S_1^n u_{n+1}\|,\dots,\|S_{p-1}^n\cdots S_1^n u_{n+1}\|,\|S_pS_{p-1}^n\cdots S_1^n u_{n+1}\|,\|S_{p-1}^{n-1}\cdots S_1^{n-1}v_n\|+\|S_pS_{p-1}^{n-1}\cdots S_1^{n-1}v_n\|+\|S_{p-1}^{n-1}\cdots S_1^{n-1}u_n\|+\|S_pS_{p-1}^{n-1}\cdots S_1^{n-1}u_n\|$ and $\{g(v_n)\}$ are bounded. □

Lemma 3.2 *Let* $\{u_n\}$ *be a sequence generated by Algorithm* 3.1. *Then*

(i) $\lim_{n\to\infty}\|u_{n+1}-u_n\|=0$.

(ii) $\lim_{n\to\infty}\|u_n-S_p^nS_{p-1}^n\cdots S_1^n u_n\|=0$.

Proof From the sequence $\{v_n\}$ defined in Algorithm 3.1, we have

$$\begin{aligned}
\|v_n-v_{n-1}\| &= \bigl\|b_nu_n+(1-b_n)S_p^nS_{p-1}^n\cdots S_1^nu_n \\
&\quad - b_{n-1}u_{n-1}-(1-b_{n-1})S_p^{n-1}S_{p-1}^{n-1}\cdots S_1^{n-1}u_{n-1}\bigr\| \\
&= \bigl\|(1-b_n)\bigl(S_p^nS_{p-1}^n\cdots S_1^nu_n - S_p^{n-1}S_{p-1}^{n-1}\cdots S_1^{n-1}u_{n-1}\bigr) \\
&\quad - (b_n-b_{n-1})S_p^{n-1}S_{p-1}^{n-1}\cdots S_1^{n-1}u_{n-1} \\
&\quad + b_n(u_n-u_{n-1})-(b_{n-1}-b_n)u_{n-1}\bigr\| \\
&\leq \|u_n-u_{n-1}\|+|b_n-b_{n-1}|\bigl\|S_p^{n-1}S_{p-1}^{n-1}\cdots S_1^{n-1}u_{n-1}-u_{n-1}\bigr\| \\
&\quad + (1-b_n)\bigl\|S_p^nS_{p-1}^n\cdots S_1^nu_n - S_p^{n-1}S_{p-1}^{n-1}\cdots S_1^{n-1}u_{n-1}\bigr\|. \qquad (3.5)
\end{aligned}$$

From the definition of S_i^n it follows that

$$\begin{aligned}
\bigl\|S_2^nS_1^nv_n-S_2^{n-1}S_1^{n-1}v_n\bigr\| &\leq \bigl\|S_2^nS_1^nv_n-S_2^nS_1^{n-1}v_n\bigr\| + \bigl\|S_2^nS_1^{n-1}v_n-S_2^{n-1}S_1^{n-1}v_n\bigr\| \\
&\leq \bigl\|S_1^nv_n-S_1^{n-1}v_n\bigr\| + \bigl\|S_2^nS_1^{n-1}v_n-S_2^{n-1}S_1^{n-1}v_n\bigr\| \\
&\leq \bigl\|\bigl(1-d_n^1\bigr)v_n+d_n^1S_1v_n-\bigl(1-d_{n-1}^1\bigr)v_n-d_{n-1}^1S_1v_n\bigr\| \\
&\quad + \bigl\|\bigl(1-d_n^2\bigr)S_1^{n-1}v_n+d_n^2S_2S_1^{n-1}v_n \\
&\quad - \bigl(1-d_{n-1}^2\bigr)S_1^{n-1}v_n-d_{n-1}^2S_2S_1^{n-1}v_n\bigr\| \\
&\leq \bigl|d_n^1-d_{n-1}^1\bigr|\bigl(\|v_n\|+\|S_1v_n\|\bigr) \\
&\quad + \bigl|d_n^2-d_{n-1}^2\bigr|\bigl(\bigl\|S_1^{n-1}v_n\bigr\|+\bigl\|S_2S_1^{n-1}v_n\bigr\|\bigr), \qquad (3.6)
\end{aligned}$$

and from (3.6), we have

$$\begin{aligned}
&\left\|S_3^n S_2^n S_1^n v_n - S_3^{n-1} S_2^{n-1} S_1^{n-1} v_n\right\| \\
&\quad\leq \left\|S_3^n S_2^n S_1^n v_n - S_3^n S_2^{n-1} S_1^{n-1} v_n\right\| + \left\|S_3^n S_2^{n-1} S_1^{n-1} v_n - S_3^{n-1} S_2^{n-1} S_1^{n-1} v_n\right\| \\
&\quad\leq \left\|S_2^n S_1^n v_n - S_2^{n-1} S_1^{n-1} v_n\right\| \\
&\qquad + \left\|\left(1-d_n^3\right) S_2^{n-1} S_1^{n-1} v_n + d_n^3 S_3 S_2^{n-1} S_1^{n-1} v_n \right. \\
&\qquad \left. - \left(1-d_{n-1}^3\right) S_2^{n-1} S_1^{n-1} v_n - d_{n-1}^3 S_3 S_2^{n-1} S_1^{n-1} v_n\right\| \\
&\quad\leq \left|d_n^1 - d_{n-1}^1\right| \left(\|v_n\| + \|S_1 v_n\|\right) + \left|d_n^2 - d_{n-1}^2\right| \left(\left\|S_1^{n-1} v_n\right\| + \left\|S_2 S_1^{n-1} v_n\right\|\right) \\
&\qquad + \left|d_n^3 - d_{n-1}^3\right| \left(\left\|S_2^{n-1} S_1^{n-1} v_n\right\| + \left\|S_3 S_2^{n-1} S_1^{n-1} v_n\right\|\right).
\end{aligned} \tag{3.7}$$

By induction on p, it follows that

$$\begin{aligned}
&\left\|S_p^n S_{p-1}^n \cdots S_1^n v_n - S_p^{n-1} S_{p-1}^{n-1} \cdots S_1^{n-1} v_n\right\| \\
&\quad\leq \left|d_n^1 - d_{n-1}^1\right| \left(\|v_n\| + \|S_1 v_n\|\right) + \left|d_n^2 - d_{n-1}^2\right| \left(\left\|S_1^{n-1} v_n\right\| + \left\|S_2 S_1^{n-1} v_n\right\|\right) \\
&\qquad + \cdots + \left|d_n^p - d_{n-1}^p\right| \left(\left\|S_{p-1}^{n-1} \cdots S_1^{n-1} v_n\right\| + \left\|S_p S_{p-1}^{n-1} \cdots S_1^{n-1} v_n\right\|\right).
\end{aligned} \tag{3.8}$$

Similarly,

$$\begin{aligned}
&\left\|S_p^n S_{p-1}^n \cdots S_1^n u_n - S_p^{n-1} S_{p-1}^{n-1} \cdots S_1^{n-1} u_n\right\| \\
&\quad\leq \left|d_n^1 - d_{n-1}^1\right| \left(\|u_n\| + \|S_1 u_n\|\right) + \left|d_n^2 - d_{n-1}^2\right| \left(\left\|S_1^{n-1} u_n\right\| + \left\|S_2 S_1^{n-1} u_n\right\|\right) \\
&\qquad + \cdots + \left|d_n^p - d_{n-1}^p\right| \left(\left\|S_{p-1}^{n-1} \cdots S_1^{n-1} u_n\right\| + \left\|S_p S_{p-1}^{n-1} \cdots S_1^{n-1} u_n\right\|\right).
\end{aligned} \tag{3.9}$$

From (3.5), (3.8) and (3.9), it follows that

$$\begin{aligned}
\|u_{n+1} - u_n\| &= \left\|a_n \rho g(v_n) + c_n v_n + \left[(1-c_n)I - a_n \mu F\right] S_p^n S_{p-1}^n \cdots S_1^n v_n \right. \\
&\quad - a_{n-1} \rho g(v_{n-1}) - c_{n-1} v_{n-1} \\
&\quad \left. - \left[(1-c_{n-1})I - a_{n-1} \mu F\right] S_p^{n-1} S_{p-1}^{n-1} \cdots S_1^{n-1} v_{n-1}\right\| \\
&= \left\|a_n \rho\left(g(v_n) - g(v_{n-1})\right) + a_n \rho g(v_{n-1}) - a_{n-1} \rho g(v_{n-1}) \right. \\
&\quad + c_n(v_n - v_{n-1}) + c_n v_{n-1} - c_{n-1} v_{n-1} \\
&\quad + \left(\left[(1-c_n)I - a_n \mu F\right] S_p^n S_{p-1}^n \cdots S_1^n v_n \right. \\
&\quad \left. - \left[(1-c_n)I - a_n \mu F\right] S_p^{n-1} S_{p-1}^{n-1} \cdots S_1^{n-1} v_{n-1}\right) \\
&\quad + \left[(1-c_n)I - a_n \mu F\right] S_p^{n-1} S_{p-1}^{n-1} \cdots S_1^{n-1} v_{n-1} \\
&\quad \left. - \left[(1-c_{n-1})I - a_{n-1} \mu F\right] S_p^{n-1} S_{p-1}^{n-1} \cdots S_1^{n-1} v_{n-1}\right\| \\
&= \left\|a_n \rho\left(g(v_n) - g(v_{n-1})\right) + (a_n - a_{n-1}) \rho g(v_{n-1}) \right. \\
&\quad + c_n(v_n - v_{n-1}) + (c_n - c_{n-1}) v_{n-1} \\
&\quad + \left(\left[(1-c_n)I - a_n \mu F\right]\left(S_p^n S_{p-1}^n \cdots S_1^n v_n - S_p^{n-1} S_{p-1}^{n-1} \cdots S_1^{n-1} v_{n-1}\right)\right) \\
&\quad \left. + \left(\left[(1-c_n)I - a_n \mu F\right] - \left[(1-c_{n-1})I - a_{n-1} \mu F\right]\right) S_p^{n-1} S_{p-1}^{n-1} \cdots S_1^{n-1} v_{n-1}\right\|
\end{aligned}$$

$$
\begin{aligned}
&\leq a_n\rho\tau\|v_n - v_{n-1}\| + |a_n - a_{n-1}|\big\|\rho g(v_{n-1})\big\| \\
&\quad + c_n\|v_n - v_{n-1}\| + |c_n - c_{n-1}|\|v_{n-1}\| \\
&\quad + (1-c_n)\left(1 - \frac{a_n\nu}{1-c_n}\right)\big\|S_p^n S_{p-1}^n \cdots S_1^n v_n - S_p^{n-1} S_{p-1}^{n-1} \cdots S_1^{n-1} v_n\big\| \\
&\quad + \big(|c_n - c_{n-1}| + |a_n - a_{n-1}|\mu F\big)\big\|S_p^{n-1} S_{p-1}^{n-1} \cdots S_1^{n-1} v_{n-1}\big\| \\
&\leq (a_n\rho\tau + c_n)\|v_n - v_{n-1}\| \\
&\quad + |a_n - a_{n-1}|\big(\big\|\rho g(v_{n-1})\big\| + \mu F\big\|S_p^{n-1} S_{p-1}^{n-1} \cdots S_1^{n-1} v_{n-1}\big\|\big) \\
&\quad + |c_n - c_{n-1}|\big(\|v_{n-1}\| + \big\|S_p^{n-1} S_{p-1}^{n-1} \cdots S_1^{n-1} v_{n-1}\big\|\big) \\
&\quad + (1-c_n)\left(1 - \frac{a_n\nu}{1-c_n}\right)\big\|S_p^n S_{p-1}^n \cdots S_1^n v_n - S_p^{n-1} S_{p-1}^{n-1} \cdots S_1^{n-1} v_n\big\| \\
&\leq \big(1 - a_n(1-\rho\tau)\big)\|u_n - u_{n-1}\| \\
&\quad + |a_n - a_{n-1}|\big(\big\|\rho g(v_{n-1})\big\| + \mu F\big\|S_p^{n-1} S_{p-1}^{n-1} \cdots S_1^{n-1} v_{n-1}\big\|\big) \\
&\quad + |b_n - b_{n-1}|\big\|S_p^{n-1} S_{p-1}^{n-1} \cdots S_1^{n-1} u_{n-1} - u_{n-1}\big\| \\
&\quad + |c_n - c_{n-1}|\big(\|v_{n-1}\| + \big\|S_p^{n-1} S_{p-1}^{n-1} \cdots S_1^{n-1} v_{n-1}\big\|\big) \\
&\quad + (1-c_n)\left(1 - \frac{a_n\nu}{1-c_n}\right)\big\|S_p^n S_{p-1}^n \cdots S_1^n v_n - S_p^{n-1} S_{p-1}^{n-1} \cdots S_1^{n-1} v_n\big\| \\
&\quad + (1-b_n)\big\|S_p^n S_{p-1}^n \cdots S_1^n u_n - S_p^{n-1} S_{p-1}^{n-1} \cdots S_1^{n-1} u_n\big\| \\
&\leq \big(1 - a_n(1-\rho\tau)\big)\|u_n - u_{n-1}\| \\
&\quad + |a_n - a_{n-1}|\big(\big\|\rho g(v_{n-1})\big\| + \mu F\big\|S_p^{n-1} S_{p-1}^{n-1} \cdots S_1^{n-1} v_{n-1}\big\|\big) \\
&\quad + |b_n - b_{n-1}|\big\|S_p^{n-1} S_{p-1}^{n-1} \cdots S_1^{n-1} u_{n-1} - u_{n-1}\big\| \\
&\quad + |c_n - c_{n-1}|\big(\|v_{n-1}\| + \big\|S_p^{n-1} S_{p-1}^{n-1} \cdots S_1^{n-1} v_{n-1}\big\|\big) \\
&\quad + \big|d_n^1 - d_{n-1}^1\big|\big(\|v_n\| + \big\|S_1 v_n\big\| + \|u_n\| + \|S_1 u_n\big\|\big) \\
&\quad + \big|d_n^2 - d_{n-1}^2\big|\big(\big\|S_1^{n-1} v_n\big\| + \big\|S_2 S_1^{n-1} v_n\big\| + \big\|S_1^{n-1} u_n\big\| + \big\|S_2 S_1^{n-1} u_n\big\|\big) \\
&\quad + \cdots + \big|d_n^p - d_{n-1}^p\big|\big(\big\|S_{p-1}^{n-1} \cdots S_1^{n-1} v_n\big\| + \big\|S_p S_{p-1}^{n-1} \cdots S_1^{n-1} v_n\big\| \\
&\quad + \big\|S_{p-1}^{n-1} \cdots S_1^{n-1} u_n\big\| + \big\|S_p S_{p-1}^{n-1} \cdots S_1^{n-1} u_n\big\|\big) \\
&\leq \big(1 - a_n(1-\rho\tau)\big)\|u_n - u_{n-1}\| \\
&\quad + |a_n - a_{n-1}|\big(\big\|\rho g(v_{n-1})\big\| + \mu F\big\|S_p^{n-1} S_{p-1}^{n-1} \cdots S_1^{n-1} v_{n-1}\big\| \\
&\quad + \big\|S_p^{n-1} S_{p-1}^{n-1} \cdots S_1^{n-1} u_{n-1} - u_{n-1}\big\|\big) \\
&\quad + |c_n - c_{n-1}|\big(\|v_{n-1}\| + \big\|S_p^{n-1} S_{p-1}^{n-1} \cdots S_1^{n-1} v_{n-1}\big\|\big) \\
&\quad + \big|d_n^1 - d_{n-1}^1\big|\big(\|v_n\| + \|S_1 v_n\| + \|u_n\| + \|S_1 u_n\|\big) \\
&\quad + \big|d_n^2 - d_{n-1}^2\big|\big(\big\|S_1^{n-1} v_n\big\| + \big\|S_2 S_1^{n-1} v_n\big\| + \big\|S_1^{n-1} u_n\big\| + \big\|S_2 S_1^{n-1} u_n\big\|\big) \\
&\quad + \cdots + \big|d_n^p - d_{n-1}^p\big|\big(\big\|S_{p-1}^{n-1} \cdots S_1^{n-1} v_n\big\| + \big\|S_p S_{p-1}^{n-1} \cdots S_1^{n-1} v_n\big\| \\
&\quad + \big\|S_{p-1}^{n-1} \cdots S_1^{n-1} u_n\big\| + \big\|S_p S_{p-1}^{n-1} \cdots S_1^{n-1} u_n\big\|\big) \\
&\leq \big(1 - a_n(1-\rho\tau)\big)\|u_n - u_{n-1}\|
\end{aligned}
$$

$$
\begin{aligned}
&+ M\bigl(|a_n - a_{n-1}| + |c_n - c_{n-1}| + \bigl|d_n^1 - d_{n-1}^1\bigr| \\
&+ \bigl|d_n^2 - d_{n-1}^2\bigr| + \cdots + \bigl|d_n^p - d_{n-1}^p\bigr|\bigr),
\end{aligned}
$$

where

$$
\begin{aligned}
M = \max\Bigl\{&\sup_{n\ge1}\bigl(\bigl\|\rho g(\nu_{n-1})\bigr\| + \mu F\bigl\|S_p^{n-1}S_{p-1}^{n-1}\cdots S_1^{n-1}\nu_{n-1}\bigr\| \\
&+ \bigl\|S_p^{n-1}S_{p-1}^{n-1}\cdots S_1^{n-1}u_{n-1} - u_{n-1}\bigr\|\bigr), \\
&\sup_{n\ge1}\bigl(\bigl\|S_p^{n-1}S_{p-1}^{n-1}\cdots S_1^{n-1}\nu_{n-1}\bigr\| + \|\nu_{n-1}\|\bigr), \\
&\sup_{n\ge1}\bigl(\|\nu_n\| + \|S_1\nu_n\| + \|u_n\| + \|S_1u_n\|\bigr), \\
&\sup_{n\ge1}\bigl(\bigl\|S_1^{n-1}\nu_n\bigr\| + \bigl\|S_2S_1^{n-1}\nu_n\bigr\| + \bigl\|S_1^{n-1}u_n\bigr\| + \bigl\|S_2S_1^{n-1}u_n\bigr\|\bigr), \\
&\sup_{n\ge1}\bigl(\bigl\|S_{p-1}^{n-1}\cdots S_1^{n-1}\nu_n\bigr\| + \bigl\|S_pS_{p-1}^{n-1}\cdots S_1^{n-1}\nu_n\bigr\| \\
&+ \bigl\|S_{p-1}^{n-1}\cdots S_1^{n-1}u_n\bigr\| + \bigl\|S_pS_{p-1}^{n-1}\cdots S_1^{n-1}u_n\bigr\|\bigr)\Bigr\}.
\end{aligned}
$$

From conditions (i) and (iv) of Algorithm 3.1 and Lemma 2.3, we have

$$
\lim_{n\to\infty}\|u_{n+1} - u_n\| = 0. \tag{3.10}
$$

From (3.2), we have

$$
\begin{aligned}
\bigl\|u_n - S_p^nS_{p-1}^n\cdots S_1^nu_n\bigr\| &\le \|u_n - u_{n+1}\| + \bigl\|u_{n+1} - S_p^nS_{p-1}^n\cdots S_1^nu_n\bigr\| \\
&\le \|u_n - u_{n+1}\| + \bigl\|a_n\rho g(\nu_n) + c_n\nu_n \\
&\quad + \bigl[(1-c_n)I - a_n\mu F\bigr]S_p^nS_{p-1}^n\cdots S_1^n\nu_n - S_p^nS_{p-1}^n\cdots S_1^nu_n\bigr\| \\
&\le \|u_n - u_{n+1}\| + a_n\bigl\|\rho g(\nu_n) - \mu FS_p^nS_{p-1}^n\cdots S_1^n\nu_n\bigr\| \\
&\quad + c_n\bigl\|\nu_n - S_p^nS_{p-1}^n\cdots S_1^n\nu_n\bigr\| \\
&\quad + \bigl\|S_p^nS_{p-1}^n\cdots S_1^n\nu_n - S_p^nS_{p-1}^n\cdots S_1^nu_n\bigr\| \\
&\le \|u_n - u_{n+1}\| + a_n\bigl\|\rho g(\nu_n) - \mu FS_p^nS_{p-1}^n\cdots S_1^n\nu_n\bigr\| \\
&\quad + c_n\bigl\|\nu_n - S_p^nS_{p-1}^n\cdots S_1^n\nu_n\bigr\| + \|\nu_n - u_n\| \\
&\le \|u_n - u_{n+1}\| + a_n\bigl\|\rho g(\nu_n) - \mu FS_p^nS_{p-1}^n\cdots S_1^n\nu_n\bigr\| \\
&\quad + c_n\bigl\|\nu_n - S_p^nS_{p-1}^n\cdots S_1^n\nu_n\bigr\| \\
&\quad + \bigl\|b_nu_n + (1-b_n)S_p^nS_{p-1}^n\cdots S_1^nu_n - u_n\bigr\| \\
&\le \|u_n - u_{n+1}\| + a_n\bigl\|\rho g(\nu_n) - \mu FS_p^nS_{p-1}^n\cdots S_1^n\nu_n\bigr\| \\
&\quad + c_n\bigl\|\nu_n - S_p^nS_{p-1}^n\cdots S_1^n\nu_n\bigr\| + (1-b_n)\bigl\|S_p^nS_{p-1}^n\cdots S_1^nu_n - u_n\bigr\|.
\end{aligned} \tag{3.11}
$$

From (3.11), we have

$$b_n\|S_p^nS_{p-1}^n\cdots S_1^nu_n-u_n\|\le\|u_n-u_{n+1}\|+a_n\|\rho g(v_n)-\mu FS_p^nS_{p-1}^n\cdots S_1^nv_n\|$$
$$+c_n\|v_n-S_p^nS_{p-1}^n\cdots S_1^nv_n\|.$$

Since from (i), (ii), (iii) and (3.10), we have

$$\lim_{n\to\infty}\|u_n-S_p^nS_{p-1}^n\cdots S_1^nu_n\|=0.$$ □

Lemma 3.3 *Let*

$$u_n=a_n\rho g(u_n)+c_nu_n+\big[(1-c_n)I-a_n\mu F\big]S_p^nS_{p-1}^n\cdots S_1^nu_n. \tag{3.12}$$

Then u_n converges strongly to $\tilde{u}\in\Xi$ as $n\to 0$.

Proof Since $\{u_n\}$ is bounded, we assume that $\{u_n\}$ converges weakly to a point $\tilde{u}\in W$. From Lemma 2.4, we have $\tilde{u}\in\Xi$. Now, for $\tilde{u}\in\Xi$, we get

$$\begin{aligned}\|u_n-\tilde{u}\|^2&=\big\|a_n\rho g(u_n)+c_nu_n+\big[(1-c_n)I-a_n\mu F\big]S_p^nS_{p-1}^n\cdots S_1^nu_n-\tilde{u}\big\|^2\\
&\le\big\langle a_n\big(\rho g(u_n)-\mu F(\tilde{u})\big)+c_n(u_n-\tilde{u})\\
&\quad+\big[(1-c_n)I-a_n\mu F\big]\big(S_p^nS_{p-1}^n\cdots S_1^nu_n-S_p^nS_{p-1}^n\cdots S_1^n\tilde{u}\big),u_n-\tilde{u}\big\rangle\\
&=\big\langle a_n\rho\big(g(u_n)-g(\tilde{u})\big),u_n-\tilde{u}\big\rangle\\
&\quad+a_n\big\langle\rho g(\tilde{u})-\mu F(\tilde{u}),u_n-\tilde{u}\big\rangle+c_n\langle u_n-\tilde{u},u_n-\tilde{u}\rangle\\
&\quad+\big[(1-c_n)I-a_n\mu F\big]\big\langle S_p^nS_{p-1}^n\cdots S_1^nu_n-S_p^nS_{p-1}^n\cdots S_1^n\tilde{u},u_n-\tilde{u}\big\rangle\\
&\le a_n\rho\tau\|u_n-\tilde{u}\|^2+a_n\big\langle\rho g(\tilde{u})-\mu F(\tilde{u}),u_n-\tilde{u}\big\rangle\\
&\quad+c_n\|u_n-\tilde{u}\|^2+\big[(1-c_n)I-a_n\mu F\big]\|u_n-\tilde{u}\|^2\\
&\le\big(1-a_n(\mu F-\rho\tau)\big)\|u_n-\tilde{u}\|^2+a_n\big\langle\rho g(\tilde{u})-\mu F(\tilde{u}),u_n-\tilde{u}\big\rangle.\end{aligned}$$

Hence,

$$\|u_n-\tilde{u}\|^2\le\frac{1}{(\mu F-\rho\tau)}\big\langle\rho g(\tilde{u})-\mu F(\tilde{u}),u_n-\tilde{u}\big\rangle. \tag{3.13}$$

Since $u_n\rightharpoonup\tilde{u}$, from (3.13) we obtain $u_n\to\tilde{u}$. □

Theorem 3.1 *The sequence $\{u_n\}$ generated by Algorithm* 3.1 *converges strongly to $z\in\Xi=\bigcap_{i=1}^p\overline{F}(S_i)$, which is also a unique solution of the HFPP*

$$\big\langle\rho g(z)-\mu F(z),u-z\big\rangle\le 0,\quad\forall u\in\Xi.$$

Proof Let $u_t\in W$ be a unique fixed point. Now, we claim that

$$\lim_{n\to\infty}\sup\big\langle\rho g(z)-\mu F(z),z-u_n\big\rangle\le 0,$$

where $z=\lim_{t\to 0}u_t$. It follows from Lemma 3.3 that $z\in\Xi$.

By using Lemma 2.6, we get

$$\begin{aligned}
\|u_n - u_t\|^2 &= \big\| a_n \rho g(u_n) + c_n u_n + \big[(1-c_n)I - a_n\mu F\big] S_p^n S_{p-1}^n \cdots S_1^n u_n - u_t \big\|^2 \\
&= \big\| a_n\big(\rho g(u_n) - \mu F(u_t)\big) + c_n(u_n - u_t) \\
&\quad + \big[(1-c_n)I - a_n\mu F\big] S_p^n S_{p-1}^n \cdots S_1^n u_n \\
&\quad - \big[(1-c_n)I - a_n\mu F\big] S_p^n S_{p-1}^n \cdots S_1^n u_t \big\|^2 \\
&\le \big\| c_n(u_n - u_t) + \big[(1-c_n)I - a_n\mu F\big] S_p^n S_{p-1}^n \cdots S_1^n u_n \\
&\quad - \big[(1-c_n)I - a_n\mu F\big] S_p^n S_{p-1}^n \cdots S_1^n u_t \big\|^2 + 2a_n\big\langle \rho g(u_n) - \mu F(u_t), u_n - u_t \big\rangle \\
&\le \left\{ c_n\|u_n - u_t\| + (1-c_n)\left\| \left(I - \frac{a_n\mu F}{1-c_n}\right) S_p^n S_{p-1}^n \cdots S_1^n u_n \right.\right. \\
&\quad \left.\left. - \left(I - \frac{a_n\mu F}{1-c_n}\right) S_p^n S_{p-1}^n \cdots S_1^n u_t \right\| \right\}^2 \\
&\quad + 2a_n\rho\big\langle g(u_n) - g(u_t), u_n - u_t \big\rangle + 2a_n\big\langle \rho g(u_t) - \mu F(u_t), u_n - u_t \big\rangle \\
&\le \left\{ c_n\|u_n - u_t\| + (1-c_n)\left(I - \frac{a_n\nu}{1-c_n}\right)\|u_n - u_t\| \right\}^2 \\
&\quad + 2a_n\rho\tau\|u_n - u_t\|\|u_n - u_t\| + 2a_n\big\langle \rho g(u_t) - \mu F(u_t), u_n - u_t \big\rangle \\
&\le \big\{ c_n\|u_n - u_t\| + (1 - c_n - a_n\nu)\|u_n - u_t\| \big\}^2 + 2a_n\rho\tau\|u_n - u_t\|^2 \\
&\quad + 2a_n\big\langle \rho g(u_t) - \mu F(u_t), u_n - u_t \big\rangle \\
&\le \big((1 - a_n\nu)^2 + 2a_n\rho\tau\big)\|u_n - u_t\|^2 + 2a_n\big\langle \rho g(u_t) - \mu F(u_t), u_n - u_t \big\rangle.
\end{aligned}$$

From the above we have

$$\big\langle \rho g(u_t) - \mu F(u_t), u_t - u_n \big\rangle \le \frac{\mathcal{A}_n(t)}{2a_n}\|u_n - u_t\|^2,$$

where $\mathcal{A}_n(t) = [1 - [(1 - a_n\nu)^2 + 2a_n\rho\tau]]$.

Further,

$$\lim_{n\to\infty}\sup\big\langle \rho g(u_t) - \mu F(u_t), u_t - u_n \big\rangle \le \frac{\mathcal{A}_n(t)}{2}\mathcal{M}, \tag{3.14}$$

where $\mathcal{M} > 0$ is a constant such that $\mathcal{M} \ge \|u_n - u_t\|^2$.

Taking the lim sup as $t \to 0$ in (3.14), we get

$$\lim_{n\to\infty}\sup\big\langle \rho g(z) - \mu F(z), z - u_n \big\rangle \le 0.$$

Now, we have to show that $u_n \to z$.

$$\begin{aligned}
\|u_{n+1} - z\|^2 &= \big\| a_n\rho g(v_n) + c_n v_n + \big[(1-c_n)I - a_n\mu F\big] S_p^n S_{p-1}^n \cdots S_1^n v_n - z \big\|^2 \\
&= \big\langle a_n\rho g(v_n) + c_n v_n + \big[(1-c_n)I - a_n\mu F\big] S_p^n S_{p-1}^n \cdots S_1^n v_n - z, u_{n+1} - z \big\rangle \\
&\le \left\langle a_n\big(\rho g(v_n) - \mu F(z)\big) + c_n(v_n - z) + (1-c_n)\left[\left(I - \frac{a_n\mu F}{1-c_n}\right) S_p^n S_{p-1}^n \cdots S_1^n v_n \right.\right.
\end{aligned}$$

$$
\begin{aligned}
&\quad - \left(I - \frac{a_n \mu F}{1 - c_n}\right) S_p^n S_{p-1}^n \cdots S_1^n z\Bigg], u_{n+1} - z\Bigg\rangle \\
&= \big\langle a_n \rho \big(g(v_n) - g(z)\big), u_{n+1} - z\big\rangle + a_n \big\langle \rho g(z) - \mu F(z), u_{n+1} - z\big\rangle \\
&\quad + c_n \langle v_n - z, u_{n+1} - z\rangle + (1 - c_n)\Bigg\langle \left(I - \frac{a_n \mu F}{1 - c_n}\right) S_p^n S_{p-1}^n \cdots S_1^n v_n \\
&\quad - \left(I - \frac{a_n \mu F}{1 - c_n}\right) S_p^n S_{p-1}^n \cdots S_1^n z, u_{n+1} - z\Bigg\rangle \\
&\le a_n \rho \tau \|v_n - z\| \|u_{n+1} - z\| + a_n \big\langle \rho g(z) - \mu F(z), u_{n+1} - z\big\rangle \\
&\quad + c_n \|v_n - z\| \|u_{n+1} - z\| + (1 - c_n - a_n \nu) \|v_n - z\| \|u_{n+1} - z\| \\
&\le (a_n \rho \tau + 1 - a_n \nu) \|v_n - z\| \|u_{n+1} - z\| + a_n \big\langle \rho g(z) - \mu F(z), u_{n+1} - z\big\rangle \\
&\le \big(1 - a_n(\nu - \rho\tau)\big) \|v_n - z\| \|u_{n+1} - z\| + a_n \big\langle \rho g(z) - \mu F(z), u_{n+1} - z\big\rangle \\
&\le \big(1 - a_n(\nu - \rho\tau)\big) \|u_n - z\| \|u_{n+1} - z\| + a_n \big\langle \rho g(z) - \mu F(z), u_{n+1} - z\big\rangle \\
&\le \frac{(1 - a_n(\nu - \rho\tau))}{2} \big(\|u_n - z\|^2 + \|u_{n+1} - z\|^2\big) \\
&\quad + a_n \big\langle \rho g(z) - \mu F(z), u_{n+1} - z\big\rangle.
\end{aligned}
$$

Further,

$$
\begin{aligned}
\left[1 - \frac{(1 - a_n(\nu - \rho\tau))}{2}\right] \|u_{n+1} - z\|^2 &\le \left[\frac{1 - a_n(\nu - \rho\tau)}{2}\right] \|u_n - z\|^2 \\
&\quad + a_n \big\langle \rho g(z) - \mu F(z), u_{n+1} - z\big\rangle, \\
\left[\frac{1 + a_n(\nu - \rho\tau)}{2}\right] \|u_{n+1} - z\|^2 &\le \left[\frac{1 - a_n(\nu - \rho\tau)}{2}\right] \|u_n - z\|^2 \\
&\quad + a_n \big\langle \rho g(z) - \mu F(z), u_{n+1} - z\big\rangle,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\|u_{n+1} - z\|^2 &\le \left[\frac{1 - a_n(\nu - \rho\tau)}{1 + a_n(\nu - \rho\tau)}\right] \|u_n - z\|^2 \\
&\quad + \left[\frac{2a_n}{1 + a_n(\nu - \rho\tau)}\right] \big\langle \rho g(z) - \mu F(z), u_{n+1} - z\big\rangle, \\
\|u_{n+1} - z\|^2 &\le \left[1 - \frac{2a_n(\nu - \rho\tau)}{1 + a_n(\nu - \rho\tau)}\right] \|u_n - z\|^2 \\
&\quad + \left[\frac{2a_n(\nu - \rho\tau)}{1 + a_n(\nu - \rho\tau)}\right] \left\{\frac{1}{\nu - \rho\tau} \big\langle \rho g(z) - \mu F(z), u_{n+1} - z\big\rangle\right\}.
\end{aligned}
$$

Let $w_n = [\frac{2a_n(\nu-\rho\tau)}{1+a_n(\nu-\rho\tau)}]$ and

$$
t_n = \left[\frac{2a_n(\nu - \rho\tau)}{1 + a_n(\nu - \rho\tau)}\right] \left\{\frac{1}{\nu - \rho\tau} \big\langle \rho g(z) - \mu F(z), u_{n+1} - z\big\rangle\right\}.
$$

We have $\sum_{n=1}^{\infty} a_n = \infty$ and $\lim_{n\to\infty} \sup\{\frac{1}{\nu-\rho\tau} \langle \rho g(z) - \mu F(z), u_{n+1} - z\rangle\} \le 0$. It follows that $\sum_{n=1}^{\infty} w_n = \infty$ and $\lim_{n\to\infty} \sup \frac{t_n}{w_n} \le 0$. Thus, all the conditions of Lemma 2.3 are fulfilled. Hence, $u_n \to z$. □

4 Examples

The following example ensures that all the conditions of Algorithm 3.1 and the convergence result are fulfilled.

Example 4.1 Let $a_n = \frac{1}{3n}$, $b_n = \frac{2n-1}{3n}$ and $c_n = \frac{1}{3n}$. Then

$$\lim_{n\to\infty} a_n = \frac{1}{3}\lim_{n\to\infty}\frac{1}{n} = 0,$$

and

$$\sum_{n=1}^{\infty} a_n = \frac{1}{3}\sum_{n=1}^{\infty}\frac{1}{n} = \infty.$$

The sequence $\{a_n\}$ satisfies condition (i) of Algorithm 3.1.

Now we compute

$$a_{n-1} - a_n = \frac{1}{3(n-1)} - \frac{1}{3n} = \frac{1}{3}\left(\frac{1}{n-1} - \frac{1}{n}\right) = \frac{1}{3n(n-1)}.$$

So,

$$\sum_{n=1}^{\infty} |a_{n-1} - a_n| < \infty.$$

Similarly, we can show

$$\sum_{n=1}^{\infty} |c_{n-1} - c_n| < \infty.$$

The sequences $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ satisfy conditions (i), (ii) and (iii).

Let $d_n^i = \frac{n}{n+i}$ for $i = 1, 2$. Then

$$\sum_{n=1}^{\infty} \left|d_{n-1}^i - d_n^i\right| < \infty.$$

Hence the sequence $\{d_n^i\}$ also satisfies condition (iv) of Algorithm 3.1.

Let $S_1, S_2 : \mathbb{R} \to \mathbb{R}$ be defined by

$$S_1(u) = \sin\frac{u}{2}$$

and

$$S_2(u) = \frac{u}{2}, \quad \forall u \in \mathbb{R},$$

and let the mapping $g : \mathbb{R} \to \mathbb{R}$ be defined by

$$g(u) = \frac{u}{2} + 1, \quad \forall u \in \mathbb{R}.$$

It is easy to verify that S_1 and S_2 are $\frac{1}{2}$-nonexpansive and g is a $\frac{1}{2}$-contraction mapping.

Table 1 The values of u_n and v_n with the initial values $u_1 = -10$ and $u_1 = 10$

	$u_1 = -10$		$u_1 = 10$	
	u_n	v_n	u_n	v_n
$n = 1$	−10.0000	−14.5154	10.0000	14.6347
$n = 2$	−7.0467	−7.0467	7.8560	7.8997
$n = 3$	−2.7768	−2.7268	5.2941	5.2981
$n = 4$	−1.0152	−1.0152	2.0321	2.0431
$n = 5$	−0.3254	−0.3252	0.8356	0.8436
$n = 6$	−0.0703	−0.0703	0.3587	0.3487
$n = 7$	−0.0458	−0.0458	0.1662	0.1689
$n = 8$	−0.0492	−0.0429	0.0889	0.0989
$n = 9$	−0.0399	−0.0399	0.0571	0.0671
$n = 10$	−0.0327	−0.0372	0.0429	0.0429
$n = 11$	−0.0298	−0.0331	0.0357	0.0375
$n = 12$	−0.0297	−0.0289	0.0315	0.0351
$n = 13$	−0.0146	−0.0279	0.0286	0.0211
$n = 14$	−0.0118	−0.0164	0.0224	0.0200
$n = 15$	−0.0109	−0.0131	0.0207	0.0198

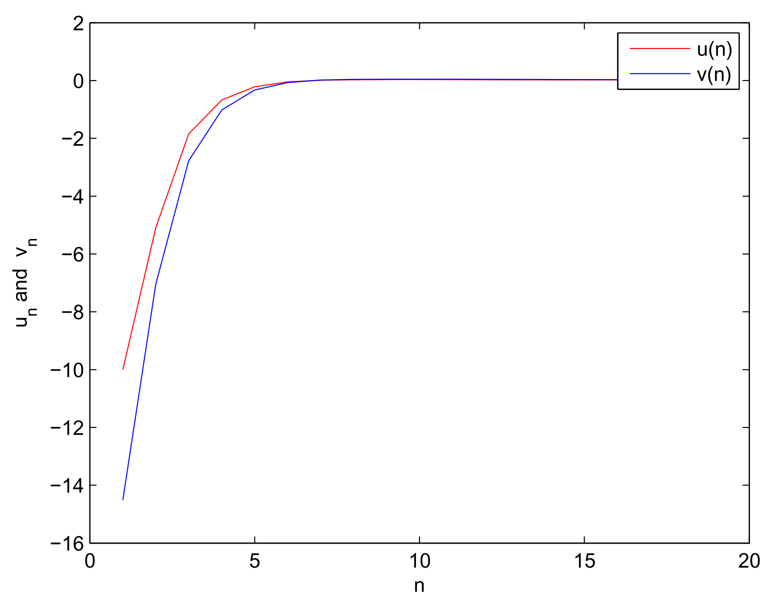

Figure 1 The convergence of u_n and v_n with the initial value $u_1 = -10$.

Further,

$$\Xi = \bigcap_{i=1}^{2} F(S_i) = \{0\}.$$

Suppose that the mapping $F : \mathbb{R} \to \mathbb{R}$ is defined by

$$F(u) = 2u, \quad \forall u \in \mathbb{R}.$$

Hence, F is 2-strongly monotone and 2-Lipschitzian.

Assume that $\rho = \frac{1}{15}$ and $\mu = \frac{1}{5}$ and they satisfy $0 < \mu < \frac{2\eta}{k^2}$ and $0 \leq \rho\tau < \nu$, where $\nu = 1 - \sqrt{1 - \mu(2\eta - \mu k^2)}$.

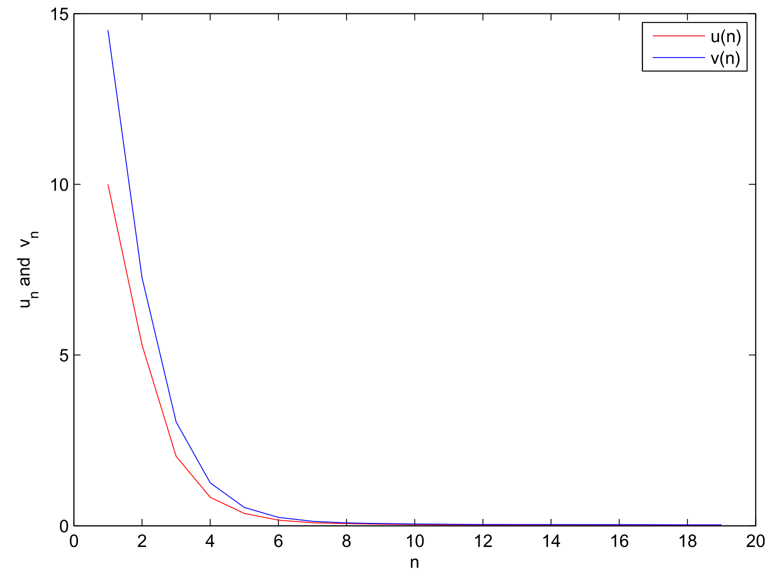

Figure 2 The convergence of u_n and v_n with the initial value $u_1 = 10$.

All codes were written in Matlab, the values of $\{v_n\}$ and $\{u_n\}$ with different n are given in Table 1.

Remark 4.1 Table 1 and Figures 1 and 2 show that the sequences $\{v_n\}$ and $\{u_n\}$ converge to 0. Also, $\{0\} \in \Xi$.

5 Conclusion

We have analyzed an iterative method for finding an approximate solution of hierarchical fixed point problem (1.1) and variational inequality problem (1.5) involving a finite family of nonexpansive mappings in a real Hilbert space. This method can be viewed as a modification and improvement of some existing methods [15, 16] for solving the variational inequality problem and the hierarchical fixed point problem. Therefore, Algorithm 3.1 is expected to be widely applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both the authors contributed equally and approved the final manuscript.

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