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On some fixed point theorems in generalized metric spaces

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Abstract

In this paper, we obtain some generalizations of fixed point results for Kannan, Chatterjea and Hardy-Rogers contraction mappings in a new class of generalized metric spaces introduced recently by Jleli and Samet (Fixed Point Theory Appl. 2015:33, 2015).

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1 Introduction

The concept of a metric space is a very important tool in many scientific fields and particulary in the fixed point theory.

In recent years, this notion has been generalized in several directions and many notions of a metric-type space was introduced (b-metric, dislocated space, generalized metric space, quasi-metric space, symmetric space, etc.).

In 2015, Jleli and Samet [1] introduced a very interesting concept of a generalized metric space, which covers different well-known metric structures including classical metric spaces, b-metric spaces, dislocated metric spaces, modular spaces, and so on.

In this paper, we establish and generalize some well-known fixed point results for nonlinear contractions in this new class of generalized metric spaces.

Let us recall that a mapping T on a metric space (X,d) is called a Kannan contraction if there exists $\alpha \in [0, \frac{1}{2}[$ such that

$$d(Tx, Ty) \leq \alpha d(x, Tx) + \alpha d(y, Ty)$$

for all $x, y \in X$.

Using this contraction notion, Kannan [2] proved the following result.

Theorem 1.1 ([2]) *Let (X,d) be a complete metric space, $\lambda \in [0, \frac{1}{2}[$, and T a self-mapping on X such that*

$$d(Tx, Ty) \leq \lambda \bigl[d(x, Tx) + d(y, Ty)\bigr] \tag{1}$$

for all $x, y \in X$. Then T has a unique fixed point.

In 1972, Chatterjea [3] obtained a similar result by considering a constant $\lambda \in [0, \frac{1}{2}[$ and a mapping $T : X \longrightarrow X$ such that

$$d(Tx, Ty) \leq \lambda\big[d(x, Ty) + d(y, Tx)\big] \tag{2}$$

for all $x, y \in X$.

In this paper, we are interested by Kannan, Chatterjea, and Hardy-Rogers contraction types (see [2–4] and [5]); we establish some results on fixed points in generalized metric spaces. We also give some examples to show the effectiveness of the obtained results.

Our results generalize and improve many fixed point theorems existing in the literature in various metric-type spaces.

2 Definitions and preliminaries

Let X be a nonempty set, and let $D : X \times X \longrightarrow [0, +\infty]$ be a given mapping. For every $x \in X$, let us define the set

$$C(D, X, x) = \Big\{\{x_n\} \subset X : \lim_{n\to\infty} D(x_n, x) = 0\Big\}$$

Definition 2.1 ([1]) D is called a generalized metric on X if it satisfies the following conditions:

(D_1) For every $(x, y) \in X \times X$, we have

$$D(x, y) = 0 \quad \Rightarrow \quad x = y.$$

(D_2) For all $(x, y) \in X \times X$, we have

$$D(x, y) = D(y, x).$$

(D_3) There exists a real constant $C > 0$ such that, for all $(x, y) \in X \times X$ and $\{x_n\} \in C(D, X, x)$, we have

$$D(x, y) \leq C \limsup_{n\to\infty} D(x_n, y).$$

The pair (X, D) is called a generalized metric space.

Remark 2.2 If the set $C(D, X, x)$ is empty for every $x \in X$, then (X, D) is a generalized metric space if and only if (D_1) and (D_2) are satisfied.

Definition 2.3 Let (X, D) be a generalized metric space, let $\{x_n\}$ be a sequence in X, and let $x \in X$. We say that $\{x_n\}$ D-converges to x in X if $\{x_n\} \in C(D, X, x)$.

Remark 2.4 Let $\{x_n\}$ be the sequence defined by $x_n = x$ for all $n \in \mathbb{N}$. If it D-converges to x, then $D(x, x) = 0$.

Definition 2.5 Let (X, D) be a generalized metric space. A sequence $\{x_n\}$ in X is called a D-Cauchy sequence if $\lim_{m,n\to\infty} D(x_n, x_{n+m}) = 0$.

The space (X, D) is said to be D-complete if every Cauchy sequence in X is D-convergent to some element in X.

In the sequel, we use the following definition of a Cauchy sequence.

Definition 2.6 Let (X,D) be a generalized metric space, and let $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is a D-Cauchy sequence if $\lim_{m,n\to\infty} D(x_n,x_m) = 0$.

Proposition 2.7 *$C(D,X,x)$ is a nonempty set if and only if $D(x,x) = 0$.*

Proof If $C(D,X,x) \neq \emptyset$, then there exists a sequence $\{x_n\} \subset X$ such that $\lim_{n\to\infty} D(x_n,x) = 0$. Using property (D_3), we obtain

$$D(x,x) \leq C \limsup_{n\to\infty} D(x_n,x),$$

and thus $D(x,x) = 0$.

Assume now that $D(x,x) = 0$. The sequence $\{x_n\} \subset X$ defined by $x_n = x$ for all $n \in \mathbb{N}$ converges to x, which ends the proof. □

3 Main results

Proposition 3.1 *Let (X,D) be a generalized metric space, and let $f : X \longrightarrow X$ be a mapping satisfying inequality* (1) *for some $\lambda \in [0, \frac{1}{2})$. Then any fixed point $\omega \in X$ of f satisfies*

$$D(\omega,\omega) < \infty \quad \Rightarrow \quad D(\omega,\omega) = 0.$$

Proof Let $\omega \in X$ be a fixed point of f such that $D(\omega,\omega) < \infty$. Using (1), we obtain

$$\begin{aligned} D(\omega,\omega) &= D(f\omega,f\omega) \\ &\leq \lambda\big(D(\omega,f\omega) + D(\omega,f\omega)\big) \\ &\leq 2\lambda D(\omega,\omega). \end{aligned}$$

Since $2\lambda \in [0,1[$, we obtain $D(\omega,\omega) = 0$. □

For every $x \in X$, we define

$$\delta(D,f,x) = \sup\big\{D\big(f^i x,f^j x\big) : i,j \in \mathbb{N}\big\}.$$

Theorem 3.2 *Let (X,D) be a D-complete generalized metric space, and let f be a self-mapping on X satisfying* (1) *for some constant $\lambda \in [0, \frac{1}{2}[$ such that $C\lambda < 1$.*

If there exists an element $x_0 \in X$ such that $\delta(D,f,x_0) < \infty$, then the sequence $\{f^n x_0\}$ converges to some $\omega \in X$. Moreover, if $D(\omega,f\omega) < \infty$, then ω is a fixed point of f. Moreover, for each fixed point ω' of f in X such that $D(\omega',\omega') < \infty$, we have $\omega = \omega'$.

Proof Let $n \in \mathbb{N}$ ($n \geq 1$). For all $i,j \in \mathbb{N}$, we have

$$D\big(f^{n+i}x_0,f^{n+j}x_0\big) \leq \lambda\big[D\big(f^{n+i-1}x_0,f^{n+i}x_0\big) + D\big(f^{n+j-1}x_0,f^{n+j}x_0\big)\big]$$

and then

$$D\big(f^{n+i}x_0,f^{n+j}x_0\big) \leq 2\lambda\delta\big(D,f,f^{n-1}x_0\big),$$

which gives

$$\delta\big(D,f,f^n x_0\big) \leq 2\lambda\delta\big(D,f,f^{n-1}x_0\big).$$

Consequently, we obtain

$$\delta\big(D,f,f^n x_0\big) \leq (2\lambda)^n\delta(D,f,x_0)$$

and

$$D\big(f^n x_0,f^m x_0\big) \leq \delta\big(D,f,f^n x_0\big) \leq (2\lambda)^n\delta(D,f,x_0) \tag{3}$$

for all integer m such that $m > n$.

Since $\delta(D,f,x_0) < \infty$ and $2\lambda \in [0,1[$, we obtain

$$\lim_{m,n\to\infty} D\big(f^n x_0,f^m x_0\big) = 0.$$

It follows that $\{f^n x_0\}$ is a D-Cauchy sequence, and thus there exists $\omega \in X$ such that

$$\lim_{n\to\infty} D\big(f^n x_0,\omega\big) = 0$$

and

$$D(f\omega,\omega) \leq C\limsup_{n\to\infty} D\big(f\omega,f^{n+1}x_0\big). \tag{4}$$

By (1) we have

$$D\big(f^{n+1}x_0,f\omega\big) \leq \lambda\big(D\big(f^{n+1}x_0,f^n x_0\big) + D(\omega,f\omega)\big). \tag{5}$$

By (3) and (5) we obtain

$$\limsup_{n\to\infty} D\big(f\omega,f^{n+1}x_0\big) \leq \lambda D(\omega,f\omega).$$

Using (4), we obtain

$$D(\omega,f\omega) \leq C\lambda D(\omega,f\omega).$$

Since $C\lambda < 1$ and $D(\omega,f\omega) < \infty$, we deduce that $D(\omega,f\omega) = 0$, which implies that $f\omega = \omega$.

If ω' is any fixed point of f such that $D(\omega',\omega') < \infty$, we obtain

$$\begin{aligned} D\big(\omega,\omega'\big) &= D\big(f\omega,f\omega'\big) \\ &\leq \lambda\big(D(f\omega,\omega) + D\big(f\omega',\omega'\big)\big) \\ &\leq \lambda\big(D(\omega,\omega) + D\big(\omega',\omega'\big)\big) \\ &\leq 0, \end{aligned}$$

which implies that $\omega' = \omega$. □

Example 3.3 Let $X = [0,1]$, and let $D : X \times X \to [0,\infty[$ be the mapping defined by

$$\begin{cases} D(x,y) = x + y & \text{if } x \neq 0 \text{ and } y \neq 0, \\ D(0,x) = D(x,0) = \frac{x}{2} & \text{for all } x \in X. \end{cases}$$

Conditions (D_1) and (D_2) are trivially satisfied. By Proposition 2.7 we need to verify condition (D_3) only for elements x of X such that $D(x,x) = 0$, which implies that $x = 0$.

Let $(x_n) \subset X$ be a sequence such that $\lim_{n\to\infty} D(x_n,0) = 0$. For all $n \in \mathbb{N}$ and $y \in X$, we have:

$$D(x_n,y) = \begin{cases} x_n + y & \text{if } x_n \neq 0, \\ \frac{y}{2} & \text{if } x_n = 0. \end{cases}$$

Then

$$\frac{y}{2} \leq D(x_n,y),$$

which implies that

$$D(0,y) = \frac{y}{2} \leq \limsup_{n\to\infty} D(x_n,y).$$

It follows that (X,D) is a generalized metric space that is not a standard metric space since the triangular inequality does not hold: If $x,y \in X - \{0\}$, then we have $D(x,y) = x + y$ and $D(x,0) + D(0,y) = \frac{x+y}{2}$, and thus

$$D(x,y) > D(x,0) + D(0,y).$$

Note that (X,D) is D-complete.

Define the mapping T on X by

$$T(x) = \frac{x}{x+2} \quad \text{for all } x \in X.$$

For any $x \in X$, we have:

$$D\bigl(T(x),T(0)\bigr) = D\left(\frac{x}{x+2},0\right) = \frac{x}{2(x+2)}$$

and

$$D\bigl(T(x),x\bigr) + D\bigl(0,T(0)\bigr) = D\left(\frac{x}{x+2},x\right) + D(0,0) = \frac{x}{x+2} + x.$$

Then

$$D\bigl(T(x),T(0)\bigr) \leq \frac{1}{3}\bigl(D\bigl(T(x),x\bigr) + D\bigl(0,T(0)\bigr)\bigr).$$

For $x, y \in X - \{0\}$, we have

$$D\big(T(x), T(y)\big) = D\left(\frac{x}{x+2}, \frac{y}{y+2}\right) = \frac{x}{2(x+2)} + \frac{y}{2(y+2)}$$

and

$$D\big(T(x), x\big) + D\big(y, T(y)\big) = D\left(\frac{x}{x+2}, x\right) + D\left(y, \frac{y}{y+2}\right) = \frac{x}{x+2} + \frac{y}{y+2} + x + y.$$

Then

$$D\big(T(x), T(y)\big) \leq \frac{1}{3}\big[D\big(T(x), x\big) + D\big(y, T(y)\big)\big].$$

The hypotheses of Theorem 3.2 are satisfied. Therefore T has a unique fixed point since D is bounded; note that $T(0) = 0$.

Example 3.4 Let $X = \{a, b, c\}$ and define T on X by $T(a) = a$, $T(b) = b$, and $T(c) = a$. There is no metric for which T is a Kannan contraction on X.

Define $D : X \times X \longrightarrow [0, +\infty]$ by

$$\begin{cases} D(a,a) = D(b,b) = 0 \quad \text{and} \quad D(b,b) = +\infty, \\ D(a,b) = D(b,a) = 1, \\ D(a,c) = D(c,a) = 2, \\ D(b,c) = D(c,b) = 3. \end{cases}$$

Then (X, D) is a complete generalized metric space, T is a Kannan contraction on (X, D) for any $\lambda \in]0, \frac{1}{2}[$, and we can apply Theorem 3.2.

Theorem 3.2 generalizes well-known results for b-metric and metric spaces.

Corollary 3.5 *Let (X, d) be a complete b-metric space with constant $s \geq 1$, and let $f : X \to X$ a mapping for which there exists $\lambda \in [0, \frac{1}{s+1}[$ such that*

$$d(fx, fy) \leq \lambda\big[d(x, fx) + d(y, fy)\big]$$

for all $x, y \in X$. Then f has a unique fixed point.

Proof Let $x_0 \in X$. For all $n \in \mathbb{N}$, we have

$$d\big(f^n x_0, f^{n+1} x_0\big) \leq \lambda\big[d\big(f^n x_0, f^{n-1} x_0\big) + d\big(f^{n+1} x_0, f^n x_0\big)\big],$$

which implies

$$d\big(f^n x_0, f^{n+1} x_0\big) \leq \frac{\lambda}{1-\lambda} d\big(f^n x_0, f^{n-1} x_0\big).$$

Let $k = \frac{\lambda}{1-\lambda}$. By induction we obtain

$$d\big(f^n x_0, f^{n+1} x_0\big) \leq k^n d(x_0, f x_0),$$

and then

$$\begin{aligned} d(f^n x_0, f^m x_0) &\leq s d(f^n x_0, f^{n+1} x_0) + \cdots + s^{m-n} d(f^{m-1} x_0, f^m x_0) \\ &\leq s(k)^n d(x_0, f x_0) + \cdots + s^{m-n}(k)^{m-1} d(x_0, f x_0) \\ &\leq s(k)^n \frac{1-(sk)^{m-n}}{1-sk} d(x_0, f x_0) \leq \frac{s}{1-sk} d(x_0, f x_0) \end{aligned}$$

for all $n, m \in \mathbb{N}$ such that $m > n$.

This implies that $\delta(D, f, x_0) < \infty$. Then by Theorem 3.2 we conclude that f has a unique fixed point. □

Corollary 3.6 (Kannan fixed point theorem [2]) *Let (X, d) be a complete metric space, and let $f : X \to X$ a mapping for which there exists $\lambda \in [0, \frac{1}{2}[$ such that*

$$d(fx, fy) \leq \lambda\big(d(x, fx) + d(y, fy)\big)$$

for all $x, y \in X$. Then f has a unique fixed point.

Corollary 3.7 *Let (X, d) be a dislocated metric space, and let $f : X \to X$ be a mapping for which there exists $\lambda \in [0, \frac{1}{2}[$ such that*

$$d(fx, fy) \leq \lambda\big[d(x, fx) + d(y, fy)\big] \tag{6}$$

for all $x, y \in X$. Then f has a unique fixed point.

In the following, we need the basic lemma.

Lemma 3.8 ([6]) *Let λ is a real number such that $0 \leq \lambda < 1$, and let $\{b_n\}$ be a sequence of positives reals numbers such that $\lim_{n\to\infty} b_n = 0$. Then, for any sequence of positives numbers $\{a_n\}$ satisfying*

$$a_{n+1} \leq \lambda a_n + b_n \quad \text{for all } n \in \mathbb{N},$$

we have $\lim_{n\to\infty} a_n = 0$.

Theorem 3.9 *Let (X, D) be a D-complete generalized metric space, $\lambda \in [0, \frac{1}{2}[$, and let f be a self-mapping on X such that*

$$D(fx, fy) \leq \lambda\big(D(y, fx) + D(x, fy)\big) \tag{7}$$

for all $x, y \in X$. If there exists a point $x_0 \in X$ such that $\delta(D, f, x_0) < \infty$, then the sequence $\{f^n x_0\}$ converges to some $\omega \in X$. Moreover, if $D(x_0, f\omega) < \infty$, then ω is a fixed point of f, and for any fixed point ω' of f such that $D(\omega, \omega') < \infty$, we have $\omega = \omega'$.

Proof Let $n \in \mathbb{N}$ ($n \geq 1$). For all integers i, j, we have

$$D(f^{n+i} x_0, f^{n+j} x_0) \leq \lambda\big[D(f^{n+i} x_0, f^{n+j-1} x_0) + D(f^{n+i-1} x_0, f^{n+j} x_0)\big],$$

which implies that

$$D(f^{n+i}x_0,f^{n+j}x_0) \leq 2\lambda\delta(D,f,f^{n-1}x_0).$$

Hence

$$\delta(D,f,f^nx_0) \leq 2\lambda\delta(D,f,f^{n-1}x_0),$$

and consequently

$$\delta(D,f,f^nx_0) \leq (2\lambda)^n\delta(D,f,x_0).$$

This inequality implies that

$$D(f^nx_0,f^mx_0) \leq \delta(D,f,f^nx_0) \leq (2\lambda)^n\delta(D,f,x_0)$$

for all integers n,m such that $m > n$. Since $\delta(D,f,x_0) < \infty$ and $2\lambda \in [0,1)$, we obtain

$$\lim_{m,n\to\infty} D(f^nx_0,f^mx_0) = 0.$$

It follows that $\{f^nx_0\}$ is a D-Cauchy sequence, and thus there exists $\omega \in X$ such that

$$\lim_{n\to\infty} D(f^nx_0,\omega) = 0.$$

By (D_3) we have

$$D(f^nx_0,\omega) \leq C\limsup_{m\to\infty} D(f^nx_0,f^mx_0) \leq (2\lambda)^nC\delta(D,f,x_0) \leq C\delta(D,f,x_0).$$

Then

$$D(f^nx_0,\omega) < \infty \quad \text{for all } n \in \mathbb{N}.$$

By (7) we have

$$D(f^{n+1}x_0,f\omega) \leq \lambda\bigl[D(f^{n+1}x_0,\omega) + D(f^nx_0,f\omega)\bigr].$$

Since $D(x_0,f\omega) < \infty$, we have $D(f^nx_0,f\omega) < \infty$ for all $n \in \mathbb{N}$. By Lemma 3.8 we obtain

$$\lim_{n\to\infty} D(f^nx_0,f\omega) = 0.$$

It follows that $f\omega = \omega$.

Let ω' be any fixed point of X. We have

$$\begin{aligned} D(\omega,\omega') &= D(f\omega,f\omega') \\ &\leq \lambda\bigl(D(f\omega,\omega') + D(f\omega',\omega)\bigr) \\ &\leq \lambda\bigl(D(\omega,\omega') + D(\omega',\omega)\bigr) \\ &\leq 2\lambda D(\omega,\omega'). \end{aligned}$$

Since $D(\omega,\omega') < \infty$, we obtain $D(\omega,\omega') = 0$, which ends the proof. $\square$

Example 3.10 Let $X = [0,1]$, and let $D : X \times X \to [0,\infty]$ be defined by

$$\begin{cases} D(x,1) = D(1,x) = \infty & \text{for all } x \in [0,1], \\ D(x,y) = x + y & \text{if } x \neq 1 \text{ and } y \neq 1. \end{cases}$$

It is easy to see that (X,D) is a D-complete generalized metric space with $C = 1$.

Consider the function $T : [0,1] \to [0,1]$ given by

$$T(x) = \frac{1}{2}x \quad \text{if } x \in [0,1[,$$
$$T(1) = 1.$$

The function T is a Chatterjea contraction with $\lambda = \frac{1}{3}$ in (X,D). By Theorem 3.9, T has a fixed point $\omega \in X$.

Note that the mapping T has two different fixed points, so we cannot apply the classical fixed point theorems for Banach, Kannan, and Chatterjea contractions since they give the uniqueness of the fixed point.

Definition 3.11 Let (X,D) be a generalized metric space. A self-mapping f on X is called a Hardy-Rogers contraction if there exists nonnegative real constants λ_i for $i = 1,2,3,4,5$ such that $\lambda = \sum_{i=1}^{i=5} \lambda_i \in [0,1[$ and

$$D(fx,fy) \leq \lambda_1 D(x,y) + \lambda_2 D(x,fx) + \lambda_3 D(y,fy) + \lambda_4 D(y,fx) + \lambda_5 D(x,fy) \tag{8}$$

for all $x,y \in X$.

Proposition 3.12 *Let (X,D) be a generalized metric space, and let $f : X \longrightarrow X$ be a Hardy-Rogers contraction. Then any fixed point $\omega \in X$ of f satisfies*

$$D(\omega,\omega) < \infty \quad \Rightarrow \quad D(\omega,\omega) = 0.$$

Proof Let $\omega \in X$ be a fixed point of f such that $D(\omega,\omega) < \infty$. We have

$$\begin{aligned} D(\omega,\omega) &= D(f\omega,f\omega) \\ &\leq \lambda_1 D(\omega,\omega) + \lambda_2 D(\omega,f\omega) + \lambda_3 D(\omega,f\omega) + \lambda_4 D(\omega,f\omega) + \lambda_5 D(\omega,f\omega) \\ &\leq \lambda D(\omega,\omega). \end{aligned}$$

Since $\lambda \in [0,1[$, we have $D(\omega,\omega) = 0$. $\square$

To prove a fixed point result for Hardy-Rogers contraction mappings, we need the following lemma.

Lemma 3.13 ([6]) *Let (a_n) be a sequence of nonnegative real numbers, and let (λ_n) be a real sequence in $[0,1]$ such that*

$$\sum_{n=0}^{\infty} \lambda_n = \infty.$$

If, for a given $\varepsilon > 0$, there exists a positive integer n_0 such that

$$a_{n+1} \le (1 - \lambda_n)a_n + \varepsilon\lambda_n \quad \text{for all } n \ge n_0,$$

then $0 \le \limsup_{n\to\infty} a_n \le \varepsilon$.

Theorem 3.14 *Let (X,D) be a D-complete generalized metric space, and let f be a self-mapping on X satisfying* (8).

Assume that $C\lambda_3 + \lambda_5 < 1$ and that there exists a point $x_0 \in X$ such that $\delta(D,f,x_0) < \infty$. Then the sequence $\{f^n x_0\}$ converges to some $\omega \in X$. If $D(x_0,f\omega) < \infty$, then ω is a fixed point of f. Moreover, if $\omega' \in X$ is another fixed point of f such that $D(\omega,\omega') < \infty$ and $D(\omega',\omega') < \infty$, then $\omega = \omega'$.

Proof Let $n \in \mathbb{N}$ $(n \ge 1)$. For all $i,j \in \mathbb{N}$, we have

$$D(f^{n+i}x_0,f^{n+j}x_0) \le \lambda\delta(D,f,f^{n-1}x_0).$$

We obtain

$$\delta(D,f,f^n x_0) \le \lambda\delta(D,f,f^{n-1}x_0)$$

and

$$\delta(D,f,f^n x_0) \le \lambda^n\delta(D,f,x_0),$$

which leads to

$$D(f^n x_0,f^m x_0) \le \delta(D,f,f^n x_0) \le \lambda^n\delta(D,f,x_0). \tag{9}$$

Since $\delta(D,f,x_0) < \infty$ and $\lambda \in [0,1)$, we obtain

$$\lim_{m,n\to\infty} D(f^n x_0,f^m x_0) = 0.$$

It follows that $\{f^n x_0\}$ is a D-Cauchy sequence, and thus there exists $\omega \in X$ such that

$$\lim_{n\to\infty} D(f^n x_0,\omega) = 0.$$

From (8) we have

$$\begin{aligned} D(f^{n+1}x_0,f\omega) \le{}& \lambda_1 D(f^n x_0,\omega) + \lambda_2 D(f^{n+1}x_0,f^n x_0) + \lambda_3 D(f\omega,\omega) \\ &+ \lambda_4 D(\omega,f^{n+1}x_0) + \lambda_5 D(f^n x_0,f\omega). \end{aligned} \tag{10}$$

Let

$$\begin{cases} a_n = D(f^n x_0, f\omega), \\ b_n = \lambda_1 D(f^n x_0, \omega) + \lambda_2 D(f^{n+1}x_0, f^n x_0) + \lambda_4 D(\omega, f^{n+1}x_0), \\ K = \lambda_3 D(f\omega, \omega). \end{cases} \tag{11}$$

By (10) we obtain

$$a_{n+1} \le \lambda_5 a_n + b_n + K.$$

Since $\lim_{n\to\infty} b_n = 0$, for every $\epsilon > 0$ such that $\epsilon > \frac{K}{1-\lambda_5}$, there exists N_ϵ such that

$$b_n \le \varepsilon(1-\lambda_5) - K \quad \text{for all } n \ge N_\varepsilon.$$

Then

$$a_{n+1} \le \lambda_5 a_n + b_n + K \le \lambda_5 a_n + \varepsilon(1-\lambda_5).$$

By Lemma 3.13 we have

$$0 \le \limsup_{n\to\infty} a_n \le \varepsilon \quad \text{for all } \varepsilon > \frac{K}{1-\lambda_5}.$$

Then

$$0 \le \limsup_{n\to\infty} a_n \le \frac{K}{1-\lambda_5}. \tag{12}$$

By (D_3) we obtain

$$D(f\omega, \omega) \le C \limsup_{n\to\infty} D\bigl(f\omega, f^{n+1}x_0\bigr) \le C \limsup_{n\to\infty} a_n \le C\frac{K}{1-\lambda_5}, \tag{13}$$

and by (11) we have

$$D(f\omega, \omega) \le \frac{C\lambda_3 D(f\omega, \omega)}{1-\lambda_5}.$$

Since $C\lambda_3 + \lambda_5 < 1$, we have $\frac{C\lambda_3}{1-\lambda_5} < 1$. Then $D(f\omega, \omega) = 0$, which implies $f\omega = \omega$.

If ω' is any fixed point of f such that $D(\omega, \omega') < \infty$ and $D(\omega', \omega') < \infty$, then (8) implies

$$\begin{aligned} D\bigl(\omega, \omega'\bigr) &= D\bigl(f\omega, f\omega'\bigr) \\ &\le \lambda_1 D\bigl(\omega, \omega'\bigr) + \lambda_2 D(\omega, f\omega) + \lambda_3 D\bigl(\omega', f\omega'\bigr) + \lambda_4 D\bigl(\omega', f\omega\bigr) + \lambda_5 D\bigl(\omega, f\omega'\bigr) \\ &\le \lambda_1 D\bigl(\omega, \omega'\bigr) + \lambda_2 D(\omega, \omega) + \lambda_3 D\bigl(\omega', \omega'\bigr) + \lambda_4 D\bigl(\omega', \omega\bigr) + \lambda_5 D\bigl(\omega, \omega'\bigr). \end{aligned}$$

Then we obtain

$$(1 - \lambda_1 - \lambda_4 - \lambda_5) D\bigl(\omega, \omega'\bigr) \le \lambda_2 D(\omega, \omega) + \lambda_3 D\bigl(\omega', \omega'\bigr).$$

From Proposition 3.12 we have $D(\omega,\omega) = D(\omega',\omega') = 0$, and then $D(\omega,\omega') = 0$, which ends the proof. □

By the symmetry of the generalized metric D the theorem is true if either $C\lambda_3 + \lambda_5 < 1$ or $C\lambda_2 + \lambda_4 < 1$.

4 Fixed point results for T-contractions

Beiranvand, Moradi, Omid, and Pazandeh [7] introduced the notion of a T-contraction and established a version of the Banach contraction principle.

Let us introduce the following definitions.

Definition 4.1 Let (X,D) be a metric space, and let T be a self-mapping on X. We say that

(a) T is continuous if

$$\lim_{n\to\infty} D(x_n,x) = 0 \quad \Rightarrow \quad \lim_{n\to\infty} D\bigl(T(x_n),T(x)\bigr) = 0$$

for all $x \in X$;

(b) T is sequentially convergent if for every sequence $\{x_n\}$ such that $\{T(x_n)\}$ converges, $\{x_n\}$ converges;

(c) T is subsequentially convergent if for every sequence $\{x_n\}$ such that $\{T(x_n)\}$ converges, $\{x_n\}$ has a convergent subsequence.

Definition 4.2 Let (X,D) be a metric space, and let T and f be two self-mappings on X. We say that

(a) f is a T-Banach contraction if there exists $k \in]0,1[$ such that

$$D\bigl(Tf(x),Tf(y)\bigr) \le kD(Tx,Ty)$$

for all $x,y \in X$;

(b) f is a T-Kannan contraction if there exists $k \in]0,\frac{1}{2}[$ such that

$$D\bigl(Tf(x),Tf(y)\bigr) \le k\bigl[D\bigl(Tx,Tf(x)\bigr) + D\bigl(Ty,Tf(y)\bigr)\bigr]$$

for all $x,y \in X$;

(c) f is a T-Chatterjea contraction if there exists $k \in]0,\frac{1}{2}[$ such that

$$D\bigl(Tf(x),Tf(y)\bigr) \le k\bigl[D\bigl(Tx,Tf(y)\bigr) + D\bigl(Ty,Tf(x)\bigr)\bigr]$$

for all $x,y \in X$;

(d) f is a T-Hardy-Rogers contraction if

$$D(Tfx,Tfy) \le \lambda_1 D(Tx,Ty) + \lambda_2 D(Tx,Tfx) + \lambda_3 D(Ty,Tfy) + \lambda_4 D(Ty,Tfx) + \lambda_5 D(Tx,Tfy)$$

for all $x,y \in X$, where λ_i, $i = 1,2,3,4,5$, are nonnegative constants such that $\lambda = \sum_{i=0}^{i=5} \lambda_i \in [0,1[$.

To show new results for T-contractions on a complete generalized metric space (X,D), we consider the mapping $D_T : X \times X \to [0,\infty]$ defined by

$$D_T(x,y) = D(Tx,Ty) \quad \text{for all } x,y \in X,$$

where T is continuous, sequentially convergent, and one-to-one.

Proposition 4.3 *We have*:

1. *For every sequence x_n in X,*

$$\lim_n D(x_n,x) = 0 \quad \Longleftrightarrow \quad \lim_n D_T(x_n,x) = 0;$$

2. *D_T is a generalized metric on X;*
3. *If (X,D) is complete, then (X,D_T) is complete.*

Proof 1. Let $\{x_n\}$ be a sequence such that

$$\lim_{n\to\infty} D_T(x_n,x) = 0.$$

By continuity we have

$$\lim_{n\to\infty} D(Tx_n,Tx) = 0.$$

Assume that $\lim_{n\to\infty} D(Tx_n,Tx) = 0$. Since T is sequentially convergent, there exists $u \in X$ such that

$$\lim_n D(x_n,u) = 0,$$

and by the continuity of T we have $\lim_n D(Tx_n,Tu) = 0$. It follows that $Tu = Tx$, and since T is one-to-one, we have $u = x$.

2. Let $x,y \in X$ be such that $D_T(x,y) = 0$. Then $D(Tx,Ty) = 0$. Since T is one-to-one, we obtain $x = y$ by (D_1).

The symmetry is obvious. Let now $x,y \in X$, and let $\{x_n\}$ be a sequence that converges to x in (X,D_T). Then $\{Tx_n\}$ converges to Tx in (X,D), and by (D_3) we have

$$D(Tx,Ty) \le C \limsup_n D(Tx_n,Ty),$$

which is equivalent to $D_T(x,y) \le C \limsup_n D_T(x_n,y)$. Hence (X,D_T) is a generalized metric space with the same constant C.

3. If $\lim_{n,m\to\infty} D_T(x_n,x_m) = 0$, then $\lim_{n,m\to\infty} D(Tx_n,Tx_m) = 0$. So $\{Tx_n\}$ is a Cauchy sequence in (X,D), which is complete. It follows that there exists $u \in X$ such that

$$\lim_{n\to\infty} D(Tx_n,u) = 0.$$

Since T is sequentially convergent, there exists $x \in X$ such that $\lim_n D(x_n,x) = 0$, which is equivalent to $\lim_n D_T(x_n,x) = 0$. □

Remark 4.4 If a mapping f is a T-Banach (resp. T-Kannan, T-Chatterjea, T-Hardy-Rogers) contraction in (X,D), then f is a Banach (resp. Kannan, Chatterjea, Hardy-Rogers) contraction in (X,D_T) with the same constants.

For every $x \in X$, we define

$$\delta_T(D,f,x) = \sup\bigl\{D\bigl(Tf^i x, Tf^j x\bigr) : i,j \in \mathbb{N}\bigr\}.$$

From Remark 4.4 and Theorem 3.3 in [1] we can deduce the following corollaries.

Corollary 4.5 *Let (X,D) be a complete metric space, and let $T,f : X \to X$ be two mappings such that T is continuous, one-to-one, and sequentially convergent. Assume that f is a T-Banach contraction. If there exists $x_0 \in X$ such that $\delta_T(D,f,x_0) < \infty$, then $\{f^n x_0\}$ converges to a fixed point ω of f. Moreover, if ω' is any fixed point of f such that $D(T\omega, T\omega') < \infty$, then $\omega = \omega'$.*

Corollary 4.6 *Let (X,D) be a complete generalized metric space, and let $T,f : X \to X$ be two mappings such that T is continuous, one-to-one, and sequentially convergent. Assume that f is a T-Kannan contraction with constant $k > 0$ such that $Ck < 1$. If there exists $x_0 \in X$ such that $\delta_T(D,f,x_0) < \infty$, then $\{f^n x_0\}$ converges to some $\omega \in X$. Moreover, if $D(Tx_0, Tf\omega) < \infty$, then ω is a fixed point of f, and for every fixed point ω' of f such that $D(T\omega, T\omega') < \infty$, we have $\omega = \omega'$.*

Corollary 4.7 *Let (X,D) be a complete generalized metric space, and let $T,f : X \to X$ be two mappings such that T is continuous, one-to-one, sequentially convergent. Assume that f is a T-Chatterjea contraction and that there exists $x_0 \in X$ such that $\delta_T(D,f,x_0) < \infty$. Then $\{f^n x_0\}$ converges to some $\omega \in X$, and if $D(Tx_0, Tf\omega) < \infty$, then ω is a fixed point of f. Moreover, if $\omega' \in X$ is another fixed point of f such that $D(T\omega, T\omega') < \infty$, then $\omega = \omega'$.*

Corollary 4.8 *Let (X,D) be a complete generalized metric space, and let $T,f : X \to X$ be two mappings such that T is continuous, one-to-one, and sequentially convergent. Assume that f is a T-Hardy-Rogers contraction with nonnegative constants $\lambda_i, i = 1,2,3,4,5$, such that $\lambda = \sum_{i=0}^{i=5} \lambda_i \in [0,1[$ and $C\lambda_3 + \lambda_5 < 1$. Assume that there exists $x_0 \in X$ such that $\delta_T(D,f,x_0) < \infty$. Then $\{f^n x_0\}$ converges to some $\omega \in X$. If $D(Tx_0, Tf\omega) < \infty$, then ω is a fixed point of f. Moreover, if $\omega' \in X$ is another fixed point of f such that $D(T\omega, T\omega') < \infty$ and $D(T\omega', T\omega') < \infty$, then $\omega = \omega'$.*

5 Some remarks on Senapati et al. results

In this section, we discuss some results by Senapati et al. [8] on extensions of Ciric and Wardowski-type fixed point theorems in generalized metric spaces. The authors introduced the following definition.

Definition 5.1 ([8], Definition 3.3) Let (X,D) be a generalized metric space, and let T be a self-mapping on X. Then T is said to be a D-admissible mapping if for all $x,y \in X$,

$$D(x,y) < \infty \quad \Longrightarrow \quad D(Tx, Ty) < \infty.$$

They proved the following lemma.

Lemma 5.2 ([8], Lemma 3.4) *Let (X,D) be a generalized metric space, and let T be a D-admissible mapping on X. Then, for every sequence $\{x_n\}$ converging to a point a point $w \in X$, we have $D(w,Tw) < \infty$.*

The proof of this lemma uses the following implication:

$$D(Tx_n,Tw) < \infty \quad (\forall n \geq n_0) \quad \Longrightarrow \quad \limsup_n D(Tx_n,Tw) < \infty.$$

Example 5.3 We consider Example 2.3 in [8] and the mapping $T : X \longrightarrow X$ defined by

$$Tx = \begin{cases} 1 & \text{if } x = 0, \\ \frac{1}{x} & \text{otherwise.} \end{cases}$$

For all $x,y \in X$, we have $Tx + Ty \leq D(Tx,Ty) \leq Tx + Ty + 1$. Then $D(Tx,Ty) < \infty$, which implies that T is admissible.

For $x_n = \frac{1}{n}$, we have $\lim_n D(x_n,0) = 0$ but $\limsup_n D(Tx_n,T0) = \infty$.

Example 5.4 Let $X = [0,+\infty[$, and let D be defined by

$$D(x,0) = D(0,x) = 0 \quad \text{and} \quad D(x,y) = \frac{1}{x} + \frac{1}{y} \quad \text{for } x \neq 0, y \neq 0.$$

Let T be the mapping defined by

$$\begin{cases} T0 = 1, \\ Tx = \frac{x}{x+1} \quad \text{for } x \neq 0. \end{cases}$$

Then T is admissible. If the sequence $(x_n)_n$ is defined by $x_1 = 1$ and $x_{n+1} = Tx_n$ for all $n \geq 2$, then we have

$$\lim_n D(x_n,0) = 0, \quad D(Tx_n,T0) = n+1 < \infty, \quad \text{and} \quad \limsup_n D(Tx_n,T0) = \infty.$$

The lemma was used to prove Theorems 3.5 and 3.7 of [8], and they deduced the following corollary:

Corollary 5.5 ([8], Corollary 3.8) *Let $T : X \longrightarrow X$ be a D-admissible self-mapping, and let (X,D) be a complete D-generalized metric space. Suppose the following conditions hold:*

(i) *for all $x,y \in X$, there exists $k \in]0,1[$ such that*

$$D(Tx,Ty) \leq k \max\left\{D(x,y), D(x,Tx), D(y,Ty), \frac{D(x,Ty) + D(y,Tx)}{2}\right\};$$

(ii) *there exists $x_0 \in X$ such that $\delta(D,T,x_0) < \infty$.*

Then $(T^n(x_0))_n$ converges to some $w \in X$, and this w is a fixed point of T. Moreover, if w' is another fixed point of T with $D(w,w') < \infty$ and $D(w',w') < \infty$, then $w = w'$.

The inequalities

$$\frac{D(x,Tx)}{1+D(x,Tx)} \leq D(x,Tx), \qquad \frac{D(x,Ty)+D(y,Tx)}{2} \leq \max\bigl(D(x,Ty),D(y,Tx)\bigr)$$

give the following implications:

$$T \text{ satisfies (i)} \quad \Longrightarrow \quad T \text{ is a } k\text{-quasi-contraction}$$

and

$$T \text{ satisfies rational inequality} \quad \Longrightarrow \quad T \text{ satisfies (i).}$$

These implications show that in [8], Theorem 3.7 is a consequence of Corollary 3.8.

Finally, in the proof of Theorem 4.2 in [8], the authors use the F-contraction defined as follows.

Definition 5.6 ([8], Definition 4.1) A self-mapping T defined on X is said to be an F-contraction if, for all $x, y \in X$,

$$D(x,y) > 0 \quad \text{and} \quad D(Tx,Ty) > 0 \quad \Longrightarrow \quad \tau + F\bigl(D(Tx,Ty)\bigr) \leq F(D(x,y)$$

for some $\tau > 0$.

We suspect that that existence of some $x_0 \in X$ such that $\delta(D,T,x_0) = c$ does not give $D(T^{n+i}x_0, T^{n+j}x_0) > 0$ for all integers n, i, j as the proofs of Theorem 4.2 and Corollary 4.3 of [8] blame.

6 Conclusion

In this paper, we gave a generalized version of Kannan, Chatterjea, and Hardy-Rogers contraction fixed point theorems and some fixed point results for T-contractions in a generalized metric space.

Our examples show that the results can be applied to prove the existence of fixed points in generalized metric spaces, whereas its classical counterpart fails to give positive answers.

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