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Fixed points of set-valued maps in locally complete spaces

Carlos Bosch[1*], César L García[1], Thomas Gilsdorf[2], Claudia Gómez-Wulschner[1] and Rigoberto Vera[1]

*Correspondence: bosch@itam.mx
[1]Departamento de Matemáticas, ITAM, Río Hondo 1, Ciudad de México, 01080, México
Full list of author information is available at the end of the article

Abstract

We prove an extension of the Pareto optimization criterion to locally complete locally convex vector spaces to guarantee the existence of fixed points of set-valued maps.

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Keywords: fixed points; set-valued map; Pareto optimization

1 Introduction

Ekeland's variational principle, a milestone in the theory of nonlinear optimization, focuses on solving an optimization problem via a perturbed optimization problem. Since its appearance many extensions and equivalent formulations have been shown. Some of them, related to our discussion below, are contained in [1–8]. In [9] Azé and Corvellec generalized, in the setting of metric spaces, a result due to Lim ([10]) on the existence of fixed points for weakly inward multivalued contractions, defined on a nonempty closed subset of a Banach space. Their argument is remarkably simple: on the one hand, it avoids the use of transfinite induction (as in Lim's paper) and, on the other hand, it uses Ekeland's variational principle as a main tool to guarantee the existence of fixed points. Ekeland's principle has been shown equivalent to other optimization statements, in particular, and of interest in this paper, is the equivalence to an optimization criterion of Pareto (see [11]) for which the existence of optima (critical points for dynamical systems) has been shown in complete metric spaces (see [12]). Our main result (Theorem 3.2) is a modification of that optimization criterion of Pareto, which shows how the existence of fixed points for set-valued maps can be extended to the setting of locally complete spaces. Among the consequences of Theorem 3.2, we provide a simple argument of Azé and Corvellec's theorem ([9], Theorem 2.3). Although our results are set in the context of locally complete spaces, let us say that related to Ekeland's type variational principles, some other forms of completeness have been use in the literature (*e.g.*, quasi-metric spaces with Q-functions, fuzzy metric spaces, or sequentially lower complete spaces). Some of these include equivalences to Ekeland's variational principle, for instance, to Caristi fixed point theorem or Takahashi minimization theorem, all of them with their natural connection to solution to equilibrium or fixed point theorems for set-valued maps (see, for instance [11, 13–19]).

2 Preliminaries

Throughout this paper (X,τ) will denote a Hausdorff locally convex space where the topology τ is generated by a saturated family of seminorms $\{\rho_j\}_{j\in J}$. If B is a subset of X which is balanced and convex, we will call B a *disk*. Let X_B be the linear span of B, endowed with the topology generated by the Minkowski gauge of B, ρ_B. When B is bounded ρ_B is a norm, and the norm topology is finer than the topology inherited from X. If (X_B,ρ_B) is a Banach space we say that B is a *Banach disk*. We say that X is a *locally complete* space if each closed, bounded disk is a Banach disk. Local complete spaces are also known as c^∞ or *convenient spaces*. The notion of local completeness has become important in all sorts of applications, for instance, in nonlinear distribution theory (see [1, 20] and the references therein), or as a context where existence and uniqueness for nonlinear integro-differential equations can be shown (see [21]). By *lsc* we refer to a lower semicontinuous functions, $f: X\to\mathbb{R}\cup\{\infty\}$, which are *proper*, that is, their effective domain, $\operatorname{dom}(f):=\{x: f(x)<\infty\}$, is nonempty.

For a closed subset $A\subset X$ we will consider set-valued maps $T: A\to 2^A$. Such maps T are known in the literature as *dynamical systems* and, in our discussion, points of interest are $x^*\in A$ such that $x^*\in Tx^*$ (known as *fixed points* of T) and, especially, *critical points* of T (points $x^*\in A$ such that $Tx^*=\{x^*\}$). Let us start with the definition and properties of some functions that will be used in the sequel. For each $j\in J$, $\phi_{1j},\phi_{2j}: X\to[0,\infty)$ will be the functions defined by

$$\phi_{1j}(x)=\rho_j\text{-diam}(Tx)=\sup\bigl\{\rho_j\bigl(y-y'\bigr): y,y'\in Tx\bigr\} \tag{1}$$

and

$$\phi_{2j}(x)=\rho_j\text{-dist}(x,Tx)=\inf\bigl\{\rho_j(x-y): y\in Tx\bigr\}. \tag{2}$$

Also, for each $j\in J$, and for every pair of subsets $A,B\subset X$, we define

$$e_j(A,B)=\sup\bigl\{\rho_j(x-B): x\in A\bigr\}, \tag{3}$$

where $\rho_j(x-B)=\inf\{\rho_j(x-b): b\in B\}$, is the ρ_j-distance from x to B. In general, we define

$$\rho_j(A-B)=\inf\bigl\{\rho_j(a-B): a\in A\bigr\}=\inf\bigl\{\rho_j(b-A): b\in B\bigr\}=\rho_j(B-A),$$

where $\rho_j(a-B)$ denotes the $\inf\{\rho_j(a-b): b\in B\}$ and $\rho_j(b-A)=\inf\{\rho_j(b-a): a\in A\}$.

It is important to note that, in general, $e_j(Tx,Tz)\neq e_j(Tz,Tx)$. For instance, take $X=\mathbb{R}^2$ equipped with the usual Euclidean norm d and consider the subsets $A=\{(a,2): -1\le a\le 1\}$ and $B=\{(x,y): x^2+y^2\le 1\}$. An easy computation shows that $e_d(A,B)=\sqrt{5}-1$ while $e_d(B,A)=3$.

Now we provide sufficient conditions on the functions e_j to get lower semicontinuity and continuity properties for the functions ϕ_{1j} and ϕ_{2j}. We start with the following.

Proposition 2.1 *If $M_j\in\mathbb{R}^+$ is such that $e_j(Tx,Tz)\le M_j\rho_j(x-z)$ for every $x,z\in X$ then $\phi_{1j}(x)\le\phi_{1j}(z)+2M_j\rho_j(x-z)$. Hence, ϕ_{1j} is ρ_j-continuous. In particular since τ is a uniformity we see that ϕ_{1j} is τ-uniformly continuous.*

Proof Let $x, z \in X$ and $y, y' \in Tx$. Then, for each $n \in \mathbb{N}$ choose $w_n \in Tz$ such that

$$\rho_j(y - Tz) < \rho_j(y - w_n) < \rho_j(y - Tz) + \frac{1}{n},$$

and similarly for y' choose the corresponding $w'_n \in Tz$. Then

$$\begin{aligned} \rho_j(y - y') &\leq \rho_j(y - w_n) + \rho_j(w_n - w'_n) + \rho_j(w'_n - y') \\ &< \rho_j(y - Tz) + \rho_j(w_n - w'_n) + \rho_j(y' - Tz) + \frac{2}{n} \\ &\leq \phi_{1j}(z) + 2e_j(Tx, Tz) + \frac{2}{n}. \end{aligned}$$

Thus, $\rho_j(y - y') \leq \phi_{1j}(z) + 2e_j(Tx, Tz)$. By taking the supremum over all y, y', we obtain

$$\phi_{1j}(x) \leq \phi_{1j}(z) + 2e_j(Tx, Tz) \leq \phi_{1j}(z) + 2M_j\rho_j(x - z).$$

A similar argument shows that

$$\phi_{1j}(z) - \phi_{1j}(x) \leq 2M_j\rho_j(z - x)$$

also holds. Hence ϕ_{1j} is ρ_j-uniformly continuous. □

For the function ϕ_{2j}, the result corresponding to Proposition 2.1 is as follows.

Proposition 2.2 *If $M_j \in \mathbb{R}^+$ is such that $e_j(Tx, Tz) \leq M_j\rho_j(x - z)$ for all $x, z \in X$ then ϕ_{2j} is ρ_j-continuous hence τ-continuous.*

Proof Let $x \in X$ and $\{x_\lambda\}$ be a net such that $x_\lambda \xrightarrow{\rho_j} x$. For $z \in Tx_\lambda$ we have

$$\begin{aligned} \phi_{2j}(x) &= \rho_j(x - Tx) \leq \rho_j(x - x_\lambda) + \rho_j(x_\lambda - Tx) \\ &\leq \rho_j(x - x_\lambda) + \rho_j(x_\lambda - z) + e_j(Tx_\lambda, Tx) \\ &\leq (1 + M_j)\rho_j(x - x_\lambda) + \rho_j(x_\lambda - z). \end{aligned}$$

By taking the infimum with respect to $z \in Tx_\lambda$ we get

$$\phi_{2j}(x) \leq (1 + M_j)\rho_j(x - x_\lambda) + \rho_j(x_\lambda - Tx_\lambda) = (1 + M_j)\rho_j(x - x_\lambda) + \phi_{2j}(x_\lambda).$$

Thus

$$\phi_{2j}(x) - \phi_{2j}(x_\lambda) \leq (1 + M_j)\rho_j(x_\lambda - x).$$

Similarly,

$$\phi_{2j}(x_\lambda) - \phi_{2j}(x) \leq (1 + M_j)\rho_j(x_\lambda - x),$$

and ϕ_{2j} is ρ_j-continuous. □

Definition 2.3 For $j \in J$, $\phi_j \colon X \to [0,\infty)$ will be the function defined by $\phi_j(x) = \phi_{1j}(x) + \phi_{2j}(x)$.

Lemma 2.4 *Let $A \subset X$ be any closed subset, and suppose that ϕ_{1j} and ϕ_{2j} are ρ_j-lsc. Suppose that $T \colon A \to 2^A$ is such that Tx is ρ_j-sequentially compact and for each $x \in A$ and each $n \in \mathbb{N}$ there exists $y_n \in Tx$ with the following properties: $\phi_j(y_n) \leq \phi_{1j}(x) + \frac{1}{n}$ and $\rho_j(x - y_n) < \phi_{2j}(x) + \frac{1}{n}$. Then there exists $y \in Tx$ such that $\phi_j(y) + \rho_j(x - y) \leq \phi_j(x)$.*

Proof Let $\{y_n\} \subset Tx$ be a sequence such that each y_n satisfies the conditions as in the statement of Lemma 2.4. Since Tx is ρ_j-sequentially compact, there is a subsequence $\{y_{n_k}\}_{k\in\mathbb{N}}$, ρ_j-convergent to some, not necessarily unique, $y \in Tx$. Then

$$\phi_j(y_{n_k}) + \rho_j(x - y_{n_k}) \leq \phi_{1j}(x) + \frac{1}{n_k} + \phi_{2j}(x) + \frac{1}{n_k} = \phi_j(x) + \frac{2}{n_k}.$$

By taking $\liminf$ over k on both sides of the inequality we have

$$\phi_j(y) + \rho_j(x - y) \leq \phi_j(x)$$

as desired. □

To close this section we define the sets that we will consider as target values for the dynamical systems in Theorem 3.2. Note that Lemma 2.4 is tailored to provide conditions for these sets to be nonempty.

Definition 2.5 For each $x \in X$ and each index $j \in J$ let

$$C_x^j = \bigl\{ y \in Tx : \phi_j(y) + \rho_j(x - y) \leq \phi_j(x) \bigr\}.$$

Lemma 2.6 *If ϕ_j is lsc then the set C_x^j is ρ_j-closed.*

Proof Let (y_n) be a sequence in C_x^j such that $y_n \xrightarrow{\rho_j} y \in X$.

Note that $\phi_j(y_n) + \rho_j(x - y_n) \leq \phi_j(x)$ implies that

$$\phi_j(y) + \rho_j(x - y) \leq \liminf \phi_j(y_n) + \lim \rho_j(x - y_n) \leq \phi_j(x).$$

Hence $y \in C_x^j$. □

Corollary 2.7 *If ϕ_j is lsc and for each $y \in C_x^j$ and $y + Z_{\rho_j} \subset C_x^j$, where $Z_{\rho_j} = \{x \in X : \rho_j(x) = 0\}$ is the zero set of ρ_j, then C_x^j is a ρ_j-closed linear subspace of X.*

Definition 2.8 Let $A \subset X$ be any closed subset. If $T \colon A \to 2^A$ is a dynamical system and ρ_j is one of the seminorms defining the topology τ, $x^* \in A$ is a ρ_j-critical point of T if $Tx^* \subset x^* + Z_{\rho_j}$.

Note that when ρ_j is a norm, to be a ρ_j-critical point means that $Tx^* = \{x^*\}$.

3 Modified Pareto case

Recall that the topology τ in the lcs X is generated by a saturated family of seminorms $\{\rho_j : j \in J\}$. Theorem 1 of [11] provides with sufficient conditions for a given dynamical system $T\colon A \to 2^A$, over a locally complete subset A of X, to have critical points. In [11], Theorem 1, one essentially sees that if for every $x \in A$ and every $u \in Tx$, $c_j\rho_j(x-u) \leq \Phi(x) - \Phi(u)$ for every $j \in J$ and for some function $\Phi\colon A \to \mathbb{R}$ lsc and bounded below (c_j are positive scalars such that $\bigcap_{j\in J}\{x \in X : c_j\rho_j(x) \leq 1\}$ is a nonzero Banach disk) then T has a critical point. In our setting, if we suppose that, for a fixed $j \in J$, A is ρ_j-complete and $\Phi\colon A \to \mathbb{R}$ is ρ_j-lsc, that is, $\rho_j(x_n - x) \to 0$ implies $\Phi(x) \leq \liminf_n \Phi(x_n)$ (which in turn implies τ-lsc of Φ) then we have the following proposition.

Proposition 3.1 *If for each $x \in A$ and each $u \in Tx$ we have $\rho_j(x-u) \leq \Phi(x) - \Phi(u)$, then T has a ρ_j-critical point, $x_j^* \in A$.*

The proof goes along the same lines as that of Theorem 1 in [11] and we omit it.

Theorem 3.2 (Main Result) *Let $T\colon A \to 2^A$ be a dynamical system and for each $j \in J$ let $T_j\colon A \to 2^A$ be the dynamical system defined by $T_j(x) = C_x^j$. Suppose that the following conditions are satisfied*:

1. *The seminorm ρ_j, from the family of seminorms defining the topology τ, is such that the subset A is ρ_j-sequentially complete.*
2. *The function ϕ_j (see definition* 2.3) *satisfies the condition $\phi_j(x) \leq \liminf \phi_j(x_n)$ whenever $\rho_j(x - x_n) \to 0$.*
3. *$C_x^j \neq \emptyset$ for all $x \in A$.*

Then there exists $x_j^* \in A$ such that $T_jx_j^* \subset x_j^* + Z_{\rho_j}$, that is, x_j^* is a ρ_j-critical point of the function T_j.

Proof The hypotheses of this theorem are essentially the same as those of Proposition 3.1, with $T = T_j$. The fact that the function ϕ_j is ρ_j-lsc implies that the set $T_jx = C_x^j$ is ρ_j-closed for each vector $x \in A$. Then, for all $y \in T_jx = C_x^j$,

$$y + Z_{\rho_j} = \overline{\{y\}}^{\rho_j} \subset T_jx.$$

On the other hand, the fact that A is ρ_j-sequentially complete implies that the set $T_jx = C_x^j$ is also ρ_j-sequentially complete. Hence, by Proposition 3.1, we see that there exists $x_j^* \in A$ such that

$$T_jx_j^* \subset x_j^* + Z_{\rho_j}$$

as desired. □

Corollary 3.3 *If $x_j^* \in T_jx_j^*$ then $T_jx_j^* = x_j^* + Z_{\rho_j}$.*

A weak version (in the sense of weak topology) of Theorem 3.2 can be obtained as follows. Consider $X' = (X, \tau)'$, the topological dual of X, and the saturated family of seminorms $\{\rho_f : f \in X'\}$, where $\rho_f(x) = |f(x)|$ and $Z_f = \ker(f)$. Assume as before that $A \subset X$ is ρ_f-complete for some fixed, but arbitrary, $f \in X'$, and let $\Phi\colon A \to \mathbb{R}$ be a lsc function. Then we have the following.

Proposition 3.4 *If each $x \in A$ and each $u \in Tx$ satisfies $|f(x-u)| \leq \Phi(x) - \Phi(u)$, then the dynamical system T has a ρ_f-critical point $x_f^* \in A$, that is,*

$$Tx_f^* \subset x_f^* + Z_{\rho_f} = x_f^* + Z_f.$$

We also have the following restatement of Theorem 3.2, where the continuous linear functions $f \in X'$ take the place of the indices $j \in J$.

Theorem 3.5 *For each $f \in X'$, let $T_f' \colon A \to 2^A$ denote the function given by $T_f'(x) = C_x^f$. Suppose there exists a seminorm, $\rho_f(x) = |f(x)|$ with $f \in X'$, such that:*
1. *The function $\phi_f = |f|$ satifies $\phi_f(x) \leq \liminf \phi_f(x_n) = \liminf_n |f(x_n)|$ (lower semicontinuity).*
2. *$C_x^f \neq \emptyset$ for all $x \in A$.*

Then there exists $x_f^* \in A$ such that $T_f' x_j^* \subset x_f^* + Z_f$.

Proof Just notice that if we consider the weak topology in place of τ on X, then we have the same hypotheses as in Theorem 3.2 above. □

4 Applications to fixed point theory

With the tools we have developed so far we can present a couple of interesting applications to fixed point theory. Let (X, τ) be a complete locally convex topological vector space and take $f \in X'$, $A \subset X$ a τ-closed subset, and $0 < M < 1$. Let $h \colon A \to A$ be any function such that

$$\left|f\bigl(h(x)\bigr) - f\bigl(h(z)\bigr)\right| \leq M\left|f(x) - f(z)\right| \quad \text{for all } x, z \in A. \tag{4}$$

Theorem 4.1 *Under the above assumptions there exists $x^* \in A$ such that, for all $k \in \mathbb{N}$,*

$$f\bigl(x^*\bigr) = f\bigl(h\bigl(x^*\bigr)\bigr) = f\bigl(h^k\bigl(x^*\bigr)\bigr). \tag{5}$$

Equivalently, for all $k \in \mathbb{N}$, $\rho_f(x^* - h^k(x^*)) = 0$. Moreover, if $x^{**} \in A$ also satisfies (5) *then $f(x^*) = f(x^{**})$.*

First note that Theorem 4.1 fails for $M = 1$. Indeed, in [22] Khamsi provides an example [22], Example 1, to answer Kirk's problem (in the negative) on the existence of fixed points for a map $T \colon X \to X$ such that, for all $x \in X$ and for some positive function η, $\eta(d(x, Tx)) \leq \phi(x) - T(\phi(x))$ (ϕ is a non-negative lsc function). Here we see that for $M = 1$ and $A = \{x_1, x_2, \dots\} \subset X = \mathbb{R}$ with $x_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$; if we take $f = I \in (\mathbb{R})'$ we have the inequality

$$\left|h(x_n) - h(x_m)\right| \leq |x_n - x_m| \quad \text{for all } n, m \in \mathbb{N}$$

and then the function $h(x_n) = x_{n+1}$ does not have a fixed point. Note that the set A is closed but it is not bounded.

Also note that if $A = X$ then as a consequence of inequality (4) we see that $\hat{h} \colon X/Z_f \to X/Z_f$, such that $\hat{h}([x]) = [h(x)]$ is a function. Furthermore, if $A = X$ and the identities in (5) hold, then the function $\hat{h}$ has a unique fixed point.

Proof of Theorem 4.1. Let $T\colon A \to 2^A$ defined by $Tx = \{x, h(x), h^2(x), \ldots\}$. Then we have

$$\left|f\left(h(x) - h(z)\right)\right| = \left|f\left(h(x)\right) - f\left(h(z)\right)\right| \leq M\left|f(x) - f(z)\right| = M\left|f(x - z)\right|$$

for every $x, z \in A$.

Consider $e_f\colon \{Tx\colon x \in X\} \times \{Tz\colon z \in X\} \to [0, \infty)$ given by

$$e_f(Tx, Tz) = \sup\left\{\left|f(y - Tz)\right|\colon y \in Tz\right\} = \sup\left\{\rho_f(y - Tz)\colon y \in Tx\right\}.$$

Observe that since $Tx \supset Th(x) \supset Th^2(x) \supset \cdots$ we have $e_f(Th^{k+m}(x), Th^k(x)) = 0$ for all $k, m \in \mathbb{N}$. Now Tx is not necessarily closed or bounded, however, if $k \leq m$,

$$\left|f\left(h^k(x)\right) - f\left(h^m(z)\right)\right| \leq M^k\left|f(x) - f\left(h^{m-k}(z)\right)\right|.$$

Thus,

$$\begin{aligned}
\inf_{m\geq k}\left\{\left|f\left(h^k x\right) - f\left(h^m z\right)\right|\right\} &\leq \inf_{m\geq k}\left\{M^k\left|f(x) - f\left(h^{m-k} z\right)\right|\right\} \\
&\leq M^k\left|f(x) - f(z)\right| \\
&= M^k\left|f(x - z)\right|.
\end{aligned}$$

As a consequence we see that the function e_f satisfies $e_f(Tx, Tz) \leq M|f(x - z)|$ for all $x, z \in A$.

Note that for the given function h and the seminorm ρ_f we can use Proposition 2.1 to see that the function $\phi_{1f}(x) = \operatorname{diam}_f Tx$ is Lipschitz and thus ρ_f-uniformly continuous. Also, since $x \in Tx$, $\phi_{2f}(x) = 0$ and $\phi_f(x) = \phi_{1f}(x)$. Therefore ϕ_f is ρ_f uniformly continuous and satisfies condition (2) in Theorem 3.2. Observe that since $x \in Tx$ we have trivially from Lemma 2.4 that $\phi_f(x) + \rho_f(x - x) = \phi_f(x)$. Also $x \in C_x^f = \{y \in Tx \mid \phi_f(y) + \rho_f(x - y) \leq \phi_f(x)\}$, that is, $C_x^f \neq \emptyset$, which is condition (3) in Theorem 3.2.

By Lemma 2.6, and since ρ_f is continuous, we see that C_x^f is ρ_f-closed, thus ρ_f-sequentially complete since X is ρ_f-sequentially complete. By Theorem 3.2 we conclude that, for the map $T_f\colon A \to 2^A$, defined via $T_f(x) = C_x^f$, there exists $x^* \in A$ such that $T_f(x^*) \subset x^* + Z_f$.

Now, since $x^* \in T_f(x^*)$, by Corollary 3.3 we see that $T_f(x^*) = x^* + Z_f$. Hence, for each $y \in Tx^*$ there exists $z \in Z_f$ $(f(z) = 0)$ such that $y = x^* + z$. From which we obtain $\operatorname{diam}_f Tx^* = 0$. That is, $f(x^*) = f(h^k(x^*))$ for all $k \in \mathbb{N}$. Equivalently $\rho_f(x^* - h^k(x^*)) = 0$ for all $k \in \mathbb{N}$.

As for the uniqueness, if $x^{**} \in X$ is such that $f(x^{**}) = f(h^k(x^{**}))$ for all $k \in \mathbb{N}$, we would have

$$\left|f\left(x^* - x^{**}\right)\right| = \left|f\left(h\left(x^*\right)\right) - f\left(h\left(x^{**}\right)\right)\right| \leq M\left|f\left(x^*\right) - f\left(x^{**}\right)\right| = M\left|f\left(x^* - x^{**}\right)\right|.$$

Thus $1 \leq M$, which contradicts our hypothesis. This concludes Theorem 4.1. □

There is an interesting connection with the condition of metrical inwardness and the corresponding fixed point theorem in Caristi [23], p.247. Metrical inwardness hold in our setting if we take a fixed function $f \in X'$ such that for each $x \in A$ there exists $u \in A$ such that

$f(x-u)$ and $f(u-hx)$ are both positive numbers (or both negative) where h is a function satisfying the conditions of Theorem 4.1.

All we did in the previous paragraphs can be repeated if we change the weak topology for a generic Hausdorff topology for a lcs (X,τ). Indeed, if we use a fixed seminorm ρ_j among those that generate the topology τ, then we have the following result corresponding to Proposition 4.1.

Proposition 4.2 *Let (X,τ) be a complete locally convex topological vector space, take ρ_j a seminorm, from those that generate the topology τ. Let $A \subset X$ a closed subset and let $0 < M < 1$. Take $h\colon A \to A$ defined as a function not necessarily linear or continuous and such that for ρ_j we have*

$$\rho_j\bigl(h(x)-h(z)\bigr) \leq M\rho_j(x-z) \quad \text{for all } x,z \in A. \tag{6}$$

Then there exists $x^* \in A$ such that

$$x^* - h^k\bigl(x^*\bigr) \in Z_{\rho_j} \quad \text{for all } k \in \mathbb{N}. \tag{7}$$

Equivalently

$$\rho_j\bigl(x^* - h\bigl(x^*\bigr)\bigr) = \rho_j\bigl(x^* - h^k\bigl(x^*\bigr)\bigr) = 0 \quad \text{for all } k \in \mathbb{N}.$$

If $x^{**} \in A$ also satisfies (7) we have, $\rho_j(x^* - x^{**}) = 0$.

A consequence of (6) for $A = X$ is that the relation $\hat{h}\colon X/Z_{\rho_j} \to X/Z_{\rho_j}$ defined by $\hat{h}([x]) = [h(x)]$ is a function. A consequence of (7) for $A = X$, is that the function $\hat{h}$ has a unique fixed point.

Proof of Proposition 4.2 To prove Proposition 4.2 let us define the function $T\colon X \to 2^X$ via $Tx = \{x, h(x), h^2(x), \ldots\}$.

Then we have $\rho_j(h(x) - h(z)) \leq M\rho_j(x-z)$ for all $x,z \in A$. Note that $Tx \supset Th(x) \supset Th^2(x) \supset \cdots$, which means that $e_f(Th^{k+m}(x), Th^k(x)) = 0$ for all $k,m \in \mathbb{N}$. Recall that Tx is not necessarily closed but it is bounded. Furthermore, if $k \leq m$,

$$\rho_j\bigl(h^k(x) - h^m(z)\bigr) \leq M^k\rho_j\bigl(x - h^{m-k}(z)\bigr),$$

and then

$$\inf_{m\geq k}\bigl\{\rho_j\bigl(h^k x - h^m z\bigr)\bigr\} \leq \inf_{m\geq k}\bigl\{M^k\rho_j\bigl(x - h^{m-k}z\bigr)\bigr\} \leq M^k\rho_j(x-z).$$

The trick now is to follow the argument in Theorem 4.1 but replacing the seminorm ρ_f by the seminorm ρ_j in order to get $x^* \in X$ such that, by Theorems 2.4, 3.2, and Corollary 3.3, $\overline{Tx^*}^{\rho_j} = x^* + Z_{\rho_j}$. Finally, since $\hat{0} \in Z_{\rho_j}$ we obtain $x^* \in Tx^*$. Uniqueness is proved in the same way as in Theorem 4.1. □

Note that in Theorem 4.1 if we had $x^* \in Tx^*$ and $x^{**} \in Tx^{**}$, that is, $x^* = h^k(x^*)$ and $x^{**} = h^j(x^{**})$, where $k,j \in \mathbb{N}$ are the minimum values that satisfy these equalities, then we would have, for each $n \in \mathbb{N}$, $x^* = h^{nk}(x^*)$, and $x^{**} = h^{nj}(x^{**})$. Thus, for $j \leq k$,

$$\left|f\left(x^* - x^{**}\right)\right| = \left|f\left(h^{nk}\left(x^*\right) - h^{nj}\left(x^{**}\right)\right| \leq M^{nj}\left|f\left(h^{nk-nj}\left(x^*\right) - x^{**}\right)\right|.$$

Since $\{h^{n(k-j)}(x^*)|n \in \mathbb{N}\}$ is bounded, since it is finite with at most k points, and $0 < M < 1$, the sequence $\{M^{nj}|f(h^{n(k-j)}(x^*) - x^{**})|\}$ converges to 0 as $n \to \infty$; in other words, $|f(x^* - x^{**})| = 0$ hence $|f(h((x^*)) - h(x^{**}))| = 0$. We conclude that $\rho_f(x^* - x^{**}) = 0 = \rho_f(h(x^*) - h(x^{**}))$. In general, $|f(h^{nj}(x^*) - h^{nj}(x^{**}))| = 0 = |f(h^{nk}(x^*) - h^{nk}(x^{**}))|$.

As another application of our results, we show how the Azé and Corvellec theorem ([9], Theorem 2.3) follows easily from Theorem 3.5. Azé and Corvellec defined the set of fixed points for T as $F_T = \{x \in X : x \in Tx\}$. For the seminorms ρ_j the set of fixed points of T is described as $F_T^j = \{x \in X \mid \rho_j(x - Tx) = 0\}$ or in the case of ρ_f, the set of fixed point for T is $F_T^f = \{x \in X \mid |f(x - Tx)| = 0\}$. Note that this sets are ρ_j-closed (respectively, ρ_f-closed). In [9], Azé and Corvellec proved that for all $x \in X$, $\rho_j(x - F_T^j) \leq \rho_j(x - Tx)$ ($|f(x - F_T^f)| \leq |f(x - Tx)|$). We discuss the case for ρ_f (the case for ρ_j is analogous). Let

$$A_f = \left\{x \in X \mid \left|f\left(x - F_T^f\right)\right| > \left|f(x - Tx)\right| = \phi_{2f}(x)\right\}.$$

We can prove, under the conditions of Propositions 2.1 and 2.2, which we assume, that $A_f = \emptyset$. Indeed, define the function $\psi_f : (X, d) \to \mathbb{R}$ by

$$\psi_f(x) = \left|f\left(x - F_T^f\right)\right| - \left|f(x - Tx)\right| = \left|f\left(x - F_T^f\right)\right| - \phi_{2f}(x).$$

Then we have

$$A_f = \left\{x \in X : \psi_f(x) > 0\right\} = \psi_f^{-1}(0, \infty).$$

Thus if the function ψ_f is lsc then A_f is an open set, and $A_f \cap F_T^f = \emptyset$. Consider the function $\gamma_f(x) = |f(x - F_T^f)| = \inf\{|f(x - z)| : z \in F_T^f\}$, which is always continuous. Then, under the conditions of Propositions 2.1 and 2.2, the function ϕ_{2f} is continuous. Thus, we see that the function $\psi_f(x) = \gamma_f(x) - \phi_{2f}(x)$ is continuous, giving us that A_f is open. If $A_f \neq \emptyset$, take $x \in A_f$ and $r > 0$ such that

$$D_r^f(x) = \left\{z \in X : \left|f(z - x)\right| \leq r\right\} \subset A_f.$$

Note that $D_r^f(x)$ is ρ_f-closed and ρ_f-bounded.

If we define $T'' : D_r^f(x) \to 2^X$ to be $T''x = T_f'x$, $T'' = T_f'|_{D_r^f(x)}$, then T_f' satisfies the hypotheses of Theorem 3.5. Thus, there exists $z^* \in D_r^f(x) \subset A_f$ such that $z^* \in T''z^*$, in other words, $z^* \in A_f \cap F_T^f = \emptyset$ and the set A_f must be empty.

As as additional consequence (and easy example) of Proposition 3.1 one can get well known results such as the contraction mapping theorem. Indeed, if (X, ρ) is a normed space and A is a nonempty compact subset of X then any contraction $f : A \to A$ has a fixed point. This follows from Proposition 3.1 by letting $T : A \to 2^A$ as $Tx = \{f(x)\}$ and

$$\Phi(x) = \sum_{k=0}^{\infty} \rho\left(f^k(x) - f^{k+1}(x)\right) = \frac{1}{1-c}\rho\left(x - f(x)\right)$$

where $0 < c < 1$ is the contraction constant for f. It is easy to show that Φ is lsc (in fact continuous) and that $\rho(x - f(x)) = \Phi(x) - \Phi(f(x))$. Hence T must have critical points which are, by definition of T, fixed points for f.

5 Conclusions

The paper can be considered as an extension of the Pareto optimization criterion to locally complete locally convex vector spaces [11] with some applications to fixed points. Local completeness is a very weak completeness property. This type of spaces is becoming the convenient setting for several applications. Here in this setting we get a fixed point theorem and as an application we obtain the results of Azé and Corvellec's [9].

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Author details
[1]Departamento de Matemáticas, ITAM, Río Hondo 1, Ciudad de México, 01080, México. [2]Department of Mathematics, Central Michigan University, Mt Pleasant, MI 48858, USA.

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