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Pseudo-metric space and fixed point theorem

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Abstract

The aim of this paper is to give a generalized version of Caristi fixed point theorems in pseudo-metric spaces. Our results generalize and improve many of well-known theorems. As an application of our results, we give a new existence theorem to the generalized nonlinear complementarity problem and a solution of differential inclusion in the distributions setting.

Keywords: Caristi-type fixed point; Brezis-Browder principle; nonlinear complementarity problem; supernormal cone; differential inclusion

1 Introduction

It is well known that the Ekeland variational principle [1] and Caristi-Kirk fixed point theorem are both equivalent. Many authors [2–7] have established a generalized version of these two results in different settings, that is, in vector-valued generalized metric space with respect to a convex cone $\mathbb{K}$ in a Banach space. Recall that a subset $\mathbb{K} \subset \mathbb{Y}$ is called a convex cone on a topological vector space $\mathbb{Y}$ if:

1. $\mathbb{K} + \mathbb{K} \subset \mathbb{K}$;
2. for every $\lambda > 0$, $\lambda\mathbb{K} \subset \mathbb{K}$;
3. $\mathbb{K} \cap (-\mathbb{K}) = \{\theta\}$, where θ denotes the zero of $\mathbb{Y}$.

A convex cone $\mathbb{K} \subset \mathbb{Y}$ generates a partial ordering on $\mathbb{Y}$ (*i.e.* a reflexive, antisymmetric, and transitive relation) by

$$x \preceq y \quad \Longleftrightarrow \quad y - x \in \mathbb{K}.$$

Thereby, since its appearance, the Brezis-Browder ordering principle [8] seems to be a strong tool to prove fixed point or minimal point theorems in an ordered set. Zermelo's theorem [9] shows that there is an equivalency between the existence of a fixed point of such a map and the monotonicity of the map. By the way, Hamel [10] studied existence theorems, namely minimal point, Caristi fixed point, and Ekeland variational principle in the topological product space $\mathbb{X} \times \mathbb{Y}$ where $\mathbb{X}$ is a separated uniform space, and $\mathbb{Y}$ is a topological vector space.

Fang [11] introduced the concept of 'F-type topological spaces' generating the topology by families of quasi-metrics and gave a generalization of Ekeland's variational principle.

Furthermore, Isac [12] proved an interesting Caristi-type theorem in the framework of locally convex space, which led him to derive an existence result of a nonlinear equation.

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Hence, the aim of this paper is to generalize some of the well-known fixed point theorems [11, 13–15] for a pseudo-metric space $\mathbb{X}$. This paper is divided into three sections after showing some basic results in preliminaries. Using in Section 3 the Brezis-Browder principle, we give generalized Caristi's fixed point theorems for set-valued maps and derive some corollaries. Section 4 is devoted to an Ekeland-type variational principle in more applied general setting, namely pseudo-metric spaces, and also discuss the relationships of our main results. Finally, following investigations by Isac, Section 5 is devoted to applications.

2 Preliminaries

Over this section, $\mathbb{Y}$ is a locally convex space, and $\mathbb{K}$ is a convex cone in $\mathbb{Y}$. A set Λ is said to be a directed set if '$\prec$' is a preorder and every pair of elements of Λ has an upper bound.

Definition 2.1 Let $\mathbb{X}$ be a nonempty set, and $(\Lambda, \prec)$ a directed set. A family of cone pseudo-metrics on $\mathbb{X}$ is a system $\{d_\alpha\}_{\alpha\in\Lambda}$ of mappings $d_\alpha : \mathbb{X}\times\mathbb{X} \to \mathbb{K}$ satisfying the following conditions for each $\alpha \in \Lambda$ and $x, y, z \in \mathbb{X}$:

(A1) $\theta \preceq d_\alpha(x,y)$, and $d_\alpha(x,x) = \theta$;
(A2) $d_\alpha(x,y) = d_\alpha(y,x)$;
(A3) $d_\alpha(x,z) \preceq d_\alpha(x,y) + d_\alpha(y,z)$;
(A4) If $\alpha \prec \beta$ then $d_\alpha(x,y) \preceq d_\beta(x,y)$.

Then the pair $(\mathbb{X}, \{d_\alpha\}_{\alpha\in\Lambda})$ is called a cone pseudo-metric space. Additionally, if

(A5) for all $\alpha \in \Lambda$ and $x, y \in \mathbb{X}$, $d_\alpha(x,y) = \theta$ implies $x = y$,

then the family of cone pseudo-metrics is said to be separating.

The concept of a cone pseudo-metric space was already defined by Włodarczyk *et al.* [16], who called it a Hausdorff cone pseudo-metric space. In this paper, we use a locally convex space as a target set for a cone pseudo-metric, which is more general that a normed space. If $(\mathbb{Y}, \tau)$ is a locally convex space, then it is known that the topology τ can be generated by a family of seminorms $\{p_i\}_{i\in I}$ [17]. A subset B of $\{p_i\}_{i\in I}$ is called a basis for $\{p_i\}_{i\in I}$ if for every $i \in I$, there exist $q \in B$ and $\lambda > 0$ such that $p_i \le \lambda q$.

We say that a family of seminorms $\{p_i\}_{i\in I}$ is separating if $\ker\{p_i\}_{i\in I} = \{\theta\}$ or has a Hausdorff basis B if $\ker B = \{\theta\}$, where

$$\ker B = \{x \in \mathbb{Y} : p(x) = 0, \forall p \in B\}.$$

The most useful class of cones in topological vector space is the class of normal cones. For more details, we refer the reader to [18].

Definition 2.2 ([13]) If $(\mathbb{Y}, \{p_i\}_{i\in I})$ is a locally convex space, then a convex cone $\mathbb{K} \subset \mathbb{Y}$ is said to be normal if there exists a basis B of $\{p_i\}_{i\in I}$ such that, for each $p \in B$ and all $x, y \in \mathbb{K}$,

$$\theta \preceq x \preceq y \quad \Longrightarrow \quad p(x) \le p(y).$$

Throughout this paper, we assume that the topology defined on $\mathbb{Y}$ is generated by the basis B [13], and we simply write $B = \{p_i\}_{i\in I}$.

Proposition 2.3 *Let* $(\mathbb{X}, \{d_\alpha\}_{\alpha\in\Lambda})$ *be a cone pseudo-metric space over a normal cone* $\mathbb{K}$.

Then the mappings $\delta_{\alpha i} : \mathbb{X} \times \mathbb{X} \to [0,\infty[$ *defined for each* $(\alpha, i) \in \Lambda \times I$ *by* $\delta_{\alpha i} = p_i \circ d_\alpha$ *is a family of pseudo-metrics on* $\mathbb{X}$.

Proof By (A1) and (A2) we have immediately $\delta_{\alpha i}(x,x) = 0$ and $\delta_{\alpha i}(x,y) = \delta_{\alpha i}(y,x)$ for every $x, y \in \mathbb{X}$.

Since for each $\alpha \in \Lambda$ and all $x, y, z \in \mathbb{X}$, we have $d_\alpha(x,y) \in \mathbb{K}$ and

$$\theta \preceq d_\alpha(x,z) \preceq d_\alpha(x,y) + d_\alpha(y,x)$$

and since $\mathbb{K}$ is a normal cone, we get, for each $i \in I$,

$$p_i\big(d_\alpha(x,z)\big) \leq p_i\big(d_\alpha(x,y) + d_\alpha(y,x)\big) \leq p_i\big(d_\alpha(x,y)\big) + p_i\big(d_\alpha(y,x)\big).$$

Then $\delta_{\alpha i}$ satisfies the triangle inequality. If we assume that $\{d_\alpha\}_{\alpha\in\Lambda}$ is a separating family, so is $\{\delta_{\alpha i}\}_{(\alpha,i)\in\Lambda\times I}$. □

If the convex cone $\mathbb{K}$ is solid ($\operatorname{int}\mathbb{K} \neq \emptyset$) and not normal and if $\mathbb{Y}$ is a locally convex space, then the Gerstewitz functional [19] $\xi_e : \mathbb{Y} \to \mathbb{R}$, where $e \in \operatorname{int}\mathbb{K}$, is defined as

$$\xi_e(x) = \inf\{\lambda \in \mathbb{R} : x \in \lambda e - \mathbb{K}\}$$

for each $x \in \mathbb{Y}$.

We have the following result.

Lemma 2.4 *For all* $\lambda \in \mathbb{R}$ *and* $x \in \mathbb{Y}$, *we have the following statements*:

(i) $\xi_e(x) \leq \lambda \Longleftrightarrow x \in \lambda e - \mathbb{K}$;

(ii) $\xi_e(x) > \lambda \Longleftrightarrow x \notin \lambda e - \mathbb{K}$;

(iii) $\xi_e(x) \geq \lambda \Longleftrightarrow x \notin \lambda e - \operatorname{int}\mathbb{K}$;

(iv) $\xi_e(x) < \lambda \Longleftrightarrow x \in \lambda e - \operatorname{int}\mathbb{K}$;

(v) $\xi_e(\cdot)$ *is positively homogeneous and continuous on* $\mathbb{Y}$;

(vi) *if* $x_1 \in x_2 + \mathbb{K}$, *then* $\xi_e(x_2) \leq \xi_e(x_1)$;

(vii) $\xi_e(x_1 + x_2) \leq \xi_e(x_1) + \xi_e(x_2)$ *for all* $x_1, x_2 \in \mathbb{Y}$.

Proof See, for instance, [7, 20–23]. □

The following result is Theorem 2.1 of Du [24].

Proposition 2.5 *Let* $(\mathbb{X}, \{d_\alpha\}_\alpha)$ *be a cone pseudo-metric space over a solid cone* $\mathbb{K}$. *Then the family of mappings* $\delta_\alpha : \mathbb{X} \times \mathbb{X} \to [0,\infty[$ *defined by* $\delta_\alpha = \xi_e \circ d_\alpha$ *is a family of pseudo-metrics on* $\mathbb{X}$.

Proof Since $\xi_e(\cdot)$ is a seminorm on $\mathbb{Y}$ by Lemma 2.4, Proposition 2.3 gives the result. □

If the cone $\mathbb{K}$ is normal and solid, then $\xi_e(\cdot)$ is a norm over $\mathbb{Y}$, and we have the following proposition.

Proposition 2.6 *If $(\mathbb{Y},\tau)$ is a Hausdorff topological space ordered by a normal solid cone $\mathbb{K}$, then $(\mathbb{Y},\tau)$ is a normable space.*

Proof See Proposition 1.10 in [18], Chapter 2. □

Next, we discuss some convergence properties of cone pseudo-metric spaces. We note that $x \ll y$ if and only if $y - x \in \operatorname{int}\mathbb{K}$, where the 'int' is the interior.

Definition 2.7 Let $(\mathbb{X},\{d_\alpha\}_\alpha)$ be a cone pseudo-metric space over a solid convex cone $\mathbb{K}\subset\mathbb{Y}$, where $\mathbb{Y}$ is a locally convex space, $x\in\mathbb{X}$, and $\{x_n\}_n$ a sequence in $\mathbb{X}$.

1. $\{x_n\}_n$ is Cauchy sequence whenever for every $\alpha\in\Lambda$ and $c\in\mathbb{Y}$ with $\theta\ll c$, there is a natural number N_0 such that

$$d_\alpha(x_n,x_m)\ll c,\quad \forall n,m\geq N_0.$$

2. $\{x_n\}_n$ converges to x whenever for every $\alpha\in\Lambda$ and $c\in\mathbb{Y}$ with $\theta\ll c$, there is a natural number N_0 such that

$$d_\alpha(x_n,x)\ll c,\quad \forall n\geq N_0.$$

3. $(\mathbb{X},\{d_\alpha\}_\alpha)$ is complete if each Cauchy sequence converges in $\mathbb{X}$.

Proposition 2.8 *Let $(\mathbb{X},\{d_\alpha\}_\alpha)$ be a cone pseudo-metric space over a solid convex cone $\mathbb{K}\subset\mathbb{Y}$, where $\mathbb{Y}$ is a locally convex space.*

Then, for each $\alpha\in\Lambda$, we get

$$d_\alpha(x_n,x)\longrightarrow\theta \quad\Longleftrightarrow\quad \delta_\alpha(x_n,x)=\xi_e\bigl(d_\alpha(x_n,x)\bigr)\longrightarrow 0.$$

Proof It is similar to the proof of Theorem 3.2 in [25]. □

Using this pseudo-metric δ_α, we keep saying that $(\mathbb{X},\{\delta_\alpha\}_\alpha)$ is a pseudo-metric space over a solid convex cone $\mathbb{K}$.

3 Fixed point theorems

Recall that the most famous ordering principle.

Theorem 3.1 (Brezis-Browder) *Let $(W,\precsim)$ be a quasi-ordered set (i.e. $\precsim$ is a reflexive and transitive relation), and let $\Psi: W\longrightarrow\mathbb{R}$ be a function satisfying the following conditions:*

(B1) *Ψ is bounded below;*
(B2) $w_1\precsim w_2\Longrightarrow\Psi(w_1)\leq\Psi(w_2)$;
(B3) *For every decreasing sequence $\{w_n\}_{n\in\mathbb{N}}\subset W$ with respect to '$\precsim$', there exists $w\in W$ such that $w\leq w_n$ for all $n\in\mathbb{N}$.*

Then, for every $w_0\in W$, there exists $\bar{w}\in W$ such that

(i) $\bar{w}\precsim w_0$;
(ii) $\hat{w}\precsim\bar{w}\Longrightarrow\Psi(\hat{w})=\Psi(\bar{w})$.

In particular, if we strengthen (B2) *to*

(B2′) $(w_1 \precsim w_2, w_1 \neq w_2) \Longrightarrow \Psi(w_1) < \Psi(w_2)$,

then

(ii′) $\hat{w} \precsim \bar{w} \Longrightarrow \hat{w} = \bar{w}$, *that is*, $\bar{w}$ *is minimal in* W *with respect to* '$\precsim$'.

Proof See Corollary 1 in [8]. □

Now we are able to give the main result of this section.

Theorem 3.2 *Let* $(\mathbb{Y}, \{p_i\}_{i\in I})$ *a complete separated locally convex space*, $(\mathbb{X}, \{\delta_\alpha\}_{\alpha\in\Lambda})$ *be a complete Hausdorff pseudo-metric space over a solid convex cone* $\mathbb{K}$, $T : \mathbb{X} \longrightarrow 2^{\mathbb{X}}$ *and* $S : \mathbb{X} \longrightarrow 2^{\mathbb{Y}}$ *two set-valued maps with nonempty values.*

Suppose that, for every $(\alpha, i) \in \Lambda \times I$ *and two constants* $c_\alpha, c_i > 0$, *there exist lower semicontinuous functions* $\varphi_{\alpha i} : \mathbb{Y} \longrightarrow [0,\infty)$, *and for each* $(x, y) \in G_S$, *there exist* $u \in Tx$ *and* $v \in Su$ *such that*

$$\max\bigl\{c_\alpha\delta_\alpha(x,u), c_ip_i(y-v)\bigr\} \le \varphi_{\alpha i}(y) - \varphi_{\alpha i}(v). \tag{1}$$

Then T *has a fixed point in* $\mathbb{X}$.

Proof Put

$$W_0 = \bigl\{(x,y) \in G_S; \forall(\alpha,i) \in \Lambda\times I, \max\bigl\{c_\alpha\delta_\alpha(x_0,x), c_ip_i(y_0-y)\bigr\} + \varphi_{\alpha i}(y) \le \varphi_{\alpha i}(y_0)\bigr\}$$

for some $(x_0, y_0) \in G_S$. Then W_0 is a nonempty closed subset of G_S. Indeed, let $(x_n, y_n)_n$ be a sequence in W_0 that converges to (x, y), that is, $\lim_{n\to\infty} p_i(y_n - y) = 0$. Since for each $(\alpha, i) \in \Lambda \times I$, the function $\varphi_{\alpha i}$ is lower semicontinuous, that is,

$$\varphi_{\alpha i}(y) \le \liminf_{n\to\infty} \varphi_{\alpha i}(y_n),$$

we have

$$\begin{aligned} c_ip_i(y_0-y) &\le c_ip_i(y_0-y_n) + c_ip_i(y_n-y) \\ &\le \varphi_{\alpha i}(y_0) - \varphi_{\alpha i}(y_n) + c_ip_i(y_n-y) \\ &\le \varphi_{\alpha i}(y_0) - \liminf_{k\to\infty} \varphi_{\alpha i}(y_k) + c_ip_i(y_n-y) \\ &\le \varphi_{\alpha i}(y_0) - \varphi_{\alpha i}(y) + c_ip_i(y_n-y). \end{aligned}$$

So, taking the limit with respect to n, we get $c_ip_i(y_0 - y) \le \varphi_{\alpha i}(y_0) - \varphi_{\alpha i}(y)$, and by similar arguments we get

$$c_\alpha\delta_\alpha(x_0,x) \le \varphi_{\alpha i}(y_0) - \varphi_{\alpha i}(y).$$

Hence, $\max\{c_\alpha\delta_\alpha(x_0, x), c_ip_i(y_0 - y)\} + \varphi_{\alpha i}(y) \le \varphi_{\alpha i}(y_0)$, so that $(x, y) \in W_0$.

Now we define a binary relation in W_0 as follows: for every (x_1, y_1) and (x_2, y_2) in W_0,

$$(x_1, y_1) \precsim (x_2, y_2) \quad \Longleftrightarrow \quad \max\bigl\{c_\alpha \delta_\alpha(x_1, x_2), c_i p_i(y_1 - y_2)\bigr\} \leq \varphi_{\alpha i}(y_2) - \varphi_{\alpha i}(y_1)$$

for each $(\alpha, i) \in \Lambda \times I$. We can show that the relation $\precsim$ is an ordering on W_0.

Next, we show that, for every decreasing sequence $(x_n, y_n)_{n\in\mathbb{N}} \subset W_0$ with respect to '$\precsim$', there exists $(x^*, y^*) \in W_0$ such that $(x^*, y^*) \precsim (x_n, y_n)$ for all $n \in \mathbb{N}$. Let $(x_n, y_n)_{n\in\mathbb{N}}$ be a $\precsim$-decreasing sequence in W_0. Then, for any $m, n \in \mathbb{N}$ such that $m \geq n$, we have

$$(x_m, y_m) \precsim (x_n, y_n) \quad \Longleftrightarrow \quad \max\bigl\{c_\alpha \delta_\alpha(x_m, x_n), c_i p_i(y_m - y_n)\bigr\} \leq \varphi_{\alpha i}(y_n) - \varphi_{\alpha i}(y_m)$$
$$\text{for each } (\alpha, i) \in \Lambda \times I,$$

which gives that the positive sequence $\{\varphi_{\alpha i}(y_n)\}_n$ is decreasing (for α and i fixed). Hence, there exists $r_{\alpha i}$ such that $\lim \varphi_{\alpha i}(y_n) = r_{\alpha i}$. Let $\varepsilon > 0$ and $(\alpha, i) \in \Lambda \times I$. There exists $N_0 \in \mathbb{N}^*$ such that, for any $n \geq N_0$, we have

$$r_{\alpha i} \leq \varphi_{\alpha i}(y_n) \leq r_{\alpha i} + \min(c_\alpha, c_i) \cdot \varepsilon$$

and then, for every $m \geq n \geq N_0$,

$$\begin{aligned} c_i p_i(y_m - y_n) &\leq \varphi_{\alpha i}(y_n) - \varphi_{\alpha i}(y_m) \\ &\leq r_{\alpha i} + \min(c_\alpha, c_i) \cdot \varepsilon - r_{\alpha i}. \end{aligned}$$

Thus,

$$c_i p_i(y_m - y_n) \leq \min(c_\alpha, c_i) \cdot \varepsilon \leq c_i \varepsilon.$$

Also, we get

$$\begin{aligned} c_\alpha \delta_\alpha(x_m, x_n) &\leq \varphi_{\alpha i}(y_n) - \varphi_{\alpha i}(y_m) \\ &\leq r_{\alpha i} + \min(c_\alpha, c_i) \cdot \varepsilon - r_{\alpha i} \end{aligned}$$

and thus

$$c_\alpha \delta_\alpha(x_m, x_n) \leq c_\alpha \varepsilon.$$

Repeating the last computation for every $(\alpha, i) \in \Lambda \times I$ and using the fact that $\{\delta_\alpha\}_{\alpha\in\Lambda}$ and $\{p_i\}_{i\in I}$ are separated families, we obtain that $\{x_n\}_n$ and $\{y_n\}_n$ are Cauchy sequences in the complete spaces $\mathbb{X}$ and $\mathbb{Y}$, respectively. Therefore, there exist $x^* \in \mathbb{X}$ and $y^* \in \mathbb{Y}$ such that

$$x_n \longrightarrow x^* \quad \text{and} \quad y_n \longrightarrow y^*.$$

Since W_0 is closed, we have that $(x^*, y^*) \in W_0$ and $y^* \in Sx^*$ by the definition of W_0.

Also, for all $(n,m) \in \mathbb{N}^2$ such that $m \geq n$, we have $(x_m, y_m) \precsim (x_n, y_n)$, so that for all $(\alpha, i) \in \Lambda \times I$,

$$\begin{aligned}\max\{c_\alpha \delta_\alpha(x_m, x_n), c_i p_i(y_m - y_n)\} &\leq \varphi_{\alpha i}(y_n) - \varphi_{\alpha i}(y_m) \\ &\leq \varphi_{\alpha i}(y_n) - \liminf_{k\to\infty} \varphi_{\alpha i}(y_k) \\ &\leq \varphi_{\alpha i}(y_n) - \varphi_{\alpha i}(y^*).\end{aligned}$$

Taking the limit with respect to m and using the fact that δ_α and p_i are continuous, we get

$$\max\{c_\alpha \delta_\alpha(x^*, x_n), c_i p_i(y^* - y_n)\} \leq \varphi_{\alpha i}(y_n) - \varphi_{\alpha i}(y^*) \quad \text{for all } (\alpha, i) \in \Lambda \times I.$$

Thus, for each $n \in \mathbb{N}$,

$$(x^*, y^*) \precsim (x_n, y_n).$$

Let $(\alpha, i) \in \Lambda \times I$ be fixed and choose $\Psi : W_0 \longrightarrow \mathbb{R}$ as follows: $\Psi(x,y) = \varphi_{\alpha i}(y)$ for each $(x,y) \in W_0$. Condition (B1) from Theorem 3.1 holds since $\varphi_{\alpha i}(y) \geq 0$. We also have

$$(x_1, y_1) \precsim (x_2, y_2) \quad \Longrightarrow \quad \varphi_{\alpha i}(y_1) \leq \varphi_{\alpha i}(y_2) \quad \text{for each } (\alpha, i) \in \Lambda \times I.$$

So $\Psi(x_1, y_1) \leq \Psi(x_2, y_2)$, and thus (B2) also holds. Then all assumptions of the Brezis-Browder principle are satisfied. Hence, for each $(x_0, y_0) \in W_0$, there exists $(\bar{x}, \bar{y}) \in W_0$ such that:

(i) $(\bar{x}, \bar{y}) \precsim (x_0, y_0)$;

(ii) if $(\hat{x}, \hat{y}) \precsim (\bar{x}, \bar{y})$, then $\Psi(\hat{x}, \hat{y}) = \Psi(\bar{x}, \bar{y})$.

We claim that $\bar{x}$ is a fixed point for T. For this $(\bar{x}, \bar{y}) \in W_0 \subset G_S$, there exists $(u, v) \in \mathbb{X} \times \mathbb{Y}$ such that $u \in T\bar{x}$ and $v \in S\bar{u}$ satisfy the following inequality for each $(\alpha, i) \in \Lambda \times I$:

$$\max\{c_\alpha \delta_\alpha(u, \bar{x}), c_i p_i(v - \bar{y})\} \leq \varphi_{\alpha i}(\bar{y}) - \varphi_{\alpha i}(v).$$

Given $(u, v) \precsim (\bar{x}, \bar{y})$, we have $\Psi(u, v) = \Psi(\bar{x}, \bar{y})$; hence, $x = \bar{x}$, and thus $\bar{x} \in T\bar{x}$, which completes the proof. □

Theorem 3.3 *Under the hypotheses of Theorem* 3.2, *suppose that the condition 'for each* $(x,y) \in G_S$, *there exist* $u \in Tx$ *and* $v \in Su$' *is replaced by 'for each* $(x,y) \in G_S$ *and for every* $u \in Tx$, *there exists* $v \in Su$'.

Then T *has a critical point, that is, there exists* $\bar{x} \in \mathbb{X}$ *such that* $\{\bar{x}\} = T\bar{x}$.

Proof By Theorem 3.2, T has a fixed point $\bar{x}$ in $\mathbb{X}$. We claim that it is a critical point. For this, let us show that assumption (B2′) of Brezis-Browder holds, and so we have (ii′). Let $(\alpha, i) \in \Lambda \times I$ be fixed and choose $\Psi : W_0 \longrightarrow \mathbb{R}$ as in the above proof: $\Psi(x,y) = \varphi_{\alpha i}(y)$ for each $(x,y) \in W_0$. Then

$$(x_1, y_1) \precsim (x_2, y_2), \quad (x_1, y_1) \neq (x_2, y_2) \quad \Longrightarrow \quad \Psi(x_1, y_1) < \Psi(x_2, y_2).$$

Indeed, suppose that $x_1 \neq x_2$. Then, for each $\alpha \in \Lambda$, we get

$$\delta_\alpha(x_1,x_2) \neq 0 \quad \Longrightarrow \quad \delta_\alpha(x_1,x_2) > 0.$$

Then

$$0 < c_\alpha \delta_\alpha(x_1,x_2) \leq \varphi_{\alpha i}(y_2) - \varphi_{\alpha i}(y_1),$$

and hence $\varphi_{\alpha i}(y_1) < \varphi_{\alpha i}(y_2) \Longleftrightarrow \Psi(x_1,y_1) < \Psi(x_2,y_2)$.

Otherwise, if $x_1 = x_2$, then by the assumption $(x_1,y_1) \neq (x_2,y_2)$ we must have $y_1 \neq y_2$, and then $\varphi_{\alpha i}(y_1) < \varphi_{\alpha i}(y_2)$. Therefore, assumption (B2′) in Theorem 3.1 is satisfied. Then $(\bar{x},\bar{y})$ is minimal point in W_0 by (ii′) of the Brezis-Browder principle.

Now we claim that $\bar{x}$ is a critical point for T. By inequality (1) we have

$$\max\{c_\alpha \delta_\alpha(u,\bar{x}), c_i p_i(v-\bar{y})\} \leq \varphi_{\alpha i}(\bar{y}) - \varphi_{\alpha i}(v)$$

for each $u \in T\bar{x}$ and $(\alpha,i) \in \Lambda \times I$, and then $(u,v) \precsim (\bar{x},\bar{y})$. Since $(\bar{x},\bar{y})$ is a minimal point in W_0, it follows that $u = \bar{x}$, and thus $T\bar{x} = \{\bar{x}\}$, which completes the proof. □

By the same process as before we can also get the same results if we replace the cone pseudo-distance $\{\delta_\alpha\}_{\alpha\in\Lambda}$ with respect to the solid cone with the real-valued pseudo-distance $\{d_\alpha\}_{\alpha\in\Lambda}$.

Proposition 3.4 *Let $(\mathbb{X},\{d_\alpha\}_{\alpha\in\Lambda})$ be a complete Hausdorff pseudo-metric space, $(\mathbb{Y},\{p_i\}_{i\in I})$ a complete separated locally convex space, and $T:\mathbb{X}\longrightarrow 2^{\mathbb{X}}$ and $S:\mathbb{X}\longrightarrow 2^{\mathbb{Y}}$ two set-valued maps with nonempty values.*

Suppose that, for every $(\alpha,i)\in\Lambda\times I$ and two constants $c_\alpha, c_i > 0$, there exist lower semi-continuous functions $\varphi_{\alpha i}:\mathbb{Y}\longrightarrow[0,\infty)$ and, for each $(x,y)\in G_S$, there exist $u\in Tx$ and $v\in Su$ (resp., for every $u\in Tx$, there exists $v\in Su$) such that:

$$\max\{c_\alpha d_\alpha(x,u), c_i p_i(y-v)\} \leq \varphi_{\alpha i}(y) - \varphi_{\alpha i}(v). \tag{2}$$

Then T has a fixed point (resp. critical point) in $\mathbb{X}$.

If the set-valued map S in Proposition 3.4 is only a single-valued map, then we have the following:

Corollary 3.5 (Isac [12]) *Let $(\mathbb{X},\{p_\alpha\}_{\alpha\in\Lambda})$ be a Hausdorff locally convex space, and $M\subset\mathbb{X}$ be a nonempty set. The set-valued map $T:\mathbb{X}\longrightarrow 2^{\mathbb{X}}$ has a critical point if and only if there exist a complete Hausdorff locally convex space $(\mathbb{Y},\{q_i\}_{i\in I})$, a subset $M_0\subseteq M$, $S:M_0\longrightarrow\mathbb{Y}$, for every couple $(\alpha,i)\in\Lambda\times I$, a function $\varphi_{\alpha i}:\overline{S(M_0)}\longrightarrow[0,\infty)$, and two constants $c_\alpha, c_i>0$ such that:*

(i) *$T(M_0)\subset M_0$, and $M_0\subset M$ is closed;*
(ii) *S is closed, and $\overline{S(M_0)}$ is complete;*
(iii) *$\varphi_{\alpha i}$ is lower semicontinuous for each $(\alpha,i)\in\Lambda\times I$;*
(iv) *$\max\{c_\alpha p_\alpha(x-y), c_i q_i(S(x)-S(y))\}\leq\varphi_{\alpha i}(S(x))-\varphi_{\alpha i}(S(y))$ for all $x\in M_0$ and all $y\in Tx$.*

Proof If T has a critical point $\bar{x} \in M$, then the assumptions of Isac's theorem are satisfied if we put $M_0 = \{\bar{x}\}$, $\mathbb{X} = \mathbb{Y}$, $\{p_\alpha\}_{\alpha\in\Lambda} = \{q_i\}_{i\in I}$, $S = I_{M_0}$, and for each $(\alpha, i) \in \Lambda \times I$, $c_\alpha = c_i = 1$ and $\varphi_{\alpha i} = 0$.

Conversely, $\{p_\alpha\}_{\alpha\in\Lambda}$ is generating family of separated seminorms on $\mathbb{X}$, and if we set

$$p_\alpha(x-y) = d_\alpha(x,y)$$

for each $\alpha \in \Lambda$, then $(M_0, \{d_\alpha\}_{\alpha\in\Lambda})$ is a complete Hausdorff pseudo-metric subspace of $\mathbb{X}$. Also, by (ii) we get that $(\overline{S(M_0)}, \{q_i\}_{i\in I})$ is a complete Hausdorff locally convex subspace of $\mathbb{Y}$, and since $T(M_0) \subset M_0$, all assumptions of Proposition 3.4 are satisfied, so that we get the result. □

Remark 3.6 Our main result does not involve any assumptions about closeness of intermediary set-valued map S, contrary to the result of Isac [12].

Corollary 3.7 (Fang [11]) *Let $T : \mathbb{X} \longrightarrow \mathbb{X}$ be a map of a complete Hausdorff locally convex space $(\mathbb{X}, \{p_\alpha\}_{\alpha\in\Lambda})$. Suppose that there exists a lower semicontinuous function $\varphi : \mathbb{X} \longrightarrow [0,\infty)$ such that, for each $x \in \mathbb{X}$ and for each $\alpha \in \Lambda$,*

$$p_\alpha(x - Tx) \le \varphi(x) - \varphi(Tx). \tag{3}$$

Then T has a fixed point.

Proof For every $x, y \in \mathbb{X}$, we even replace $p_\alpha(x-y) = d_\alpha(x,y)$ and take single-valued maps T' and S with $Sx = \{x\}$ and $T'x = \{Tx\}$ for all $x \in \mathbb{X}$. Then inequality (3) implies inequality (2) of Proposition 3.4, and the result follows. □

We get the next obvious two corollaries.

Corollary 3.8 (Downing and Kirk [15]) *Let $\mathbb{X}$ and $\mathbb{Y}$ be complete metric spaces, and $T : \mathbb{X} \longrightarrow \mathbb{X}$ an arbitrary mapping. Suppose that there exist a closed mapping $S : \mathbb{X} \longrightarrow \mathbb{Y}$, a lower semicontinuous mapping $\varphi : S(\mathbb{X}) \longrightarrow [0,\infty)$, and a constant $c > 0$ such that, for each $x \in \mathbb{X}$,*

$$\max\bigl\{d_{\mathbb{X}}(x,Tx), c d_{\mathbb{Y}}\bigl(S(x), S(Tx)\bigr)\bigr\} \le \varphi\bigl(S(x)\bigr) - \varphi\bigl(S(Tx)\bigr).$$

Then there exists $x \in \mathbb{X}$ such that $Tx = x$.

Corollary 3.9 (Caristi [14]) *Let $(\mathbb{X}, d)$ be a complete metric space, and let $\varphi : \mathbb{X} \longrightarrow [0,\infty)$ be a lower semicontinuous function. If a mapping $T : \mathbb{X} \longrightarrow \mathbb{X}$ satisfies for each $x \in \mathbb{X}$ the condition*

$$d(x,Tx) \le \varphi(x) - \varphi(Tx),$$

then T has a fixed point in $\mathbb{X}$.

We conclude this section with an application of Theorem 3.2.

Theorem 3.10 *Let $(\mathbb{Y}, \{p_i\}_{i\in I})$ be a complete separated locally convex space, $(\mathbb{X}, \{\delta_\alpha\}_{\alpha\in\Lambda})$ be a complete Hausdorff pseudo-metric space over a solid cone $\mathbb{K}$, $S : \mathbb{X} \longrightarrow 2^{\mathbb{Y}}$ be set-valued map, and for every $(\alpha, i) \in \Lambda \times I$, $\varphi_{\alpha i} : \mathbb{Y} \longrightarrow [0,\infty)$ be lower semicontinuous function.*

Suppose that, for each $(x, y) \in G_S$, there exists $(x_0, y_0) \in G_S$ such that

1. $x_0 \neq x$;
2. $\varphi_{\alpha i}(y_0) + \max\{c_\alpha\delta_\alpha(x, x_0), c_i p_i(y - y_0)\} \le \varphi_{\alpha i}(y)$ *for every* $(\alpha, i) \in \Lambda \times I$.

Then there exist $(\bar{x}, \bar{y}) \in G_S$ and $(\alpha_0, i_0) \in \Lambda \times I$ such that $\varphi_{\alpha_0 i_0}(\bar{y}) = \inf_{t\in\mathbb{Y}} \varphi_{\alpha_0 i_0}(t)$.

Proof By contradiction suppose that, for each $(x, y) \in G_S$ and for every $(\alpha, i) \in \Lambda \times I$, we have

$$\varphi_{\alpha i}(y) > \inf_{t\in\mathbb{Y}} \varphi_{\alpha i}(t).$$

By assumptions, there exists $(x_0, y_0) \in G_S$ such that 1 and 2 hold. Set

$$\begin{aligned} E(x, y) = \Big\{&(z, t) \in G_S : z \neq x, \text{and } \forall(\alpha, i) \in \Lambda \times I, \\ &\varphi_{\alpha i}(t) + \max\big\{c_\alpha\delta_\alpha(x, z), c_i p_i(y - t)\big\} \le \varphi_{\alpha i}(y)\Big\}. \end{aligned}$$

For all $(x, y) \in G_S$, we have $(x_0, y_0) \in E(x, y)$ and $(x, y) \notin E(x, y)$. For all $x \in \mathbb{X}$, we put $G_S(x) = \{y \in \mathbb{Y} : (x, y) \in G_S\}$. Define the set-valued map T by

$$Tx = \bigcup_{y\in G_S(x)} \big\{z \in \mathbb{X} : \exists t \in Sz \text{ such that } (z, t) \in E(x, y)\big\}$$

for $x \in \mathbb{X}$. For all $(x, y) \in G_S$ and $(\alpha, i) \in \Lambda \times I$, there exist $z \in Tx$ and $t \in Sz$ such that

$$\max\big\{c_\alpha\delta_\alpha(x, z), c_i p_i(y - t)\big\} \le \varphi_{\alpha i}(y) - \varphi_{\alpha i}(t).$$

Then by Theorem 3.2, T admits a point $\bar{x}$ such that $\bar{x} \in T\bar{x}$. For this $\bar{x}$, we get that, for some $\bar{y}_1, \bar{y}_2 \in \mathbb{Y}$, $(\bar{x}, \bar{y}_1) \in E(\bar{x}, \bar{y}_2)$, which is absurd. □

4 Variational principle

Theorem 4.1 *Let $(\mathbb{Y}, \{p_i\}_{i\in I})$ be a complete separated locally convex space, $(\mathbb{X}, \{\delta_\alpha\}_{\alpha\in\Lambda})$ be a complete Hausdorff pseudo-metric space over a solid cone $\mathbb{K}$, $S : \mathbb{X} \longrightarrow 2^{\mathbb{Y}}$ be a set-valued map, and, for every $(\alpha, i) \in \Lambda \times I$, $\varphi_{\alpha i} : \mathbb{Y} \longrightarrow [0,\infty)$ be a lower semicontinuous function.*

Then, for each $\varepsilon > 0$ and $(x_0, y_0) \in G_S$ satisfying

$$\varphi_{\alpha i}(y_0) \le \inf \varphi_{\alpha i} + \varepsilon, \quad \forall(\alpha, i) \in \Lambda \times I,$$

there exists $(\bar{x}, \bar{y}) \in G_S$ such that:

(i) *for each $(\alpha, i) \in \Lambda \times I$, $\varphi_{\alpha i}(\bar{y}) \le \varphi_{\alpha i}(y_0)$;*

(ii) *for each $(x, y) \in G_S$ with $x \neq \bar{x}$, there exist $(\alpha, i) \in \Lambda \times I$ and two constants $c_\alpha, c_i > 0$ such that*

$$\varphi_{\alpha i}(\bar{y}) < \varphi_{\alpha i}(y) + \varepsilon \max\big\{c_\alpha\delta_\alpha(x, \bar{x}), c_i p_i(y - \bar{y})\big\}.$$

Proof Let $\varepsilon > 0$ and $(x_0, y_0) \in G_S$. Put

$$W_0 = \bigl\{(x,y) \in G_S; \forall(\alpha,i) \in \Lambda \times I, \varphi_{\alpha i}(y) + \varepsilon \max\bigl\{c_\alpha \delta_\alpha(x,x_0), c_i p_i(y-y_0)\bigr\} \le \varphi_{\alpha i}(y_0)\bigr\}.$$

It is a nonempty and closed subset of G_S since the family $\{\varphi_{\alpha i}\}_{\alpha i}$ is lower semicontinuous.

For all $x \in \mathbb{X}$, we put $W_0(x) = \{y \in \mathbb{Y} : (x,y) \in W_0\}$. Next, we define the set-valued map $T : \mathbb{X} \longrightarrow 2^{\mathbb{X}}$ by

$$\begin{aligned} Tx = \bigcup_{y \in W_0(x)} & \bigl\{\hat{x} \in \mathbb{X}; \exists \hat{y} \in S\hat{x}, \forall(\alpha,i) \in \Lambda \times I, \\ & \varphi_{\alpha i}(\hat{y}) + \varepsilon \max\bigl\{c_\alpha \delta_\alpha(\hat{x},x), c_i p_i(\hat{y}-y)\bigr\} \le \varphi_{\alpha i}(y)\bigr\}. \end{aligned}$$

Obviously, T satisfies inequality (1) of Theorem 3.2 with $\phi_{\alpha i} = \frac{1}{\varepsilon}\varphi_{\alpha i}$ so that T has a fixed point, that is, there exists $(\bar{x},\bar{y}) \in W_0$ such that $\bar{x} \in T\bar{x}$ with

$$(\bar{x},\bar{y}) \in W_0 \quad \Longrightarrow \quad \varphi_{\alpha i}(\bar{y}) \le \varphi_{\alpha i}(y_0),$$

and if $(\hat{x},\hat{y}) \in G_S$ with $(\hat{x},\hat{y}) \precsim (\bar{x},\bar{y})$, then $\hat{x} = \bar{x}$, which is equivalent to the assertion that, for each $(x,y) \in G_S$ with $x \neq \bar{x}$, there exist $(\alpha,i) \in \Lambda \times I$ and two constants $c_\alpha, c_i > 0$ such that

$$\varphi_{\alpha i}(\bar{y}) < \varphi_{\alpha i}(y) + \varepsilon \max\bigl\{c_\alpha \delta_\alpha(x,\bar{x}), c_i p_i(y-\bar{y})\bigr\}.$$

The proof is complete. □

Remark 4.2 We claim that Theorem 4.1 implies Theorem 3.2. Indeed, let $(x_0,y_0) \in G_S$ be given and take $\varepsilon = 1$. By Theorem 4.1 there exists $(\bar{x},\bar{y}) \in G_S$ such that assertions (i) and (ii) hold. Since (i), we have $(\bar{x},\bar{y}) \in W_0$. We claim that $\bar{x}$ is a fixed point of T. Assuming the contrary, by inequality (1) we get the existence of some $(x,y) \in G_S$ such that $x \in T\bar{x}$, $x \neq \bar{x}$, and

$$\max\bigl\{c_\alpha \delta_\alpha(x,\bar{x}), c_i p_i(y-\bar{y})\bigr\} \le \varphi_{\alpha i}(\bar{y}) - \varphi_{\alpha i}(y) \quad \text{for every } (\alpha,i) \in \Lambda \times I.$$

This contradicts (ii). Hence, $\bar{x}$ is a fixed point.

The above considerations show that Theorem 4.1 and Theorem 3.2 are equivalent.

Since the Caristi theorem (Corollary 3.9) is a particular case of our main result and the Ekeland variational principle is equivalent to Caristi's theorem, Theorem 4.1 is a generalization of the variational principle of Ekeland:

Corollary 4.3 (Ekeland [1]) *Let $(\mathbb{X},d)$ be a complete metric space, and $\varphi : \mathbb{X} \longrightarrow [0,\infty)$ be a lower semicontinuous function. Let $\varepsilon > 0$, and let a point $u \in \mathbb{X}$ be such that $\varphi(u) \le \inf \varphi + \varepsilon$. Then there exists a point $v \in \mathbb{X}$ such that*:

(i) $\varphi(v) \le \varphi(u)$;
(ii) $\varphi(v) < \varphi(w) + \varepsilon d(w;v)$ *for any* $w \in \mathbb{X}$; $w \neq v$.

5 Applications

In this section, we propose two applications.

5.1 General nonlinear complementarity problem

In a Hilbert space $(\mathbb{X}, \langle\cdot,\cdot\rangle)$, the dual cone $\mathbb{K}'$ of a convex cone $\mathbb{K}$ with respect to the duality $\langle\mathbb{X}', \mathbb{X}\rangle$ is defined by

$$\mathbb{K}' = \{y \in \mathbb{X} : \langle y, x\rangle \geq 0, \forall x \in \mathbb{K}\},$$

and the polar of $\mathbb{K}$ is $\mathbb{K}^0 = -\mathbb{K}'$.

Next, we suppose that $\mathbb{K}$ is a closed convex cone in $\mathbb{X}$. It is shown in [26] that the projection operator onto $\mathbb{K}$, denoted by $P_{\mathbb{K}}$, is well defined and satisfies, for all $x \in \mathbb{X}$,

$$\|x - P_{\mathbb{K}}(x)\| = \min_{y\in\mathbb{K}} \|x - y\|.$$

The next two results can be found in [26].

Theorem 5.1 *For every $x \in \mathbb{X}$, $P_{\mathbb{K}}$ has the following properties*:

1. $\langle P_{\mathbb{K}}(x) - x, y\rangle \geq 0$ *for every* $y \in \mathbb{K}$;
2. $\langle P_{\mathbb{K}}(x) - x, P_{\mathbb{K}}(x)\rangle = 0$.

Theorem 5.2 *For all $x, y, z \in \mathbb{X}$, the following statements are equivalent*:

1. $z = x + y$, $x \in \mathbb{K}$, $y \in \mathbb{K}^0$, *and* $\langle x, y\rangle = 0$;
2. $x = P_{\mathbb{K}}(z)$ *and* $y = P_{\mathbb{K}^0}(z)$.

Following Isac [26, 27], we give a new application of our main result to the so called general nonlinear complementarity problem (GNCP).

Let $S : \mathbb{K} \to 2^{\mathbb{X}}$ be a set-valued mapping. As is known [28], the GNCP with S and $\mathbb{K}$, denoted by GNCP$(S, \mathbb{K})$, is

$$\text{GNCP}(S,\mathbb{K}): \quad \begin{cases} \text{find } (\hat{x}, \hat{y}) \in \mathbb{K} \times \mathbb{X} \\ \text{s.t. } \hat{y} \in S(\hat{x}) \cap \mathbb{K}' \text{ and } \langle \hat{x}, \hat{y}\rangle = 0. \end{cases}$$

Before we obtain some existence results for GNCP$(S, \mathbb{K})$ by using existence results obtained in the previous sections, we give a useful theorem, which improves Theorem 4 in [26].

Theorem 5.3 *The problem* GNCP$(S, \mathbb{K})$ *has a solution if and only if the set-valued map defined, for all $x \in \mathbb{X}$, by*

$$Tx = \{z \in \mathbb{X}, z \in P_{\mathbb{K}}(x) - S(P_{\mathbb{K}}(x))\}$$

has a fixed point in $\mathbb{X}$. Moreover, if x_0 is a fixed point of T, then $\hat{x} = P_{\mathbb{K}}(x_0)$ is a solution of the problem GNCP$(S, \mathbb{K})$.

Proof Suppose that T has a fixed point x_0, that is,

$$x_0 \in P_{\mathbb{K}}(x_0) - S\big(P_{\mathbb{K}}(x_0)\big).$$

Then there exists $\hat{y} \in S(P_{\mathbb{K}}(x_0))$ such that

$$x_0 = P_{\mathbb{K}}(x_0) - \hat{y}.$$

Then if we denote by $\hat{x} = P_{\mathbb{K}}(x_0)$, then it is clear that $\hat{x} \in \mathbb{K}$, and by item 1 of Theorem 5.1 we get for all $x \in \mathbb{K}$,

$$\langle \hat{y}, x \rangle = \langle \hat{x} - x_0, x \rangle \geq 0,$$

then $\hat{y} \in \mathbb{K}'$. Therefore, by item 2 of Theorem 5.1 $\langle \hat{y}, \hat{x} \rangle = \langle \hat{x} - x_0, \hat{x} \rangle = 0$, which implies that $(\hat{x}, \hat{y})$ is a solution of $\mathrm{GNCP}(S, \mathbb{K})$.

Conversely, if $(\hat{x}, \hat{y})$ is a solution of $\mathrm{GNCP}(S, \mathbb{K})$, then denoting

$$x_0 = \hat{x} - \hat{y},$$

by Theorem 5.2 we get $\hat{x} = P_{\mathbb{K}}(x_0)$, and since $\hat{y} \in S(\hat{x}) \cap \mathbb{K}'$, we get $\hat{y} \in S(P_{\mathbb{K}}(x_0))$. Hence, $x_0 \in P_{\mathbb{K}}(x_0) - S(P_{\mathbb{K}}(x_0))$, and thus $x_0 \in Tx_0$. This completes the proof. □

Now we formulate an existence result for the $\mathrm{GNCP}(S, \mathbb{K})$ problem.

Theorem 5.4 *Let $(\mathbb{X}, \langle \cdot, \cdot \rangle)$ be a Hilbert space, and $\mathbb{K}$ be a closed convex cone in $\mathbb{X}$. Let $\{\varphi_i\}_{i \in I}$ be a family of lower semicontinuous functions from $\mathbb{X}$ to $\mathbb{R}_+$, and $a_i > 0$ and $b_i > 0$ be two families of positive real numbers. Suppose that the set-valued maps T and S defined before satisfy the supplementary condition*:

For all $i \in I$ and $(x, y) \in G_S$, there exist $z \in Tx \cap \mathbb{K}$ and $t \in S(z)$ such that

$$\max\big\{a_i \|x - z\|_{\mathbb{X}}, b_i \|y - t\|_{\mathbb{X}}\big\} \leq \varphi_i(y) - \varphi_i(t).$$

Then $\mathrm{GNCP}(S, \mathbb{K})$ *has a solution*.

Proof It suffices to replace T by T' defined from $\mathbb{K}$ into $2^{\mathbb{K}}$ as $T'(x) = T(x) \cap \mathbb{K}$ and apply Theorem 5.3 and Proposition 3.4. □

Example 5.5 Let $\mathbb{X} = \mathbb{R}$, $\mathbb{K} = \mathbb{R}_+$, and, for all $i \in I$, $a_i = b_i = 1$, $\varphi_i(x) = |x|$ for $x \in \mathbb{X}$, and $S(x) = [0, x]$ for all $x \in \mathbb{K}$. Then the GNCP problem becomes:

$$\mathrm{GNCP}(S, \mathbb{R}_+): \quad \begin{cases} \text{find } (\hat{x}, \hat{y}) \in \mathbb{R}_+ \times \mathbb{R} \\ \text{s.t. } \hat{y} \in [0, \hat{x}] \text{ and } \hat{x}\hat{y} = 0. \end{cases}$$

It is obvious that $T(x) = [0, x]$ for each $(x, y) \in G_S$. It is clear that, for all $x \geq 0$ and $y \in [0, x]$, we get

$$|x - y| + |y| \leq |x| \quad \Leftrightarrow \quad |x - y| \leq \varphi_i(x) - \varphi_i(y),$$

and choosing $z \in T(x)$ and $t \in S(z)$, we have:

1. for $x = y$, we choose $z = 0$ and $t = 0$, and then we have

$$\max\{|x|,|y|\} \leq \varphi_i(y);$$

2. for $y < x$, we choose $z = x - y + t$ and $t \leq \min\{x - y, y\}$, so that $|x - z| = |y - t|$, and then we get

$$|y - t| \leq \varphi_i(y) - \varphi_i(t).$$

Finally, by 1 and 2 we get

$$\max\{a_i|x - z|, b_i|y - t|\} \leq \varphi_i(y) - \varphi_i(t).$$

Then all assumptions of Theorem 5.4 hold, and hence problem $\mathrm{GNCP}(S,\mathbb{R}_+)$ has a solution, and the set of solutions is

$$\mathrm{Sol}\bigl(\mathrm{GNCP}(S,\mathbb{R}_+)\bigr) = \bigl\{(x,0); x \geq 0\bigr\}.$$

5.2 Differential inclusion in a nuclear space

Let $\mathbb{R}^d$ (with fixed $d \in \mathbb{N}^*$), set $\mathcal{D}(\mathbb{R}^d)$ to be the space of all complex-valued infinitely differentiable functions on $\mathbb{R}^d$ with compact support, and define the differential operator for each multiindex $\alpha \in \mathbb{N}^d$ with $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d)$ by

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\,\partial x_2^{\alpha_2}\cdots\partial x_d^{\alpha_d}},$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_d$. The space $\mathcal{D}(\mathbb{R}^d)$ is endowed by a locally convex topology defined by the family of separated seminorms

$$\|\varphi\|_N = \sup\bigl\{\bigl|D^\alpha\varphi(x)\bigr|; x \in \mathbb{R}^d \text{ and } |\alpha| \leq N\bigr\}.$$

Recall that a subset $B \subset \mathcal{D}(\mathbb{R}^d)$ is bounded if for some compact $K \subset \mathbb{R}^d$, we have $B \subset \mathcal{D}(K)$ and there are numbers $M_N < \infty$ such that every $\varphi \in B$ satisfies the inequalities

$$\|\varphi\|_N \leq M_N, \quad N = 0, 1, 2, \ldots.$$

It is worth noting that $\mathcal{D}(\mathbb{R}^d)$ endowed with the limit inductive topology of $\{\mathcal{D}(K_n)\}_n$ is a complete nonmetric space, where $(K_n)_{n\in\mathbb{N}}$ is an exhaustive sequence of compact subsets, that is, for every $n \in \mathbb{N}$, K_n included in the interior of K_{n+1}, and $\mathbb{R}^d = \cup_n K_n$; for more details, see [29].

Now, let $\mathcal{D}'(\mathbb{R}^d)$ be the strong dual of $\mathcal{D}(\mathbb{R}^d)$, also endowed with the locally convex topology generated by an uncountable separated family of seminorms over the bounded subset of $\mathcal{D}(\mathbb{R}^d)$ denoted by τ, that is,

$$p_B(f) = \sup_{\varphi\in B}\bigl|\langle f,\varphi\rangle\bigr|, \quad B \subset \mathcal{D}\bigl(\mathbb{R}^d\bigr) \quad \text{bounded}.$$

Definition 5.6 In a Hausdorff locally convex space $(\mathbb{X}, \{p_i\}_{i\in\Lambda})$, a convex cone $\mathbb{K} \subset \mathbb{X}$ is supernormal [13] if for each $i \in \Lambda$, there exists a continuous linear form $f_i \in \mathbb{K}'$ (dual cone) such that, for each $x \in \mathbb{K}$, we have

$$p_i(x) \leq f_i(x).$$

$\mathcal{D}'(\mathbb{R}^d)$ endowed with τ-topology is a nuclear space [17], and we have the following:

Proposition 5.7 *In a nuclear space $\mathbb{X}$, a convex cone $\mathbb{K} \subset \mathbb{X}$ is τ-supernormal if and only if it is τ-normal.*

It is shown in [17] that the cone $\mathbb{K}$ defined by

$$\mathbb{K} = \bigl\{\Lambda \in \mathcal{D}'\bigl(\mathbb{R}^d\bigr); \langle\Lambda, \varphi\rangle \geq 0, \forall\varphi \in \mathcal{C}\bigr\}$$

is τ-normal cone, where $\mathcal{C} = \{\varphi \in \mathcal{D}(\mathbb{R}^d); \varphi(x) \geq 0, \forall x \in \mathbb{R}^d\}$, and hence $\mathbb{K}$ is τ-supernormal.

Next, we propose to solve the partial differential inclusion problem;

$$(\mathcal{P}): \quad \begin{cases} \text{find a locally integrable function } u \in L^1_{\text{loc}}(\mathbb{R}^d) \text{ such that} \\ D^\alpha u \in F(u) \text{ a.e. on } \mathbb{R}^d, \end{cases}$$

where $\alpha \in \mathbb{N}^d$ a multiindex, and $F : L^1_{\text{loc}}(\mathbb{R}^d) \longrightarrow 2^{L^1_{\text{loc}}(\mathbb{R}^d)}$.

Given $u \in L^1_{\text{loc}}(\mathbb{R}^d)$, it is shown in [29] that u defines a regular distribution, denoted $\Lambda_u \in \mathcal{D}'(\mathbb{R}^d)$, as follows:

$$\Lambda_u(\varphi) = \int_{\mathbb{R}^d} u(x)\varphi(x)\,dx$$

for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$.

Also, if $u \in L^1_{\text{loc}}(\mathbb{R}^d)$, we know that $\Lambda_{D^\alpha u} = D^\alpha \Lambda_u$, and hence we propose to solve problem $(\mathcal{P})$ in regular distributions setting and consider the differentiability in the weak sense. Problem $(\mathcal{P})$ is transformed by the canonical isomorphism

$$\mathcal{G} : L^1_{\text{loc}}\bigl(\mathbb{R}^d\bigr) \longrightarrow \mathcal{G}\bigl(\mathcal{D}'\bigl(\mathbb{R}^d\bigr)\bigr)$$

to

$$\bigl(\mathcal{P}'\bigr): \quad \begin{cases} \text{find a regular distribution } \Lambda_u \in \mathcal{D}'(\mathbb{R}^d) \text{ such that} \\ D^\alpha \Lambda_u \in \mathcal{F}(\Lambda_u) \text{ a.e. on } \mathbb{R}^d, \end{cases}$$

where $\mathcal{F}$ is the set-valued map defined from $\mathcal{D}'(\mathbb{R}^d)$ into $2^{\mathcal{D}'(\mathbb{R}^d)}$ by

$$\Lambda_v \in \mathcal{F}(\Lambda_u) \quad \Leftrightarrow \quad v \in F(u).$$

Now, passing to the second part of our developments, there is no chance that problem $(\mathcal{P}')$ has a solution, so we will give a sufficient condition on the set-valued map F in order that the problem has at least one solution. For this, we define two subsets $\mathcal{I}$ and $\mathcal{J}$ of $\mathcal{D}'(\mathbb{R}^d)$ by

$$\mathcal{I} = \left\{\Lambda_f; f \in L^1_{\mathrm{loc}}\left(\mathbb{R}^d\right), \Lambda_f(\varphi) = \int_{\mathbb{R}^d} f(x) D^\alpha \varphi(x)\, dx \text{ for each } \varphi \in \mathcal{D}\left(\mathbb{R}^d\right)\right\};$$
$$\forall \Lambda_u \in \mathcal{D}'\left(\mathbb{R}^d\right): \quad \mathcal{J}(\Lambda_u) = \left\{\Lambda_f \in \mathcal{I}; u(x) \geq (-1)^{|\alpha|} f(x), \forall x \in \mathbb{R}^d\right\},$$

and for each regular distribution $\Lambda_u \in \mathcal{D}'(\mathbb{R}^d)$, we define the set-valued maps $\mathcal{R}$ and $\mathcal{T}$ as follows:

$$\mathcal{R}(\Lambda_u) = \left\{\Lambda_v \in \mathcal{D}'\left(\mathbb{R}^d\right); \forall \varphi \in \mathcal{C}, \langle \Lambda_u - \Lambda_v, \varphi\rangle \geq 0\right\};$$
$$\mathcal{T}(\Lambda_u) = \left\{\Lambda_v \in \mathcal{R}(\Lambda_u); D^\alpha v \in F(u) \text{ a.e. on } \mathbb{R}^d\right\}.$$

It is obvious that $\mathcal{R}(\Lambda_u)$ is nonempty since $\Lambda_u \in \mathcal{R}(\Lambda_u)$, and for $\mathcal{T}(\Lambda_u)$, we need the next lemma.

Lemma 5.8 *If for each $\Lambda_u \in \mathcal{D}'(\mathbb{R}^d)$, $\mathcal{F}(\Lambda_u) \cap \mathcal{J}(\Lambda_u) \neq \emptyset$, then $\mathcal{T}(\Lambda_u)$ is a nonempty subset of $\mathcal{D}'(\mathbb{R}^d)$.*

Proof Let f be a locally integrable function, and let $\Lambda_u \in \mathcal{D}'(\mathbb{R}^d)$. Then the function

$$\varphi \mapsto \int_{\mathbb{R}^d} f(x) D^\alpha \varphi(x)\, dx \quad \text{is an element of } \mathcal{F}(\Lambda_u),$$

and a simple calculation leads to

$$\begin{aligned}\int_{\mathbb{R}^d} f(x) D^\alpha \varphi(x)\, dx &= (-1)^{|\alpha|} \int_{\mathbb{R}^d} D^\alpha f(x) \varphi(x)\, dx \\ &= \int_{\mathbb{R}^d} D^\alpha\left[(-1)^{|\alpha|} f(x)\right] \varphi(x)\, dx.\end{aligned}$$

Put $v(x) = (-1)^{|\alpha|} f(x)$ for $x \in \mathbb{R}^d$. Then $v \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, and

$$\int_{\mathbb{R}^d} D^\alpha v(x) \varphi(x)\, dx = \Lambda_{D^\alpha v}(\varphi),$$

which leads to $\Lambda_{D^\alpha v} \in \mathcal{F}(\Lambda_u)$. Thus, $D^\alpha v \in F(u)$.

For each $\varphi \in \mathcal{C}$, we have

$$\begin{aligned}\Lambda_u(\varphi) - \Lambda_v(\varphi) &= \Lambda_u(\varphi) - (-1)^{|\alpha|} \Lambda_f(\varphi) \\ &= \int_{\mathbb{R}^d} \left[u(x) - (-1)^{|\alpha|} f(x)\right] \geq 0.\end{aligned}$$

Hence, $\Lambda_v \in \mathcal{T}(\Lambda_u)$. □

As an interesting application of the main result, we can state and prove the following existence theorem.

Theorem 5.9 *If $\mathbb{K}$ and $\mathcal{R}$ are as before and $\mathcal{T}$ satisfies the assumption in the previous lemma, then problem $(\mathcal{P}')$ has a solution.*

Proof By assumption, for each $\Lambda_u \in \mathcal{D}'(\mathbb{R}^d)$, there exists $\Lambda_v \in \mathcal{T}(\Lambda_u)$ such that:

(i) $D^\alpha \Lambda_v \in \mathcal{F}(\Lambda_u)$, and

(ii) $\Lambda_v \in \mathcal{R}(\Lambda_u)$.

Then, for every $\varphi \in \mathcal{D}(\mathbb{R}^d)$, we have

$$\langle \Lambda_u - \Lambda_v, \varphi \rangle \geq 0 \quad \Longleftrightarrow \quad (\Lambda_u - \Lambda_v)(\varphi) \geq 0,$$

which implies that $(\Lambda_u - \Lambda_v) \in \mathbb{K}$; since $\mathbb{K}$ is a supernormal cone, for each bounded subset B of $\mathcal{D}(\mathbb{R}^d)$, there exists $f_B \in \mathbb{K}'$ such that

$$p_B(\Lambda_u - \Lambda_v) \leq f_B(\Lambda_u - \Lambda_v) \quad \Longleftrightarrow \quad p_B(\Lambda_u - \Lambda_v) \leq f_B(\Lambda_u) - f_B(\Lambda_v).$$

All assumptions of our former result in Proposition 3.4 hold. Therefore, $\mathcal{T}$ has a fixed point $\Lambda_{u^\star} \in \mathcal{D}'(\mathbb{R}^d)$, that is,

$$\Lambda_{u^\star} \in \mathcal{T}(\Lambda_{u^\star}) \quad \Leftrightarrow \quad D^\alpha u^\star \in F\bigl(u^\star\bigr). \qquad \square$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All the authors contributed equally. All authors read and approved the final manuscript.

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