Nieto et al. *Fixed Point Theory and Applications* (2016) 2016:98
DOI 10.1186/s13663-016-0590-8

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

Random fixed point theorems in partially ordered metric spaces

Juan J Nieto[1*], Abdelghani Ouahab[2] and Rosana Rodríguez-López[1]

*Correspondence:
juanjose.nieto.roig@usc.es
[1]Departamento de Análise Matemática, Estatística e Optimización, Facultade de Matemáticas, Universidade de Santiago de Compostela, Santiago de Compostela, 15782, Spain
Full list of author information is available at the end of the article

Abstract
We present the random version in partially ordered metric spaces of the classical Banach contraction principle and some of its generalizations to ordered metric spaces.
The results are used to prove the existence of solutions for random differential equations with boundary conditions.

MSC: 47H10; 47H30; 54H25

Keywords: fixed point; partially ordered set; random operator; random differential equation

1 Introduction

In 1922, the Polish mathematician Stefan Banach established a remarkable fixed point theorem, the famous contraction principle, which is one of the most important results of analysis. It is the most widely applied fixed point result in different areas of mathematics and applications. It requires the structure of a complete metric space with contractive condition on the map which is easy to test in many situations. It has been generalized in many different directions. Moreover, the proof of the Banach contraction principle gives a sequence of approximate solutions and useful information as regards the rate of convergence toward the fixed point. This is very important since a fundamental principle both in mathematics and computer science is iteration. Particularly, fixed point iterations and monotone iterative techniques are the core methods when solving a large class of abstract and applied mathematical problems and play an important role in many algorithms.

The existence of fixed points for self-mappings in partially ordered sets has been considered in [1, 2], where some applications to matrix equations are presented. This result was extended by Nieto *et al.* [3] and Nieto and Rodríguez-López [4, 5] in partially ordered sets and applied to study ordinary differential equations.

Probabilistic functional analysis is an important mathematical area of research due to its applications to probabilistic models in applied problems. Random operator theory is needed for the study of various classes of random equations. Indeed, in many cases, the mathematical models or equations used to describe phenomena in biology, physics, engineering, and systems sciences contain certain parameters or coefficients which have specific interpretations, but whose values are unknown. Therefore, it is more realistic to consider such equations as random operator equations. These equations are much more difficult to handle mathematically than deterministic equations. Important contributions to the study of the mathematical aspects of such random equations have been presented in [6, 7] among others. The problem of fixed points for random mappings was initiated by the Prague school of probability research. The first results were studied in 1955-1956 by Špaček and Hanš in the context of Fredholm integral equations with random kernel. In a separable metric space, random fixed point theorems for contraction mappings were proved by Hanš [8, 9], Hanš and Špaček [10] and Mukherjee [11, 12]. Then random fixed point theorems of Schauder or Krasnosel'skii type were given by Mukherjea (*cf.* Bharucha-Reid [6], p. 110), Bharucha-Reid [13] and Itoh [14]. Now it has become a full fledged research area and a vast amount of mathematical activities have been carried out in this direction (see, for examples, [15–18]). The existence of a random fixed point for mappings in partially ordered metric spaces and partially ordered probabilistic metric spaces was studied, for example, in [19, 20].

The goal of this paper is to establish a random version of some fixed point theorems in partially ordered and ordered L-spaces. Finally, we apply our results to random differential equations.

2 Some basic concepts on order and *L*-spaces

Let X be a nonempty set and let $F : X \to X$ be an operator. We denote the successive iterations of F as $F^0 = 1_X$, $F^1 = F$, $F^n = F \circ F^{n-1}$, for $n \in \mathbb{N}$, $n \geq 2$.

Definition 2.1 If $(X, \preceq)$ is a partially ordered set and $f : X \to X$, we say that f is monotone nondecreasing if

$$x \preceq y, \quad x, y \in X \quad \Longrightarrow \quad F(x) \preceq F(y).$$

Remark 2.1 The above definition coincides with the notion of a nondecreasing function in the case where $X = \mathbb{R}$ and $\preceq$ represents the usual total order in $\mathbb{R}$.

Let X be a metric space and $F : X \to X$ be an operator.

Definition 2.2 The set of all nonempty invariant subsets of F is

$$I(F) = \{Y \subset X : F(Y) \subseteq Y\}.$$

We denote by $s(X)$ the set of sequences in X, that is,

$$s(X) = \{(x_n)_{n\in\mathbb{N}} : x_n \in X, n \in \mathbb{N}\}.$$

Fréchet introduced in [21] the notion of L-space as follows.

Definition 2.3 An L-space is a triple $(X, c(X), \lim)$, where X is a set, $c(X) \subseteq s(X)$ is a family of sequences of elements of X, and $\lim : c(X) \to X$ is a mapping having the following two properties:

(i) If $x_n = x$, for all $n \in \mathbb{N}$, then $(x_n)_{n\in\mathbb{N}} \in c(X)$ and $\lim(x_n)_{n\in\mathbb{N}} = x$.

(ii) If $(x_n)_{n\in\mathbb{N}} \in c(X)$ and $\lim(x_n)_{n\in\mathbb{N}} = x$, then for all subsequences $(x_{n_i})_{i\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ we have $(x_{n_i})_{i\in\mathbb{N}} \in c(X)$ and $\lim(x_{n_i})_{i\in\mathbb{N}} = x$.

The elements of $c(X)$ are called convergent sequences and $\lim(x_n)_{n\in\mathbb{N}} = x$ is the limit of the sequence, also written $x_n \to x$ as $n \to \infty$. An L-space is denoted by $(X, \to)$.

Definition 2.4 A mapping $F : X \to X$ is said to be orbitally continuous if $x \in X$, and $F^{n_i}(x) \to a$ as $i \to \infty$ implies that $F^{n_i+1}(x) \to F(a)$ as $i \to \infty$.

Definition 2.5 Let $(X, \preceq)$ be a partially ordered set and

$$X_{\preceq} := \{(x,y) \in X \times X : x \preceq y \text{ or } y \preceq x\}.$$

For each $x, y \in X$ with $x \preceq y$, we denote

$$[x,y]_{\preceq} := \{z \in X : x \preceq z \preceq y\}.$$

Definition 2.6 For X a nonempty set, $(X, \to, \preceq)$ is an ordered L-space if:

(i) $(X, \to)$ is an L-space.

(ii) $(X, \preceq)$ is a partially ordered set.

(iii) If $(x_n)_{n\in\mathbb{N}} \to x$, $(y_n)_{n\in\mathbb{N}} \to y$, and $x_n \preceq y_n$, for each $n \in \mathbb{N}$, then $x \preceq y$.

For L-spaces equipped with a partial ordering $\preceq$, we obtain the following result, which requires us to define first the concept of orbitally $\preceq$-continuous function.

Definition 2.7 Let $(X, \to)$ be an L-space equipped with a partial ordering $\preceq$. We say that $F : (X, \to) \to (X, \to)$ is orbitally $\preceq$-continuous if $x \in X$ and

$$F^{n_i}(x) \to a \quad \text{as } i \to \infty,$$
$$\bigl(F^{n_i}(x), a\bigr) \in X_{\preceq}, \quad \forall i \in \mathbb{N},$$

imply that

$$F^{n_i+1}(x) \to F(a) \quad \text{as } i \to \infty.$$

3 Random variable

If X is a metric space, we shall use $B(X)$ to denote the Borel σ-algebra on X. Let $(\Omega, \mathcal{F})$ be a measurable space. The expression $\mathcal{F} \otimes B(X)$ denotes the smallest σ-algebra on $\Omega \times X$ which contains all the sets $Q \times S$, where $Q \in \mathcal{F}$ and $S \in B(X)$. Given a topological space E, we denote $\mathcal{P}(E) = \{Y \subset E\colon Y \neq \emptyset\}$, $\mathcal{P}_{cl}(E) = \{Y \in \mathcal{P}(E)\colon Y \text{ closed}\}$. Let $F : X \to \mathcal{P}(Y)$ be a multivalued map. A single-valued map $f : X \to Y$ is said to be a selection of F (and we write $f \subset F$) whenever $f(x) \in F(x)$ for every $x \in X$. Let (Ω, Σ) be a measurable space and $F : \Omega \to \mathcal{P}(X)$ a multivalued mapping, F is called measurable if $F_+^{-1}(Q) = \{\omega \in \Omega : F(\omega) \subset Q\}$ is measurable for every $Q \in \mathcal{P}_{cl}(X)$; equivalently, for every U open subset of X, the set $F_-^{-1}(U) = \{\omega \in \Omega : F(\omega) \cap U \neq \emptyset\}$ is measurable.

Proposition 3.1 ([22]) *Let $(\Omega, \mathcal{U})$ be a measurable space and $F : \Omega \to \mathcal{P}_{cl}(X)$ be a multivalued map. If, for every $C \in \mathcal{P}(X)$, we have $F_-^{-1}(C) \in \mathcal{U}$, then F is measurable.*

Definition 3.1 Recall that a mapping $f : \Omega \times X \to X$ is said to be a random operator if, for any $x \in X$, $f(\cdot, x)$ is measurable.

Definition 3.2 A random fixed point of f is a measurable function $y : \Omega \to X$ such that

$$y(\omega) = f\bigl(\omega, y(\omega)\bigr) \quad \text{for all } \omega \in \Omega.$$

Equivalently, it is a measurable selection for the multivalued map $\operatorname{Fix} F : \Omega \to \mathcal{P}(X)$ defined by

$$\operatorname{Fix} F(\omega) = \bigl\{x \in X : x = f(\omega, x)\bigr\}, \quad \omega \in \Omega.$$

Theorem 3.1 ([23], Theorem 6.1) *Let X be a separable metric space, Y a metric space and $f : \Omega \times X \to Y$ a continuous random operator. Then f is a measurable random operator.*

Theorem 3.2 *Let (Ω, Σ) be a measurable space, Y be a separable metric space, and $\phi : \Omega \to \mathcal{P}_{cl}(Y)$ be a measurable multivalued function. Then ϕ has a measurable selection.*

Let X, Y be two locally compact metric spaces and $f : \Omega \times X \to Y$. By $C(X, Y)$, we denote the space of continuous functions from X into Y endowed with the compact-open topology.

Lemma 3.1 ([24]) *f is a Carathéodory function if and only if $\omega \to r(\omega)(\cdot) = f(\omega, \cdot)$ is a measurable function from Ω to $C(X, Y)$.*

4 Random fixed point theorems

Theorem 4.1 *Let $(\Omega, \mathcal{F})$ be a measurable space, $(X, d, \preceq)$ be a separable complete partially ordered metric space, and $F : \Omega \times X \to X$ be a continuous random operator such that, for each $\omega \in \Omega$, the function $F(\omega, \cdot)$ is a monotone (either order-preserving or order-reversing) operator. Suppose that the following assertions hold:*

($\mathcal{H}_1$) *For each $\omega \in \Omega$, there exists $k(\omega) \in [0, 1)$ such that*

$$d\bigl(F(\omega, x_1), F(\omega, x_2)\bigr) \le k(\omega) d(x_1, x_2) \quad \textit{for each } x_1, x_2 \in X, x_1 \succeq x_2.$$

($\mathcal{H}_2$) *There exists a random variable $x_0 : \Omega \to X$ with*

$$x_0(\omega) \preceq F\bigl(\omega, x_0(\omega)\bigr), \quad \forall \omega \in \Omega$$

or

$$x_0(\omega) \succeq F\bigl(\omega, x_0(\omega)\bigr), \quad \forall \omega \in \Omega.$$

Then there exists a random variable $x : \Omega \to X$ which is a random fixed point of F.

Proof If, for each $\omega \in \Omega$, $F(\omega, x_0(\omega)) = x_0(\omega)$, then x_0 is a random fixed point of F. Suppose that, for some $\omega \in \Omega$, $F(\omega, x_0(\omega)) \neq x_0(\omega)$. We define the sequence $y_0(\omega) = x_0(\omega)$, $y_n(\omega) =$

$F(\omega, y_{n-1}(\omega))$, $\omega \in \Omega, n \in \mathbb{N}$. From the condition $(\mathcal{H}_2)$, we have, for each $\omega \in \Omega$ and $n \in \mathbb{N}$, one of the following relations:

$$y_n(\omega) \preceq y_{n+1}(\omega) \quad \text{or} \quad y_n(\omega) \succeq y_{n+1}(\omega).$$

Using $(\mathcal{H}_1)$, we get

$$d\big(y_n(\omega), y_{n+1}(\omega)\big) \leq \big[k(\omega)\big]^n d\big(x_0(\omega), F\big(\omega, x_0(\omega)\big)\big), \quad \omega \in \Omega,$$

and, as a consequence,

$$d\big(y_n(\omega), y_{n+m}(\omega)\big) \leq \big(\big[k(\omega)\big]^n + \big[k(\omega)\big]^{n+1} + \cdots + \big[k(\omega)\big]^{n+m-1}\big) d\big(y_1(\omega), x_0(\omega)\big).$$

So $(y_n(\omega))_{n\in\mathbb{N}}$ is a Cauchy sequence, for every $\omega \in \Omega$. Let $y_*(\omega) = \lim_{n\to\infty} y_n(\omega)$, $\omega \in \Omega$. Since $y_0(\cdot)$ is measurable, then $y_1(\cdot)$ is measurable. Hence, by induction, we can easily prove that, for each $n \in \mathbb{N}$, the function $\omega \to y_n(\omega)$ is measurable. This implies that $y_*(\cdot)$ is measurable. Now, we show that $y_*(\omega) = F(\omega, y_*(\omega))$. It is clear that, for each $\omega \in \Omega$, $d(y_*(\omega), F(\omega, y_*(\omega))) = \lim_{n\to\infty} d(y_n(\omega), F(\omega, y_n(\omega)))$. Then

$$d\big(y_n(\omega), F\big(\omega, y_n(\omega)\big)\big) = d\big(F\big(\omega, y_{n-1}(\omega)\big), F\big(\omega, y_n(\omega)\big)\big) \leq k(\omega) d\big(y_{n-1}(\omega), y_n(\omega)\big),$$

for each $\omega \in \Omega$. Hence, for each $\omega \in \Omega$,

$$d\big(y_n(\omega), F\big(\omega, y_n(\omega)\big)\big) \leq k(\omega) d\big(y_{n-1}(\omega), y_*(\omega)\big) + k(\omega) d\big(y_*(\omega), y_n(\omega)\big) \to 0 \quad \text{as } n \to \infty.$$

Thus,

$$y_*(\omega) = F\big(\omega, y_*(\omega)\big) \quad \text{for each } \omega \in \Omega. \qquad \square$$

Next, we consider the following additional condition proposed in [1]:

$(\mathcal{H}_3)$ Every pair of elements of X has a lower bound or an upper bound.

Remark 4.1 ([5]) The above condition $(\mathcal{H}_3)$ is equivalent to:

$(\overline{\mathcal{H}}_3)$ For every $x, y \in X$, there exists $z \in X$ that is comparable to x and y.

Remark 4.2 In the statement of Theorem 4.1, if we impose, additionally, the condition $(\mathcal{H}_3)$ (equivalently, $(\overline{\mathcal{H}}_3)$), then we can conclude the existence of a random variable $x : \Omega \to X$ which is the unique random fixed point of F.

Indeed, following the ideas in [1], we prove that, if we take any random variable $\bar{x}_0 : \Omega \to X$ and we define the sequence

$$\bar{y}_0(\omega) = \bar{x}_0(\omega), \qquad \bar{y}_n(\omega) = F\big(\omega, \bar{y}_{n-1}(\omega)\big), \quad \omega \in \Omega, n \in \mathbb{N},$$

we get $(\bar{y}_n(\omega))_{n\in\mathbb{N}} \to y_*(\omega)$, as $n \to \infty$, for every $\omega \in \Omega$, where y_* is the fixed point of F obtained in the proof of Theorem 4.1. If $\bar{x}_0(\omega)$ is comparable to $x_0(\omega)$ for every $\omega \in \Omega$, it is obvious, since $F(w,\bar{x}_0(w))$ is comparable to $F(w,x_0(w))$ for every $\omega\in\Omega$, so that $\bar{y}_n(\omega)$ is comparable to $y_n(\omega)$ for every $\omega\in\Omega$. Hence

$$\begin{aligned} d\big(y_n(\omega),\bar{y}_n(\omega)\big) &= d\big(F\big(\omega,y_{n-1}(\omega)\big),F\big(\omega,\bar{y}_{n-1}(\omega)\big)\big) \\ &\leq k(\omega)d\big(y_{n-1}(\omega),\bar{y}_{n-1}(\omega)\big) \leq \big[k(\omega)\big]^n d\big(y_0(\omega),\bar{y}_0(\omega)\big), \quad \omega\in\Omega, \end{aligned}$$

and $(\bar{y}_n(\omega))_{n\in\mathbb{N}}$ tends to $y_*(\omega)$, as $n\to\infty$, for every $w\in\Omega$.

On the other hand, for an arbitrary random variable $\bar{x}_0:\Omega\to X$ then, for each $\omega\in\Omega$, there exists $z(\omega)\in X$ that is comparable to $x_0(\omega)$ and $\bar{x}_0(\omega)$ simultaneously, then if we define

$$z_0(\omega)=z(\omega), \qquad z_n(\omega)=F\big(\omega,z_{n-1}(\omega)\big), \quad \omega\in\Omega, n\in\mathbb{N},$$

then $y_n(\omega)$ is comparable to $z_n(\omega)$, for every $\omega\in\Omega$, and $\bar{y}_n(\omega)$ is comparable to $z_n(\omega)$, for every $\omega\in\Omega$. Therefore,

$$\begin{aligned} d\big(y_n(\omega),\bar{y}_n(\omega)\big) &\leq d\big(y_n(\omega),z_n(\omega)\big)+d\big(z_n(\omega),\bar{y}_n(\omega)\big) \\ &\leq \big[k(\omega)\big]^n d\big(y_0(\omega),z(\omega)\big)+\big[k(\omega)\big]^n d\big(z(\omega),\bar{x}_0(\omega)\big), \quad \omega\in\Omega, \end{aligned}$$

which proves that $(\bar{y}_n(\omega))_{n\in\mathbb{N}}\to y_*(\omega)$, as $n\to\infty$, for every $\omega\in\Omega$.

In the following result, we replace the monotonicity of $F(\omega,\cdot)$ by a more general condition.

Theorem 4.2 *Let $(X,\preceq)$ be a partially ordered set, (X,d) be a complete separable metric space, and $(\Omega,\mathcal{U})$ be a measurable space. Let $F:\Omega\times X\to X$ be a joint measurable operator such that, for every $\omega\in\Omega$, $F(\omega,\cdot)$ maps comparable elements into comparable elements, that is,*

$$x\preceq y \quad\Longrightarrow\quad \textit{for every } \omega\in\Omega, \quad \big[F(\omega,x)\preceq F(\omega,y) \textit{ or } F(\omega,y)\preceq F(\omega,x)\big].$$

($\mathcal{H}_4$) *If $(x_n)_{n\in\mathbb{N}}$ is a sequence in X whose consecutive terms are comparable and $(x_n)_{n\in\mathbb{N}}\to x$, then there exists a subsequence $(x_{n_p})_{p\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that every term is comparable to the limit x.*

Suppose that $(\mathcal{H}_1)$ and $(\mathcal{H}_3)$ hold. If there exists $x_0:\Omega\to X$, a random variable with $x_0(\omega)$ comparable to $F(\omega,x_0(\omega))$ for every $\omega\in\Omega$, then F has a unique random fixed point.

Proof If $x_0(\omega)=F(\omega,x_0(\omega))$, for every $\omega\in\Omega$, then the proof is finished. Suppose that $F(\omega,x_0(\omega))\neq x_0(\omega)$, for some $\omega\in\Omega$. We define the sequence $y_0(\cdot)=x_0(\cdot)$ and $y_n(\cdot)=F(\cdot,y_{n-1}(\cdot))$, for $n\in\mathbb{N}$.

Since, for each $\omega\in\Omega$, we can compare $y_0(\omega)=x_0(\omega)$ to $y_1(\omega)=F(\omega,x_0(\omega))$, by the hypothesis on F, we can compare $y_1(\omega)=F(w,y_0(\omega))$ to $y_2(\omega)=F(w,y_1(\omega))$, so that, for each $\omega\in\Omega$ and each $n\in\mathbb{N}$, we see that $y_n(\omega)$ is comparable to $y_{n-1}(\omega)$, for every $\omega\in\Omega$ and

$n \in \mathbb{N}$. Then from $(\mathcal{H}_1)$, we get

$$d\big(y_n(\omega), y_{n+1}(\omega)\big) \le \big[k(\omega)\big]^n d\big(x_0(\omega), F\big(\omega, x_0(\omega)\big)\big), \quad \omega \in \Omega.$$

This implies that $(y_n(\omega))_{n\in\mathbb{N}}$ is a Cauchy sequence in X, for each $\omega \in \Omega$. Since X is a complete metric space, then, for each $\omega \in \Omega$, there exists $y_*(\omega) \in X$ such that $\lim_{n\to\infty} y_n(\omega) = y_*(\omega)$. From the definition of y_n and since F is joint measurable, $y_* : \Omega \to X$ is a measurable function. Finally, we prove that y_* is a random fixed point of F, that is, $F(\omega, y_*(\omega)) = y_*(\omega)$, for every $\omega \in \Omega$. From $(\mathcal{H}_4)$, for each $\omega \in \Omega$, there exists a subsequence $(y_{n_p}(\omega))_{p\in\mathbb{N}}$ of $(y_n(\omega))_{n\in\mathbb{N}}$ such that every term is comparable to $y_*(\omega)$. Therefore,

$$d\big(y_*(\omega), F\big(\omega, y_*(\omega)\big)\big) \le d\big(y_*(\omega), y_{n_p}(\omega)\big) + k(\omega) d\big(y_{n_p-1}(\omega), y_*(\omega)\big) \to 0 \quad \text{as } p \to \infty.$$

Hence

$$y_*(\omega) = F\big(\omega, y_*(\omega)\big) \quad \text{for each } \omega \in \Omega.$$

The uniqueness follows similarly to the proof in Remark 4.2. □

Very recently, the random fixed point theory in vector metric spaces was studied by Sinacer *et al.* in [25]. Next, we present the random version of the Perov fixed point theorem in partially ordered generalized metric spaces in the following sense.

Definition 4.1 Let X be a nonempty set. By a vector-valued metric on X, we mean a map $d : X \times X \to \mathbb{R}^k$ with the following properties:

(i) $d(u, v) \ge 0$ for all $u, v \in X$; if $d(u, v) = 0$ then $u = v$;
(ii) $d(u, v) = d(v, u)$ for all $u, v \in X$;
(iii) $d(u, v) \le d(u, w) + d(w, v)$ for all $u, v, w \in X$.

We call the pair (X, d) a generalized metric space with

$$d(x, y) := \begin{pmatrix} d_1(x, y) \\ \vdots \\ d_k(x, y) \end{pmatrix}.$$

Notice that d is a generalized metric space on X if and only if d_i, $i = 1, \dots, k$, are metrics on X. The inequalities in Definition 4.1 are understood componentwise.

By the same method as used in Theorem 4.1, we can easily prove the following result.

Theorem 4.3 *Let $(\Omega, \mathcal{U})$ be a measurable space, X be a real separable complete generalized metric space with partial ordering $\preceq$ and $F : \Omega \times X \to X$ be a continuous random operator such that, for each $\omega \in \Omega$, the function $F(\omega, \cdot)$ is a monotone* (*either order-preserving or order-reversing*) *operator. Suppose that the following assertions hold*:

$(\overline{\mathcal{H}_1})$ *for every $\omega \in \Omega$, there exists $M(\omega) \in \mathcal{M}_{k\times k}(\mathbb{R}_+)$, a random variable matrix such that, for every $\omega \in \Omega$, $\|M(\omega)\| < 1$ and*

$$d\big(F(\omega, x_1), F(\omega, x_2)\big) \le M(\omega) d(x_1, x_2) \quad \textit{for each } x_1, x_2 \in X, x_1 \preceq x_2$$

and $(\mathcal{H}_2)$, *then there exists a random variable* $x : \Omega \to X$ *that is a random fixed point of* F (*the unique random fixed point of* F *if the condition* $(\mathcal{H}_3)$ *holds*).

Proof Similarly to the proof of Theorem 4.1 and with an analogous construction, using $(\overline{\mathcal{H}_1})$, we get

$$d\big(y_n(\omega), y_{n+1}(\omega)\big) \preceq \big[M(\omega)\big]^n d\big(x_0(\omega), F\big(\omega, x_0(\omega)\big)\big) \quad \text{for every } \omega \in \Omega,$$

and, therefore, for each $\omega \in \Omega$,

$$\begin{aligned} d\big(y_n(\omega), y_{n+m}(\omega)\big) &\preceq \big(\big[M(\omega)\big]^n + \big[M(\omega)\big]^{n+1} + \cdots + \big[M(\omega)\big]^{n+m-1}\big) d\big(y_1(\omega), x_0(\omega)\big) \\ &= \big[M(\omega)\big]^n \big(I + M(\omega) + \cdots + \big[M(\omega)\big]^{m-1}\big) d\big(y_1(\omega), x_0(\omega)\big). \end{aligned}$$

Since $\|M(\omega)\| < 1$, the series $\sum_{n=0}^{\infty} \|M(\omega)\|^n$ is convergent and, in consequence, the series $\sum_{n=0}^{\infty} \|[M(\omega)]^n\|$ is also convergent and, therefore, $\sum_{n=0}^{\infty} [M(\omega)]^n$ is convergent. Since all the coefficients of $M(\omega)$ are nonnegative, we get

$$\begin{aligned} d\big(y_n(\omega), y_{n+m}(\omega)\big) &\preceq \big(\big[M(\omega)\big]^n + \big[M(\omega)\big]^{n+1} + \cdots + \big[M(\omega)\big]^{n+m-1}\big) d\big(y_1(\omega), x_0(\omega)\big) \\ &= \big[M(\omega)\big]^n \sum_{l=0}^{\infty} \big[M(\omega)\big]^l d\big(y_1(\omega), x_0(\omega)\big). \end{aligned}$$

Taking norms in the above inequality and using $\|M(\omega)\| < 1$ or, in other words, that the matrix $[M(\omega)]^n$ converges to the null matrix as $n \to \infty$, it is easy to deduce that $(y_n(\omega))_{n\in\mathbb{N}}$ is a Cauchy sequence, for every $\omega \in \Omega$. We take $y_*(\omega) = \lim_{n\to\infty} y_n(\omega)$, $\omega \in \Omega$. Similarly to the proof of Theorem 4.1, y_* is measurable. It is obvious, for each $\omega \in \Omega$, that $d(y_*(\omega), F(\omega, y_*(\omega))) = \lim_{n\to\infty} d(y_n(\omega), F(\omega, y_n(\omega)))$. It comes from

$$\begin{aligned} &\big|d_i\big(y_*(\omega), F\big(\omega, y_*(\omega)\big)\big) - d_i\big(y_n(\omega), F\big(\omega, y_n(\omega)\big)\big)\big| \\ &\quad \le d_i\big(y_*(\omega), y_n(\omega)\big) + d_i\big(F\big(\omega, y_n(\omega)\big), F\big(\omega, y_*(\omega)\big)\big), \end{aligned}$$

for $\omega \in \Omega$ and $i = 1, \ldots, k$. The proof is finished analogously to the proof of Theorem 4.1 by replacing $k(\omega)$ by $M(\omega)$. □

Now, we present the random fixed point theorem in partially ordered spaces for expansive operators.

Lemma 4.1 *Let* $(X, d, \preceq)$ *be a partially ordered metric space, and* $F : X \to X$ *be an operator. Suppose that the following assertions hold*:

$(\mathcal{C}_1)$ *There exists* $k > 1$ *such that*

$$d\big(F(x_1), F(x_2)\big) \ge kd(x_1, x_2) \quad \textit{for each } x_1, x_2 \in X, x_1 \succeq x_2$$

and

$$X \subseteq F(X).$$

Then F has inverse operator $F^{-1}: X \to X$. If F^{-1} preserves (or reverses) the order of elements, then

$$d\big(F^{-1}(x_1), F^{-1}(x_2)\big) \le \frac{1}{k} d(x_1, x_2) \quad \textit{for each } x_1, x_2 \in X, x_1 \succeq x_2.$$

Proof Let $x, y \in X$ be such that $F(x) = F(y)$, then $F(x) \preceq F(y)$ and $F(y) \preceq F(x)$, hence

$$0 = d\big(F(x), F(y)\big) \ge kd(x,y) \quad \Longrightarrow \quad x = y.$$

This implies that F has inverse operator F^{-1}. In the case when F^{-1} preserves the order, we show that

$$d\big(F^{-1}(x_1), F^{-1}(x_2)\big) \le \frac{1}{k} d(x_1, x_2) \quad \text{for each } x_1, x_2 \in X, x_1 \succeq x_2.$$

Indeed, let $x_1, x_2 \in X$ with $x_1 \succeq x_2$, then $F^{-1}(x_1) \succeq F^{-1}(x_2)$. By $(\mathcal{C}_1)$, we have

$$d\big(F\big(F^{-1}(x_1)\big), F\big(F^{-1}(x_2)\big)\big) \ge kd\big(F^{-1}(x_1), F^{-1}(x_2)\big).$$

This implies that

$$d\big(F^{-1}(x_1), F^{-1}(x_2)\big) \le \frac{1}{k} d(x_1, x_2).$$

The case when F^{-1} reserves the order is similar. □

We can easily prove the following result.

Corollary 4.1 *Let $(X, \preceq)$ be a partially ordered set and $F: X \to X$ be a bijective mapping. Suppose that, for every $x, y \in X$, we can compare x to y (that is, the ordering is total). Then F is monotone if and only if F^{-1} is monotone (of the same type).*

Proof Suppose that F is order-preserving. Let $x, y \in X$ such that $x \preceq y$. Then $F^{-1}x$ and $F^{-1}y$ are comparable. We prove that $F^{-1}x \preceq F^{-1}y$. In other case, if $F^{-1}x \preceq F^{-1}y$ is not true, then $F^{-1}x \succeq F^{-1}y$, hence $x = F(F^{-1}x) \succeq F(F^{-1}y) = y$. This contradicts the assumption. □

We state the following auxiliary result.

Theorem 4.4 ([5]) *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $N: X \to X$ be a monotone function (nondecreasing or nonincreasing) such that there exists $k \in [0,1)$ with $d(N(x), N(y)) \le kd(x,y)$, $x \preceq y$. Suppose that N is continuous. If there exists $x_0 \in X$ with $x_0 \preceq N(x_0)$ or $x_0 \succeq N(x_0)$, then N has a unique fixed point.*

Theorem 4.5 *Let $(\Omega, \mathcal{F})$ be a measurable space, $(X, d, \preceq)$ be a separable complete partially ordered metric space, and $F: \Omega \times X \to X$ be a continuous random operator such that, for each $\omega \in \Omega$, the function $F(\omega, \cdot)$ is a monotone (either order-preserving or order-reversing) operator. Suppose that the following assertions hold: $(\mathcal{H}_2)$ and*

$(\mathcal{C}_2)$ *for each $\omega \in \Omega$, there exists $k(\omega) > 1$ such that*

$$d\big(F(\omega,x_1),F(\omega,x_2)\big) \ge k(\omega)d(x_1,x_2) \quad \text{for each } x_1,x_2 \in X, x_1 \succeq x_2.$$

Suppose also that, for each $\omega \in \Omega$, we have $X \subseteq F(\omega,X)$.

If, for every $\omega \in \Omega$, the operator $F_\omega^{-1}(\cdot) = [F(\omega,\cdot)]^{-1}$ is monotone (either order-preserving or order-reversing) and continuous, then there exists a random variable $x:\Omega \to X$ that is a random fixed point of F (the unique random fixed point of F in the case of validity of the condition $(\mathcal{H}_3)$).

Proof Let $\omega \in \Omega$, then from Lemma 4.1, $F_\omega^{-1}(\cdot) = [F(\omega,\cdot)]^{-1}$ exists and

$$d\big(F_\omega^{-1}(x),F_\omega^{-1}(y)\big) \le \frac{1}{k}d(x,y), \quad x \succeq y.$$

Using $(\mathcal{H}_2)$, we can conclude from Theorem 4.4 that, for every $\omega \in \Omega$, there exists a unique $x_\omega \in X$ fixed point of $[F(\omega,\cdot)]^{-1}$. Hence,

$$F_\omega^{-1}(x_\omega) = x_\omega, \quad \omega \in \Omega \quad \Longrightarrow \quad F(\omega,x_\omega) = x_\omega, \quad \omega \in \Omega.$$

Define a multivalued mapping $G:\Omega \to \mathcal{P}(X)$ by

$$G(\omega) = \big\{x \in X : x = F(\omega,x)\big\}.$$

Since $F(\omega,\cdot)$ is a continuous mapping, for every $\omega \in \Omega$, we have $G(\omega) \in \mathcal{P}_{cl}(X)$. Let $C \in \mathcal{P}_{cl}(X)$, then

$$G_-^{-1}(C) = \big\{\omega \in \Omega : G(\omega) \cap C \neq \emptyset\big\} = \big\{\omega \in \Omega : \text{ there exists } x \in C \text{ with } x = F(\omega,x)\big\}.$$

Since X is a separable metric space, there exists $\{x_i : i \in \mathbb{N}\} \subseteq C$ such that

$$\overline{\{x_i : i \in \mathbb{N}\}} = C,$$
$$G_-^{-1}(C) = \bigcap_{i=1}^{\infty}\bigcup_{n=1}^{\infty}\left\{\omega \in \Omega : d\big(x_i,F(\omega,x_i)\big) < \frac{1}{n}\right\}.$$

Therefore, $G_-^{-1}(C)$ is measurable. From Theorem 3.2, there exists a measurable selection $x:\Omega \to X$ of G, that is, $x(\omega) \in G(\omega)$, for every $\omega \in \Omega$, or $x(\omega) = F(\omega,x(\omega))$, for every $\omega \in \Omega$, so that x is a random fixed point of F. □

5 Random fixed point theorems in *L*-spaces

In the following results, we consider the notation $F_\omega = F(\omega,\cdot)$, for $\omega \in \Omega$.

Theorem 5.1 *Let $(X,\to)$ be an L-space with a partial ordering $\preceq$, and $(\Omega,\mathcal{U})$ be a measurable space. Let $F:\Omega \times X \to X$ be a joint measurable operator. Suppose that:*

$(\mathcal{H}_5)$ *For every $x,y:\Omega \to X$, random variables, there exists a random variable $z:\Omega \to X$ such that, for every $\omega \in \Omega$, $(x(\omega),z(\omega)), (y(\omega),z(\omega)) \in X_\preceq$.*

($\mathcal{H}_6$) *There exist random variables $x_0, x_* : \Omega \to X$ such that the sequence defined as*

$$y_0(\omega) := x_0(\omega), \quad \omega \in \Omega, \qquad y_n(\omega) := F_\omega^n\big(x_0(\omega)\big) = F\big(\omega, y_{n-1}(\omega)\big), \quad \omega \in \Omega, n \in \mathbb{N},$$

satisfies

$$\big(y_n(\omega)\big)_{n\in\mathbb{N}} \to x_*(\omega) \quad \textit{as } n \to \infty, \textit{ for every } \omega \in \Omega,$$

and, for each $\omega \in \Omega$ fixed, there exists $(y_{n_k}(\omega))_{k\in\mathbb{N}}$, a subsequence of $(y_n(\omega))_{n\in\mathbb{N}}$ such that

$$\big(y_{n_k}(\omega), x_*(\omega)\big) \in X_\preceq \quad \textit{for each } k \in \mathbb{N}.$$

($\mathcal{H}_7$) *$F : \Omega \times X \to X$ is such that $F_\omega = F(\omega, \cdot)$ is orbitally $\preceq$-continuous for every $\omega \in \Omega$.*
($\mathcal{H}_8$) *If $x, y : \Omega \to X$ are two random variables such that, for every $\omega \in \Omega$, we have*

$$\big(x(\omega), y(\omega)\big) \in X_\preceq \quad \textit{and} \quad \big(F_\omega^n\big(x(\omega)\big)\big)_{n\in\mathbb{N}} \to x_*(\omega) \quad \textit{as } n \to \infty,$$

then

$$\big(F_\omega^n\big(y(\omega)\big)\big)_{n\in\mathbb{N}} \to x_*(\omega) \quad \textit{as } n \to \infty, \textit{ for every } \omega \in \Omega.$$

Then F has a unique random fixed point. In fact, x_* is the unique random fixed point of F.

Proof If the random variable $x_* : \Omega \to X$ is such that $F(\omega, x_*(\omega)) = x_*(\omega)$ for every $\omega \in \Omega$, then it is clear that x_* is a random fixed point of F. Consider that, for some $\omega \in \Omega$, $F(\omega, x_*(\omega)) \neq x_*(\omega)$. From ($\mathcal{H}_6$), for every $\omega \in \Omega$, we have

$$\big(y_n(\omega)\big)_{n\in\mathbb{N}} = \big(F_\omega^n\big(x_0(\omega)\big)\big)_{n\in\mathbb{N}} \to x_*(\omega) \quad \text{as } n \to \infty.$$

Besides, if we fix $\omega \in \Omega$, there exists $(y_{n_k}(\omega))_{k\in\mathbb{N}}$, a subsequence of $(y_n(\omega))_{n\in\mathbb{N}}$ such that

$$\big(y_{n_k}(\omega), x_*(\omega)\big) \in X_\preceq \quad \text{for each } k \in \mathbb{N}. \tag{5.1}$$

Therefore, by ($\mathcal{H}_7$), due to

$$\big(F_\omega^{n_k}\big(x_0(\omega)\big)\big)_{k\in\mathbb{N}} \to x_*(\omega) \quad \text{as } k \to \infty,$$

and (5.1), we obtain

$$\big(y_{n_k+1}(\omega)\big)_{k\in\mathbb{N}} = \big(F_\omega^{n_k+1}\big(x_0(\omega)\big)\big)_{k\in\mathbb{N}} \to F_\omega\big(x_*(\omega)\big) = F\big(\omega, x_*(\omega)\big) \quad \text{as } k \to \infty.$$

Since $(y_{n_k+1}(\omega))_{k\in\mathbb{N}} = (F_\omega^{n_k+1}(x_0(\omega)))_{k\in\mathbb{N}}$ is a subsequence of $(y_n(\omega))_{n\in\mathbb{N}} = (F_\omega^n(x_0(\omega)))_{n\in\mathbb{N}}$,

$$\big(y_{n_k+1}(\omega)\big)_{k\in\mathbb{N}} \to x_*(\omega) \quad \text{as } k \to \infty,$$

and, hence,

$$F\big(\omega, x_*(\omega)\big) = x_*(\omega).$$

Since this identity is valid for every $\omega \in \Omega$, we have proved that x_* is a random fixed point of F.

Let $x : \Omega \to X$ be an arbitrary random variable, and we distinguish two cases:

- Consider that, for every $\omega \in \Omega$, we have

  $$\big(x(\omega), x_0(\omega)\big) \in X_{\preceq}.$$

  By $(\mathcal{H}_6)$, we obtain

  $$\big(y_n(\omega)\big)_{n\in\mathbb{N}} = \big(F_\omega^n\big(x_0(\omega)\big)\big)_{n\in\mathbb{N}} \to x_*(\omega) \quad \text{as } n \to \infty, \text{for every } \omega \in \Omega,$$

  and, by $(\mathcal{H}_8)$, we get

  $$\big(F_\omega^n\big(x(\omega)\big)\big)_{n\in\mathbb{N}} \to x_*(\omega) \quad \text{as } n \to \infty, \text{for every } \omega \in \Omega.$$

- Consider that, for some $\omega \in \Omega$, $(x(\omega), x_0(\omega)) \notin X_{\preceq}$. From $(\mathcal{H}_5)$, there exists a random variable $z : \Omega \to X$ such that, for every $\omega \in \Omega$, $(x(\omega), z(\omega))$, $(x_0(\omega), z(\omega)) \in X_{\preceq}$. Since

  $$\big(F_\omega^n\big(x_0(\omega)\big)\big)_{n\in\mathbb{N}} \to x_*(\omega) \quad \text{as } n \to \infty, \text{ for every } \omega \in \Omega,$$

  by $(\mathcal{H}_8)$, we obtain

  $$\big(F_\omega^n\big(z(\omega)\big)\big)_{n\in\mathbb{N}} \to x_*(\omega) \quad \text{as } n \to \infty, \text{ for every } \omega \in \Omega$$

  and

  $$\big(F_\omega^n\big(x(\omega)\big)\big)_{n\in\mathbb{N}} \to x_*(\omega) \quad \text{as } n \to \infty, \text{ for every } \omega \in \Omega.$$

This justifies that x_* is the unique random fixed point of F. □

Theorem 5.2 *Let $(X, \to)$ be an L-space with a partial ordering $\preceq$, and $(\Omega, \mathcal{U})$ be a measurable space. Let $F : \Omega \times X \to X$ be a joint measurable operator. Suppose that conditions $(\mathcal{H}_6)$ and $(\mathcal{H}_7)$ hold and that the following conditions hold*:

$(\overline{\mathcal{H}}_5)$ *For every $x, y \in X$, there exists $z \in X$ such that (x, z), $(y, z) \in X_{\preceq}$ (that is, the condition $(\overline{\mathcal{H}}_3)$).*

$(\overline{\mathcal{H}}_8)$ *If $x, y \in X$ and $\omega \in \Omega$ fixed are such that $(x, y) \in X_{\preceq}$ and $(F_\omega^n(x))_{n\in\mathbb{N}} \to x_*(\omega)$, as $n \to \infty$, then*

$$\big(F_\omega^n(y)\big)_{n\in\mathbb{N}} \to x_*(\omega) \quad \textit{as } n \to \infty.$$

Then F has a unique random fixed point. In fact, x_* is the unique random fixed point of F.

Proof To prove that x_* is a random fixed point, we proceed similarly to the proof of Theorem 5.1. To prove the uniqueness, we consider $x : \Omega \to X$ an arbitrary random variable and we take $\omega \in \Omega$ fixed. Again, we distinguish two cases:

- Suppose that $(x(\omega), x_0(\omega)) \in X_{\preceq}$. By $(\mathcal{H}_6)$,

$$\big(y_n(\omega)\big)_{n\in\mathbb{N}} = \big(F_\omega^n\big(x_0(\omega)\big)\big)_{n\in\mathbb{N}} \to x_*(\omega) \quad \text{as } n \to \infty,$$

  and, by $(\overline{\mathcal{H}}_8)$, we get

$$\big(F_\omega^n\big(x(\omega)\big)\big)_{n\in\mathbb{N}} \to x_*(\omega) \quad \text{as } n \to \infty.$$

- On the other hand, consider that $(x(\omega), x_0(\omega)) \notin X_{\preceq}$. Then, from $(\overline{\mathcal{H}}_5)$, there exists $z(\omega) \in X$ such that $(x(\omega), z(\omega))$, $(x_0(\omega), z(\omega)) \in X_{\preceq}$.

  Since

$$\big(F_\omega^n\big(x_0(\omega)\big)\big)_{n\in\mathbb{N}} \to x_*(\omega) \quad \text{as } n \to \infty,$$

  by $(\overline{\mathcal{H}}_8)$, we obtain

$$\big(F_\omega^n\big(z(\omega)\big)\big)_{n\in\mathbb{N}} \to x_*(\omega) \quad \text{as } n \to \infty$$

  and

$$\big(F_\omega^n\big(x(\omega)\big)\big)_{n\in\mathbb{N}} \to x_*(\omega) \quad \text{as } n \to \infty.$$

Since $\omega \in \Omega$ is arbitrary, this justifies that x_* is the unique random fixed point of F. □

6 Boundary value problem

Nonlocal boundary value problems arise in many applied sciences. For example, the vibrations of a guy wire of uniform cross section and composed of N parts of different densities, and some problems in the theory of elastic stability, can be modeled by multi-point boundary value problems (see [26, 27]). The existence of solutions for systems of local and nonlocal boundary value problems (BVPs) has received increased attention by researchers; see, for example, the papers of Agarwal, O'Regan, and Wong [28–30], Henderson, Ntouyas, and Purnaras [31], Precup [32, 33] and the references therein. Very recently, the coupled systems of BVPs with local and nonlocal conditions were studied by Bolojan-Nica *et al.* [34] and Precup [35] among others.

In this section, we are interested in getting a random solution to the following coupled system:

$$\begin{cases} x''(t,\omega) = f(t, x(t,\omega), y(t,\omega), \omega), & t \in (0,1), \omega \in \Omega, \\ y''(t,\omega) = g(t, x(t,\omega), y(t,\omega), \omega), & t \in (0,1), \omega \in \Omega, \\ x(0,\omega) = 0, \qquad x(1,\omega) = L_1(\int_0^1 x(t,\omega)\,d\alpha(t)), & \omega \in \Omega, \\ y(0,\omega) = 0 \qquad y(1,\omega) = L_2(\int_0^1 y(t,\omega)\,d\beta(t)), & \omega \in \Omega, \end{cases} \tag{6.1}$$

where $L_1, L_2 \in C(\mathbb{R}, \mathbb{R})$ and $\int_0^1 x(t,\omega)\,d\alpha(t)$, $\int_0^1 y(t,\omega)\,d\beta(t)$ denote, respectively, the Riemann-Stieltjes integrals of x and y with respect to α and β.

Lemma 6.1 *Let $f,g : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be two continuous functions and consider the problem*

$$\begin{cases} x''(t) = f(t,x(t),y(t)), & t \in (0,1), \\ y''(t) = g(t,x(t),y(t)), & t \in (0,1), \\ x(0) = 0, & x(1) = L_1(\int_0^1 x(t)\,d\alpha(t)), \\ y(0) = 0, & y(1) = L_2(\int_0^1 y(t)\,d\beta(t)), \end{cases} \tag{6.2}$$

where $L_1, L_2 \in C(\mathbb{R},\mathbb{R})$ and $\int_0^1 x(t)\,d\alpha(t)$, $\int_0^1 y(t)\,d\beta(t)$ denote Riemann-Stieltjes integrals. The pair $(x,y) \in C([0,1],\mathbb{R}) \times C([0,1],\mathbb{R})$ is a solution to problem (6.2) *if and only if*

$$\begin{cases} x(t) = \int_0^1 \bar{K}(t,s) f(s,x(s),y(s))\,ds + L_1(\int_0^1 x(s)\,d\alpha(s))t, & t \in (0,1), \\ y(t) = \int_0^1 \bar{K}(t,s) g(s,x(s),y(s))\,ds + L_2(\int_0^1 y(s)\,d\beta(s))t, & t \in (0,1), \end{cases}$$

where

$$\bar{K}(t,s) = \begin{cases} -t(1-s), & 0 \le t \le s \le 1, \\ -s(1-t), & 0 \le s \le t \le 1. \end{cases}$$

In particular, we can take the set $C([a,b],\mathbb{R})$ of continuous functions $u : [a,b] \to \mathbb{R}$ equipped with the partial ordering

$$x, y \in C([a,b],\mathbb{R}), \quad x \le y \quad \iff \quad x(t) \le y(t) \quad \text{for every } t \in [a,b],$$

and, in $C([a,b],\mathbb{R}) \times C([a,b],\mathbb{R})$, we define the following partial ordering:

$$(x,y),(x_*,y_*) \in C([a,b],\mathbb{R}) \times C([a,b],\mathbb{R}), \quad (x,y) \le (x_*,y_*) \quad \iff \quad x \le x_*, y \le y_*.$$

Theorem 6.1 *Let $f,g : [0,1] \times \mathbb{R} \times \mathbb{R} \times \Omega \to \mathbb{R}$ be two Carathéodory functions. Assume that the following conditions hold*:

($\mathcal{L}_1$) *There exist $p_1, p_2, p_3, p_4 : \Omega \to \mathbb{R}$, random variables such that*

$$\left| f(t,x,y,\omega) - f(t,\widetilde{x},\widetilde{y},\omega) \right| \le p_1(\omega)|x - \widetilde{x}| + p_2(\omega)|y - \widetilde{y}|$$

and

$$\left| g(t,x,y,\omega) - g(t,\widetilde{x},\widetilde{y},\omega) \right| \le p_3(\omega)|x - \widetilde{x}| + p_4(\omega)|y - \widetilde{y}|,$$

for each $t \in [0,1]$, $x, y, \widetilde{x}, \widetilde{y} \in \mathbb{R}$ with $(x,y) \le (\widetilde{x},\widetilde{y})$ and $\omega \in \Omega$.

($\mathcal{L}_2$) *There exist $0 < K_1 < 1$, $0 < K_2 < 1$ two positive real constants such that*

$$\left| L_1(x) - L_1(y) \right| \le K_1|x - y|, \qquad \left| L_2(x) - L_2(y) \right| \le K_2|x - y|,$$

for each $x, y \in \mathbb{R}$.

($\mathcal{L}_3$) *For each $\omega \in \Omega$ fixed, the functions $f(t,\cdot,\cdot,\omega)$, $g(t,\cdot,\cdot,\omega)$ (for every $t \in [0,1]$), L_1 and L_2 are monotonic of the same type (that is, all of them are nondecreasing or all of them are nonincreasing).*

($\mathcal{L}_4$) *One of the following conditions holds:*

$$\begin{aligned} &0 \le f(t,0,0,\omega), \qquad 0 \le g(t,0,0,\omega), \quad \forall t \in [0,1], \forall \omega \in \Omega, \\ &0 \le L_1(0), \qquad 0 \le L_2(0) \end{aligned}$$

or

$$\begin{aligned} &0 \ge f(t,0,0,\omega), \qquad 0 \ge g(t,0,0,\omega), \quad \forall t \in [0,1], \forall \omega \in \Omega, \\ &0 \ge L_1(0), \qquad 0 \ge L_2(0). \end{aligned}$$

Suppose that, for every $\omega \in \Omega$, the matrix

$$\widetilde{M}(\omega) = \begin{pmatrix} p_1(\omega) + K_1 & p_2(\omega) \\ p_3(\omega) & p_4(\omega) + K_2 \end{pmatrix}$$

has norm less than 1. *Then problem* (6.1) *has a unique random solution.*

Proof Consider the operator $N : C([0,1],\mathbb{R}) \times C([0,1],\mathbb{R}) \times \Omega \to C([0,1],\mathbb{R}) \times C([0,1],\mathbb{R})$,

$$(x,y,\omega) \mapsto N(x,y,\omega) := \bigl(N_1(\cdot,x,y,\omega), N_2(\cdot,x,y,\omega)\bigr),$$

where

$$N_1(t,x,y,\omega) = \int_0^1 \bar{K}(t,s) f\bigl(s,x(s),y(s),\omega\bigr)\,ds + L_1\biggl(\int_0^1 x(s)\,d\alpha(s)\biggr)t, \quad t \in [0,1]$$

and

$$N_2(t,x,y,\omega) = \int_0^1 \bar{K}(t,s) g\bigl(s,x(s),y(s),\omega\bigr)\,ds + L_2\biggl(\int_0^1 y(s)\,d\beta(s)\biggr)t, \quad t \in [0,1].$$

First, we show that N is a random operator on $C([0,1],\mathbb{R}) \times C([0,1],\mathbb{R})$. Given $(x,y) \in C([0,1],\mathbb{R}) \times C([0,1],\mathbb{R})$, since f and g are Carathéodory functions, then $\omega \to f(t,x(t),y(t),\omega)$ and $\omega \to g(t,x(t),y(t),\omega)$ are measurable maps in view of Lemma 3.1. Further, the integral is a limit of a finite sum of measurable functions, therefore, the maps

$$\omega \to N_1(t,x,y,\omega), \qquad \omega \to N_2(t,x,y,\omega)$$

are measurable. As a result, N is a random operator from $C([0,1],\mathbb{R}) \times C([0,1],\mathbb{R}) \times \Omega$ into $C([0,1],\mathbb{R}) \times C([0,1],\mathbb{R})$.

Clearly, the random fixed points of N are solutions to (6.1) and conversely. Indeed, given a random variable $(x,y) : \Omega \to C([0,1],\mathbb{R}) \times C([0,1],\mathbb{R})$, we get

$$N\bigl(x(\omega),y(\omega),\omega\bigr) = \bigl(x(\omega),y(\omega)\bigr), \quad \forall \omega \in \Omega,$$

is equivalent to

$$x(\omega)(t) = \int_0^1 \bar{K}(t,s)f\big(s,x(\omega)(s),y(\omega)(s),\omega\big)\,ds + L_1\bigg(\int_0^1 x(\omega)(s)\,d\alpha(s)\bigg)t, \quad t \in [0,1],$$

$$y(\omega)(t) = \int_0^1 \bar{K}(t,s)g\big(s,x(\omega)(s),y(\omega)(s),\omega\big)\,ds + L_2\bigg(\int_0^1 y(\omega)(s)\,d\beta(s)\bigg)t, \quad t \in [0,1],$$

so that the corresponding solution to (6.1) is defined as $x(t,\omega) = x(\omega)(t)$, $y(t,\omega) = y(\omega)(t)$, for $t \in [0,1]$ and $\omega \in \Omega$.

Next, we show that N satisfies all the conditions of Theorem 4.3 on $C([0,1],\mathbb{R}) \times C([0,1],\mathbb{R})$. The proof will be given in several steps:

Step 1. For $\omega \in \Omega$ fixed, we check that $N(\cdot,\cdot,\omega) = (N_1(\cdot,\cdot,\omega), N_2(\cdot,\cdot,\omega))$ is continuous.

Take $\omega \in \Omega$ fixed. Let (x_n,y_n) be a sequence in $C([0,1],\mathbb{R}) \times C([0,1],\mathbb{R})$ such that $(x_n,y_n) \to (x,y) \in C([0,1],\mathbb{R}) \times C([0,1],\mathbb{R})$, as $n \to \infty$. Then, for $t \in [0,1]$,

$$\begin{aligned}
&\big|N_1(t,x_n,y_n,\omega) - N_1(t,x,y,\omega)\big| \\
&\quad\le \int_0^1 \big|f\big(s,x_n(s),y_n(s),\omega\big) - f\big(s,x(s),y(s),\omega\big)\big|\,ds \\
&\qquad + \bigg|L_1\bigg(\int_0^1 x_n(s)\,d\alpha(s)\bigg)t - L_1\bigg(\int_0^1 x(s)\,d\alpha(s)\bigg)t\bigg|.
\end{aligned}$$

Then, by $(\mathcal{L}_2)$,

$$\begin{aligned}
&\big\|N_1(\cdot,x_n,y_n,\omega) - N_1(\cdot,x,y,\omega)\big\|_\infty \\
&\quad\le \int_0^1 \big|f\big(s,x_n(s),y_n(s),\omega\big) - f\big(s,x(s),y(s),\omega\big)\big|\,ds + K_1\int_0^1 \big|x_n(s) - x(s)\big|\,d\alpha(s) \\
&\quad\le \int_0^1 \big|f\big(s,x_n(s),y_n(s),\omega\big) - f\big(s,x(s),y(s),\omega\big)\big|\,ds + K_1\int_0^1 \|x_n - x\|_\infty\,d\alpha(s).
\end{aligned}$$

Since f is a Carathéodory function, by the Lebesgue dominated convergence theorem, we get

$$\big\|N_1(\cdot,x_n,y_n,\omega) - N_1(\cdot,x,y,\omega)\big\|_\infty \to 0 \quad \text{as } n \to \infty.$$

Similarly,

$$\big\|N_2(\cdot,x_n,y_n,\omega) - N_2(\cdot,x,y,\omega)\big\|_\infty \to 0 \quad \text{as } n \to \infty.$$

Thus, N is a continuous random operator.

Step 2. For each $\omega \in \Omega$, the function $N(\cdot,\cdot,\omega)$ is a monotone operator.

Let $\omega \in \Omega$ be fixed. Let $(x,y), (\bar{x},\bar{y}) \in C([0,1],\mathbb{R}) \times C([0,1],\mathbb{R})$ be such that $(x,y) \le (\bar{x},\bar{y})$, that is,

$$x(t) \le \bar{x}(t), \qquad y(t) \le \bar{y}(t), \quad \forall t \in [0,1].$$

If, for every $t \in [0,1]$, $f(t,\cdot,\cdot,\omega)$, $g(t,\cdot,\cdot,\omega)$, L_1 and L_2 are nondecreasing operators, then

$$f\big(t,x(t),y(t),\omega\big) \le f\big(t,\bar{x}(t),\bar{y}(t),\omega\big), \quad \forall t \in [0,1],$$

$$g\big(t,x(t),y(t),\omega\big) \le g\big(t,\bar{x}(t),\bar{y}(t),\omega\big), \quad \forall t \in [0,1],$$
$$L_1\left(\int_0^1 x(t)\,d\alpha(t)\right) \le L_1\left(\int_0^1 \bar{x}(t)\,d\alpha(t)\right),$$

and

$$L_2\left(\int_0^1 y(t)\,d\beta(t)\right) \le L_2\left(\int_0^1 \bar{y}(t)\,d\beta(t)\right).$$

This implies that

$$N_1(t,x,y,\omega) \le N_1(t,\bar{x},\bar{y},\omega), \quad \forall t \in [0,1]$$

and

$$N_2(t,x,y,\omega) \le N_2(t,\bar{x},\bar{y},\omega), \quad \forall t \in [0,1].$$

Hence

$$N(x,y,\omega)(t) \le N(\bar{x},\bar{y},\omega)(t), \quad \forall t \in [0,1].$$

On the other hand, if, for every $t \in [0,1]$, $f(t,\cdot,\cdot,\omega)$, $g(t,\cdot,\cdot,\omega)$, L_1 and L_2 are nonincreasing operators, then

$$N(x,y,\omega)(t) \ge N(\bar{x},\bar{y},\omega)(t), \quad \forall t \in [0,1].$$

Step 3. We show that N satisfies the following property: for every $\omega \in \Omega$ and every $(x,y), (\widetilde{x},\widetilde{y}) \in C([0,1],\mathbb{R}) \times C([0,1],\mathbb{R})$ such that $(x,y) \preceq (\widetilde{x},\widetilde{y})$, we have

$$d\big(N(x,y,\omega), N(\widetilde{x},\widetilde{y},\omega)\big) \le \widetilde{M}(\omega)\, d\big((x,y),(\widetilde{x},\widetilde{y})\big).$$

Consider $\omega \in \Omega$ fixed. Let $(x,y), (\widetilde{x},\widetilde{y}) \in C([0,1],\mathbb{R}) \times C([0,1],\mathbb{R})$ be such that $(x,y) \preceq (\widetilde{x},\widetilde{y})$, then

$$\begin{aligned}
&\big|N_1(t,x,y,\omega) - N_1(t,\widetilde{x},\widetilde{y},\omega)\big| \\
&\quad \le \left|\int_0^1 \bar{K}(t,s)\big[f\big(s,x(s),y(s),\omega\big) - f\big(s,\widetilde{x}(s),\widetilde{y}(s),\omega\big)\big]\,ds\right| \\
&\qquad + \left|L_1\left(\int_0^1 x(t)\,d\alpha(t)\right) - L_1\left(\int_0^1 \widetilde{x}(t)\,d\alpha(t)\right)\right| \\
&\quad \le p_1(\omega)\|x-\widetilde{x}\|_\infty + p_2(\omega)\|y-\widetilde{y}\|_\infty + K_1\|x-\widetilde{x}\|_\infty.
\end{aligned}$$

Then

$$\big\|N_1(\cdot,x,y,\omega) - N_1(\cdot,\widetilde{x},\widetilde{y},\omega)\big\|_\infty \le \big(p_1(\omega)+K_1\big)\|x-\widetilde{x}\|_\infty + p_2(\omega)\|y-\widetilde{y}\|_\infty.$$

Similarly, we obtain

$$\big\|N_2(\cdot,x,y,\omega) - N_2(\cdot,\widetilde{x},\widetilde{y},\omega)\big\|_\infty \le p_3(\omega)\|x-\widetilde{x}\|_\infty + \big(p_4(\omega)+K_2\big)\|y-\widetilde{y}\|_\infty.$$

Hence

$$\begin{pmatrix} \|N_1(\cdot,x,y,\omega)-N_1(\cdot,\widetilde{x},\widetilde{y},\omega)\|_\infty \\ \|N_2(\cdot,x,y,\omega)-N_2(\cdot,\widetilde{x},\widetilde{y},\omega)\|_\infty \end{pmatrix} \le \begin{pmatrix} p_1(\omega)+K_1 & p_2(\omega) \\ p_3(\omega) & p_4(\omega)+K_2 \end{pmatrix} \begin{pmatrix} \|x-\widetilde{x}\|_\infty \\ \|y-\widetilde{y}\|_\infty \end{pmatrix},$$

that is,

$$d\big(N(x,y,\omega),N(\widetilde{x},\widetilde{y},\omega)\big) \le \widetilde{M}(\omega)\, d\big((x,y),(\widetilde{x},\widetilde{y})\big),$$

where

$$\widetilde{M}(\omega) = \begin{pmatrix} p_1(\omega)+K_1 & p_2(\omega) \\ p_3(\omega) & p_4(\omega)+K_2 \end{pmatrix}$$

and

$$d\big((x_1,x_2),(y_1,y_2)\big) = \begin{pmatrix} \|x_1-y_1\|_\infty \\ \|x_2-y_2\|_\infty \end{pmatrix},$$

for $(x_1,x_2),(y_1,y_2) \in C([0,1],\mathbb{R}) \times C([0,1],\mathbb{R})$.

Step 4. By $(\mathcal{L}_4)$, we can easily show that

$$0 \le N_1(\cdot,0,0,\omega) \quad \text{and} \quad 0 \le N_2(\cdot,0,0,\omega), \quad \forall\omega \in \Omega$$

or

$$0 \ge N_1(\cdot,0,0,\omega) \quad \text{and} \quad 0 \ge N_2(\cdot,0,0,\omega), \quad \forall\omega \in \Omega,$$

that is, $N(0,0,\omega) \ge (0,0)$ $\forall\omega \in \Omega$, or $N(0,0,\omega) \le (0,0)$ $\forall\omega \in \Omega$.

This means that, for the null random variable defined as $(0,0) : \Omega \to C([0,1],\mathbb{R}) \times C([0,1],\mathbb{R})$, $(0,0)(\omega) = (0,0)$, $\forall\omega \in \Omega$, one of the following conditions holds: $N((0,0)(\omega),\omega) \ge (0,0)(\omega)$ $\forall\omega \in \Omega$, or $N((0,0)(\omega),\omega) \le (0,0)(\omega)$ $\forall\omega \in \Omega$. Note that the uniqueness condition $(\mathcal{H}_3)$ also holds.

Then, from Theorem 4.3, there exists a unique random solution to problem (6.1). □

Competing interests
None of the authors have any competing interests in the manuscript.

Authors' contributions
The three authors contributed equally in this paper.

Author details
[1]Departamento de Análise Matemática, Estatística e Optimización, Facultade de Matemáticas, Universidade de Santiago de Compostela, Santiago de Compostela, 15782, Spain. [2]Laboratory of Mathematics, Sidi-Bel-Abbès University, PoBox 89, Sidi-Bel-Abbès, 22000, Algeria.

Acknowledgements
The authors would like to thank the anonymous referees for their careful reading of the manuscript and pertinent comments; their constructive suggestions substantially improved the quality of the work.

Received: 28 June 2016 Accepted: 13 October 2016 Published online: 26 October 2016

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