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System of generalized mixed equilibrium problems, variational inequality, and fixed point problems

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Abstract

The purpose of this paper is to introduce a new iterative algorithm for finding a common element of the set of solutions of a system of generalized mixed equilibrium problems, the set of common fixed points of a finite family of pseudo contraction mappings, and the set of solutions of the variational inequality for an inverse strongly monotone mapping in a real Hilbert space. We establish results on the strong convergence of the sequence by the proposed scheme to a common element of the above three solution sets. These results extend and improve some corresponding results in this area. Finally, we give a numerical example which supports our main theorem.

MSC: 47H10; 47H09; 65K10

Keywords: generalized mixed equilibrium problem; variational inequality problem; strictly pseudo contraction mapping

1 Introduction

Let Θ be a bifunction from $K \times K$ into the set of real numbers, R, where K is a nonempty closed convex subset of a real Hilbert space H. The equilibrium problem is to find a point $x \in K$ such that

$$\Theta(x,y) \geq 0, \quad \forall y \in K. \tag{1.1}$$

We denote the set of solutions of (1.1) by $\mathrm{EP}(\Theta)$. The equilibrium problem includes the fixed point problem, the variational inequality problem, the optimization problem, the saddle point problem, the Nash equilibrium problem and so on, as its special cases [1, 2].

The generalized mixed equilibrium problem is to find a point $x \in K$ such that

$$\Theta(x,y) + \langle Ax, y-x\rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in K, \tag{1.2}$$

where φ is a function on K into R and A is a nonlinear mapping from K to H. The set of solutions of a generalized mixed equilibrium problem is denoted by $\mathrm{GMEP}(\Theta, A, \varphi)$.

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If we consider $\Theta = 0$ and $\varphi = 0$ in (1.2), then we have the classical variational inequality problem which is to find a point $x \in K$ such that

$$\langle Ax, y - x\rangle \geq 0, \quad \forall y \in K. \tag{1.3}$$

The solution set of (1.3) is denoted by $\mathrm{VI}(K, A)$.

To proceed we need to recall some definitions and concepts.

Definition 1.1 Let K be a nonempty closed convex subset of a real Hilbert space H.

(i) A mapping $S : K \to K$ is called nonexpansive if $\|Sx - Sy\| \leq \|x - y\|$, for all $x, y \in K$.

(ii) A mapping $T : K \to K$ is called k-strict pseudo contractive mapping, if for all $x, y \in K$ there exists a constant $0 \leq k < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\big\|(I - T)x - (I - T)y\big\|^2, \quad \forall x, y \in K, \tag{1.4}$$

where I is the identity mapping on K.

(iii) A mapping $A : H \to H$ is called monotone if for each $x, y \in H$,

$$\langle Ax - Ay, x - y\rangle \geq 0.$$

(iv) A mapping $A : H \to H$ is called β-inverse strongly monotone if there exists $\beta > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \beta\|Ax - Ay\|^2, \quad \forall x, y \in H.$$

(v) The mapping $A : K \to H$ is L-Lipschitz continuous if there exists a positive real number L such that $\|Ax - Ay\| \leq L\|x - y\|$ for all $x, y \in H$. If $0 < L < 1$, then the mapping A is a contraction with constant L.

Clearly a nonexpansive mapping is a 0-strict pseudo contractive mapping [3]. Note that in a Hilbert space, (1.4) is equivalent to the following inequality:

$$\langle Tx - Ty, x - y\rangle \leq \|x - y\|^2 - \frac{1-k}{2}\big\|(x - y) - (Tx - Ty)\big\|^2, \quad \forall x, y \in K. \tag{1.5}$$

We denote $F(T) = \{x \in K : Tx = x\}$, the set of fixed points of T. It can be shown that, for a k-strict pseudo contractive mapping $T : K \to K$, the mapping $I - T$ is demiclosed, *i.e.*, if $\{x_n\}$ is a sequence in K with $x_n \rightharpoonup q$ and $x_n - Tx_n \to 0$, then $q \in F(T)$ (refer to [4]). The symbols $\rightharpoonup$ and $\to$ denote weak and strong convergence, respectively.

A set valued mapping $Q : H \to 2^H$ is called monotone if for all $x, y \in H$, $f \in Q(x)$ and $g \in Q(y)$ imply $\langle x - y, f - g\rangle \geq 0$. A monotone mapping $Q : H \to 2^H$ is maximal if the graph $G(Q)$ of Q is not properly contained in the graph of any other monotone mapping. It is well known that a monotone mapping Q is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$ for every $(y, g) \in G(Q)$ implies $f \in Q(x)$ [5].

For any $x \in H$ there exists a unique point in K denoted by $P_K x$ such that $\|x - P_K x\| \leq \|x - y\|$ for all $y \in K$. It is well known that the operator $P_K : H \to K$, which is called the metric projection, is a nonexpansive mapping and has the properties that, for each $x \in H$,

$P_K x \in K$ and $\langle x - P_K x, P_K x - y\rangle \geq 0$, for all $y \in K$. It is also known that $\|P_K x - P_K y\|^2 \leq \langle x - y, P_K x - P_K y\rangle$, for all $x, y \in K$ [6]. In the context of the variational inequality problem, we obtain

$$q \in \mathrm{VI}(K, A) \quad \text{if and only if} \quad q = P_K(q - \lambda Aq), \quad \forall \lambda > 0. \tag{1.6}$$

Let I be an index set. For each $i \in I$, let Θ_i be a real valued bifunction on $K \times K$, A_i a nonlinear mapping, and $\varphi_i : K \to R$ a function. The system of generalized mixed equilibrium problems as an extension of problems (1.1), (1.2), and (1.3) is to find a point $x \in K$ such that

$$\Theta_i(x, y) + \langle A_i x, y - x\rangle + \varphi_i(y) - \varphi_i(x) \geq 0, \quad \forall y \in K, \forall i \in I. \tag{1.7}$$

Note that $\bigcap_{i\in I} \mathrm{GMEP}(\Theta_i, A_i, \varphi_i)$ is the solution set of (1.7).

Vast range of problems which arise in economics, finance, image reconstruction, transportation, network and so on, appear as a special case of problem (1.7); see for example [7–10]. This problem also covers various forms of feasibility problems. So, it seems reasonable to study the system of generalized mixed equilibrium problems. There are many authors who introduced some iterative processes for finding the solution set of these problems or common solution of someone with others, for instance see [2, 11–13] and the references therein. In 2010, Peng *et al.* [14] introduced the following iterative algorithm for finding a common element of fixed points of a family of infinite nonexpansive mappings and the set of solutions of a system of finite family of equilibrium problems:

$$\begin{cases} z_1 = z \in H, \\ u_n = T_{\beta_n}^{F_m} T_{\beta_n}^{F_{m-1}} \cdots T_{\beta_n}^{F_2} T_{\beta_n}^{F_1} z_n, \\ v_n = P_K(I - s_n A)u_n, \\ z_{n+1} = \alpha_n \gamma f(W_n z_n) + (I - \alpha_n B) W_n P_K(I - r_n A)v_n. \end{cases}$$

Under suitable conditions, they presented and proved a strong convergence theorem for finding an element of $\Omega = \bigcap_{i=1}^{\infty} F(T_i) \cap \mathrm{VI}(K, A) \cap \bigcap_{k=1}^{m} \mathrm{EP}(\mathrm{F}_k)$. In 2013, Cai and Bu [11] proposed an iterative method as follows:

$$\begin{cases} x_1 = x \in K, \\ z_n = T_{r_{M,n}}^{(F_M,\varphi_M)}(I - r_{M,n}B_M) T_{r_{M-1,n}}^{(F_{M-1},\varphi_{M-1})}(I - r_{M-1,n}B_{M-1}) \cdots T_{r_{1,n}}^{(F_1,\varphi_1)}(I - r_1 B_1)x_n, \\ u_n = P_K(I - \lambda_{N,n}A_N) P_K(I - \lambda_{N-1,n}A_{N-1}) \cdots P_K(I - \lambda_{1,n}A_1)z_n, \\ x_{n+1} = \alpha_n f(S_n x_n) + \beta_n x_n + (I - \beta_n - \alpha_n) W(\tau_n) S u_n. \end{cases}$$

They proved that under appropriate conditions, the sequences $\{x_n\}$, $\{z_n\}$, and $\{u_n\}$ converge strongly to $z = P_\Omega f(z)$, where $\Omega = F(W) \cap \bigcap_{i=1}^{\infty} F(T_i) \cap \bigcap_{k=1}^{m} \mathrm{GMEP}(\mathrm{F}_k, \varphi_k, B_k) \cap \bigcap_{j=1}^{N} \mathrm{VI}(K, A_j)$ and f is a contractive mapping. The iterative method for solving a system of equilibrium problem has studied by many other authors; for example [7, 14, 15] and so on. Note that, for finding common fixed point of a finite family of mapping and solution set of other problems, authors usually have been using the so-called W-mapping [11, 16, 17]. For example Thianwan [16] proposed the following method for finding a common element of the set of solutions of an equilibrium problem, the set of common fixed points of a finite family of nonexpansive mappings, and the set of solutions of the variational inequality of an α-inverse strongly monotone mapping in a real Hilbert space:

$$\begin{cases} \phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \\ w_n = \alpha_n x_n + (1-\alpha_n) W_n P_K(u_n - \lambda_n A u_n), \\ K_{n+1} = \{z \in K_n : \|w_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{K_{n+1}}(x_0). \end{cases}$$

He showed that under suitable conditions, the above algorithm strongly converges to $\bigcap_{i=1}^{N} F(T_i) \cap \mathrm{EP}(\phi) \cap \mathrm{VI}(K, A)$, where for each $i = 1, \ldots, N$, T_i is a nonexpansive mapping and A is an α-inverse strongly monotone mapping.

In this paper, we present an iterative algorithm for finding a common solution of a system of finite generalized mixed equilibrium problems, a variational inequality problem for an inverse strongly monotone mapping and common fixed points of a finite family of strictly pseudo contractive mappings. We show that the algorithm strongly converges to a solution of the problem under certain conditions. Our results modify, improve and extend corresponding results of Takahashi and Takahashi [18], Zhang *et al.* [19], Shehu [20], Thianwan [16], and others. The rest of the paper is organized as follows. Section 2 briefly explains the necessary mathematical background. Section 3 presents the main results. A numerical example is provided in the final section.

2 Preliminaries

It is well known that in a (real) Hilbert space H

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \tag{2.1}$$

for all $x, y \in H$ [12]. Furthermore, it is easy to see that

$$\left\| \sum_{i=1}^{m} x_i \right\|^2 = \sum_{i,j=1}^{m} \langle x_i, x_j\rangle. \tag{2.2}$$

Lemma 2.1 ([13]) *Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be three nonnegative real sequences satisfying*

$$a_{n+1} \leq (1 - t_n)a_n + b_n + c_n$$

with $\{t_n\} \subset [0,1]$, $\sum_{n=1}^{\infty} t_n = \infty$, $b_n = o(t_n)$, and $\sum_{n=1}^{\infty} c_n < \infty$. Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.2 ([21]) *Let H be a (real) Hilbert space and $\{x_n\}_{n=1}^{N}$ be a bounded sequence in H. Let $\{a_n\}_{n=1}^{N}$ be a sequence of real numbers such that $\sum_{n=1}^{N} a_n = 1$. Then*

$$\left\| \sum_{i=1}^{N} a_i x_i \right\|^2 \leq \sum_{i=1}^{N} a_i \|x_i\|^2.$$

Lemma 2.3 ([22]) *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space and β_n be a sequence of real numbers such that $0 < \liminf_{n\to\infty} \beta_n < \limsup_{n\to\infty} \beta_n < 1$ for all $n \geq 0$. Suppose that $x_{n+1} = (1-\beta_n)z_n + \beta_n x_n$ for all $n \geq 0$ and $\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then $\lim_{n\to\infty} \|z_n - x_n\| = 0$.*

Let us assume that the bifunction Θ satisfies the following conditions:

(A1) $\Theta(x,x) = 0$, $\forall x \in K$;

(A2) Θ is monotone on K, *i.e.*, $\Theta(x,y) + \Theta(y,x) \leq 0$, $\forall x,y \in K$;

(A3) for all $x,y,z \in K$, $\lim_{t\to 0^+} \Theta(tz + (1-t)x, y) \leq \Theta(x,y)$;

(A4) for all $x \in K$, $y \mapsto \Theta(x,y)$ is convex and lower semicontinuous.

Lemma 2.4 ([1]) *Let K be a nonempty closed convex subset of Hilbert space H and Θ be a real valued bifunction on $K \times K$ satisfying* (A1)-(A4). *Let $r > 0$ and $x \in H$, then there exists $z \in K$ such that*

$$\Theta(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in K.$$

Lemma 2.5 ([2]) *Suppose all conditions in Lemma* 2.4 *are satisfied. For any given $r > 0$, define a mapping $T_r : H \to K$ as*

$$T_r x = \left\{ z \in K : \Theta(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in K \right\},$$

for all $x \in H$. Then the following conditions hold:

1. *T_r is single valued*;
2. *T_r is firmly nonexpansive, i.e.,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle, \quad \forall x,y \in H;$$

3. $F(T_r) = \mathrm{EP}(\Theta)$;
4. $\mathrm{EP}(\Theta)$ *is a closed and convex set.*

Remark 2.6 For the generalized mixed equilibrium problem (1.2), if the nonlinear operator A is a monotone, Lipschitz continuous mapping, φ is a convex and lower semicontinuous function, and the real valued bifunction Θ admits the conditions (A1)-(A4), then it is easy to show that $G(x,y) = \Theta(x,y) + \langle Ax, y - x\rangle + \varphi(y) - \varphi(x)$ also satisfies the conditions (A1)-(A4), and the generalized mixed equilibrium (1.2) is still the following equilibrium problem:

$$\text{find} \quad x \in K \quad \text{such that} \quad G(x,y) \geq 0, \quad \forall y \in K.$$

3 Main results

As is well known, the strict pseudo contraction mappings have more useful applications than nonexpansive mappings like in solving inverse problems [23]. In addition, various problems reduced to find the common element of the fixed point set of a family of nonlinear mappings such as image restoration (see for example [24]). For construction an algorithm which can used to obtain the fixed point set of a family of strictly pseudo contractive mappings we need to introduce the following proposition.

In the sequel, $I = \{1,2,\ldots,m\}$ and $J = \{1,2,\ldots,l\}$ are two index sets.

Proposition 3.1 *Let $T_j : K \to K$, $j \in J$, be k_j-strict pseudo contractive mappings. Define $S : K \to K$ by $S = \gamma_0 I + \gamma_1 T_1 + \cdots + \gamma_l T_l$, where the $\{\gamma_j\}$, $j \in J$, are in* $(0,1)$ *and, for each $n \in N$,*

$\sum_{j=0}^{l} \gamma_j = 1$. *If* $\gamma_0 \in [k, 1)$ *such that* $k = \max\{k_1, \dots, k_l\}$, *then* S *is a nonexpansive mapping and* $F(S) = \bigcap_{j\in J} F(T_j)$.

Proof By the definition of the mapping S, we have

$$\begin{aligned}\|Sx - Sy\|^2 &= \gamma_0^2 \|x - y\|^2 + \Big\|\sum_{j\in J} \gamma_j (T_j x - T_j y)\Big\|^2 \\ &\quad + 2\gamma_0 \sum_{j\in J} \gamma_j \langle x - y, T_j x - T_j y\rangle. \end{aligned} \tag{3.1}$$

On the other hand, from (1.4) and (2.2) we have

$$\begin{aligned}\Big\|\sum_{j\in J} \gamma_j (T_j x - T_j y)\Big\|^2 &= \sum_{j,i\in J} \gamma_j \gamma_i \langle T_j x - T_j y, T_i x - T_i y\rangle \\ &\leq \sum_{j,i\in J} \gamma_j \gamma_i \|T_j x - T_j y\| \|T_i x - T_i y\| \\ &\leq \frac{1}{2} \sum_{j,i\in J} \gamma_j \gamma_i \big(\|T_j x - T_j y\|^2 + \|T_i x - T_i y\|^2\big) \\ &\leq \frac{1}{2} \sum_{j,i\in J} \gamma_j \gamma_i \big[\|x - y\|^2 + k_j \big\|(x - y) - (T_j x - T_j y)\big\|^2 \\ &\quad + \|x - y\|^2 + k_i \big\|(x - y) - (T_i x - T_i y)\big\|^2\big] \\ &= \sum_{j,i\in J} \gamma_j \gamma_i \|x - y\|^2 \\ &\quad + \sum_{j\in J} \gamma_j \sum_{i\in J} \gamma_i k_j \big\|(x - y) - (T_j x - T_j y)\big\|^2. \end{aligned} \tag{3.2}$$

Furthermore, (1.5) implies that, for each $j \in J$,

$$\langle x - y, T_j x - T_j y\rangle \leq \|x - y\|^2 - \frac{1 - k_j}{2} \big\|(x - y) - (T_j x - T_j y)\big\|^2. \tag{3.3}$$

By substituting (3.2) and (3.3) in (3.1), we have

$$\begin{aligned}\|Sx - Sy\|^2 &\leq \Big(\gamma_0^2 + \sum_{j,i\in J} \gamma_j \gamma_i + 2\sum_{j\in J} \gamma_0 \gamma_j\Big) \|x - y\|^2 \\ &\quad + \sum_{j\in J} \gamma_j \sum_{i\in J} \gamma_i k_j \big\|(x - y) - (T_j x - T_j y)\big\|^2 \\ &\quad - \sum_{j\in J} \gamma_0 \gamma_j (1 - k_j) \big\|(x - y) - (T_j x - T_j y)\big\|^2 \\ &= \|x - y\|^2 - \sum_{j\in J} \gamma_j \Big[\gamma_0 (1 - k_j) - \sum_{i\in J} \gamma_i k_j\Big] \big\|(x - y) - (T_j x - T_j y)\big\|^2 \\ &= \|x - y\|^2 - \sum_{j\in J} \gamma_j \big[\gamma_0 (1 - k_j) - (1 - \gamma_0) k_j\big] \big\|(x - y) - (T_j x - T_j y)\big\|^2 \end{aligned}$$

$$
\begin{aligned}
&= \|x-y\|^2 - \sum_{j\in J}\gamma_j(\gamma_0-k_j)\big\|(x-y)-(T_jx-T_jy)\big\|^2 \\
&\le \|x-y\|^2 - \sum_{j\in J}\gamma_j(\gamma_0-k)\big\|(x-y)-(T_jx-T_jy)\big\|^2 \\
&\le \|x-y\|^2. \qquad (3.4)
\end{aligned}
$$

Then S is a nonexpansive mapping. Now, by the definition of S we obtain $I-S=\sum_{j\in J}\gamma_j(I-T_j)$ and clearly $F(S)=\bigcap_{j\in J}F(T_j)$. □

Theorem 3.2 *Let $\Theta_i : K\times K\to R$, $i\in I$, be bifunctions satisfying* (A1)-(A4). *Suppose that, for each $i\in I$, the B_i are θ_i-inverse strongly monotone mappings, the C_i are monotone and Lipschitz continuous mappings from K into H, and the φ_i are convex and lower semicontinuous functions from K into R. Let $T_j : K\to K$, $j\in J$, be k_j-strict pseudo contractive mappings and $A : K\to H$ be a σ-inverse strongly monotone mapping. Let $f : K\to K$ be an ε-contraction mapping and $\{v_n\}$ be a convergent sequence in K with limit point v. Suppose that $\Omega=\bigcap_{i\in I}\mathrm{GMEP}(\Theta_i,B_i,C_i,\varphi_i)\cap\bigcap_{j\in J}F(T_j)\cap\mathrm{VI}(A,K)$ is nonempty. For any initial guess $x_1\in K$, define the sequence $\{x_n\}$ by*

$$
\begin{cases}
\Theta_i(u_{n,i},y)+\langle C_iu_{n,i}+B_ix_n,y-u_{n,i}\rangle+\varphi_i(y)-\varphi_i(u_{n,i}) \\
\quad +\frac{1}{r_{n,i}}\langle y-u_{n,i},u_{n,i}-x_n\rangle\ge 0,\quad \forall y\in K,\forall i\in I, \\
y_n=\alpha_nv_n+(I-\alpha_n(I-f))P_K(\sum_{i\in I}\delta_{n,i}u_{n,i}-\lambda_nA\sum_{i\in I}\delta_{n,i}u_{n,i}), \\
x_{n+1}=\beta_nx_n+(1-\beta_n)(\gamma_0I+\sum_{j\in J}\gamma_jT_j)P_K(y_n-\lambda_nAy_n),
\end{cases}
\qquad (3.5)
$$

where for all $n\in N$, $\{\lambda_n\},\{r_{n,i}\}_{i\in I}\subseteq(0,\infty)$, and $\{\alpha_n\},\{\beta_n\},\{\delta_{n,i}\}_{i\in I},\{\gamma_j\}_{j\in J}\subseteq(0,1)$ are sequences satisfying the following control conditions:

1. $\lim_{n\to\infty}\alpha_n=0$, $\sum_{n=1}^{\infty}\alpha_n=\infty$;
2. $0<\liminf_{n\to\infty}\beta_n\le\limsup_{n\to\infty}\beta_n<1$;
3. *for some $a,b\in(0,2\sigma)$, $\lambda_n\in[a,b]$ and* $\lim_{n\to\infty}|\lambda_{n+1}-\lambda_n|=0$;
4. *for some $d>0$, $0<d\le\delta_{n,i}\le 1$, $\sum_{i\in I}\delta_{n,i}=1$ and $\sum_{n=1}^{\infty}|\delta_{n+1,i}-\delta_{n,i}|<\infty$;*
5. *for some $c>0$, $k\le\gamma_0\le c<1$ such that $k=\max_{j\in J}\{k_j\}$ and $\sum_{j\in J}\gamma_j=1$;*
6. *for some $\tau_i,\rho_i\in(0,2\theta_i)$, $r_{n,i}\in[\tau_i,\rho_i]$ and $\sum_{n=1}^{\infty}|r_{n+1,i}-r_{n,i}|<\infty$, $i\in I$.*

Then the sequences $\{x_n\}$ converges strongly to $z\in\Omega$, where $z=P_\Omega(v+f(z))$.

Proof For $x,y\in K$ and $i\in I$, put $G_i(x,y)=\Theta_i(x,y)+\langle C_ix,y-x\rangle+\varphi_i(y)-\varphi_i(x)$. By Remark 2.6, G_i satisfies the conditions (A1)-(A4) and so the algorithm (3.5) can be rewritten as follows:

$$
\begin{cases}
G_i(u_{n,i},y)+\langle B_ix_n,y-u_{n,i}\rangle+\frac{1}{r_{n,i}}\langle y-u_{n,i},u_{n,i}-x_n\rangle\ge 0,\quad \forall y\in K, i\in I, \\
y_n=\alpha_nv_n+(I-\alpha_n(I-f))P_K(\sum_{i\in I}\delta_{n,i}u_{n,i}-\lambda_nA\sum_{i\in I}\delta_{n,i}u_{n,i}), \\
x_{n+1}=\beta_nx_n+(1-\beta_n)(\gamma_0I+\sum_{j\in J}\gamma_jT_j)P_K(y_n-\lambda_nAy_n).
\end{cases}
\qquad (3.6)
$$

Claim 1 *The sequences $\{x_n\}$, $\{y_n\}$, $\{u_n\}$, $\{t_n\}$, and $\{k_n\}$ are bounded where, for each $n\in N$, $u_n=\sum_{i\in I}\delta_{n,i}u_{n,i}$, $t_n=P_K(y_n-\lambda_nAy_n)$, and $k_n=P_K(u_n-\lambda_nAu_n)$.*

To prove the claim from (3.6) we have

$$
G_i(u_{n,i},y)+\frac{1}{r_{n,i}}\big\langle y-u_{n,i},u_{n,i}-(I-r_{n,i}B_i)x_n\big\rangle\ge 0,\quad \forall y\in K, i\in I. \qquad (3.7)
$$

Then, by using Lemma 2.5, for each $i \in I$, we have $u_{n,i} = T_{r_{n,i}}(x_n - r_{n,i}B_i x_n)$, and, for any $q \in \Omega$, $q = T_{r_{n,i}}(q - r_{n,i}B_i q)$. Thus

$$\begin{aligned}
\|u_{n,i} - q\|^2 &= \big\| T_{r_{n,i}}(x_n - r_{n,i}B_i x_n) - T_{r_{n,i}}(q - r_{n,i}B_i q)\big\|^2 \\
&\le \big\|(x_n - r_{n,i}B_i x_n) - (q - r_{n,i}B_i q)\big\|^2 \\
&\le \|x_n - q\|^2 + r_{n,i}^2\|B_i x_n - B_i q\|^2 - 2r_{n,i}\langle x_n - q, B_i x_n - B_i q\rangle \\
&\le \|x_n - q\|^2 + r_{n,i}^2\|B_i x_n - B_i q\|^2 - 2r_{n,i}\theta_i\|B_i x_n - B_i q\|^2 \\
&= \|x_n - q\|^2 + r_{n,i}(r_{n,i} - 2\theta_i)\|B_i x_n - B_i q\|^2 \\
&\le \|x_n - q\|^2.
\end{aligned} \tag{3.8}$$

So, we have

$$\|u_n - q\|^2 \le \sum_{i\in I}\delta_{n,i}\|u_{n,i} - q\|^2 \le \sum_{i\in I}\delta_{n,i}\|x_n - q\|^2 = \|x_n - q\|^2. \tag{3.9}$$

By the definition of t_n and k_n we have

$$\begin{aligned}
\|t_n - q\| &\le \big\|(y_n - \lambda_n A y_n) - (q - \lambda_n A q)\big\| \\
&\le \|y_n - q\|
\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}
\|k_n - q\| &\le \big\|(u_n - \lambda_n A u_n) - (q - \lambda_n A q)\big\| \\
&\le \|u_n - q\|.
\end{aligned} \tag{3.11}$$

Since $\lim_{n\to\infty} v_n = v$, $\{v_n\}$ is bounded,

$$\begin{aligned}
\|y_n - q\| &\le \alpha_n\|v_n - q\| + \alpha_n\big\|f(k_n) - q\big\| + (1-\alpha_n)\|k_n - q\| \\
&\le \alpha_n M_1 + \alpha_n\varepsilon\|k_n - q\| + \alpha_n\big\|f(q) - q\big\| + (1-\alpha_n)\|x_n - q\| \\
&= \big[1 - \alpha_n(1-\varepsilon)\big]\|x_n - q\| + \alpha_n\big(M_1 + \big\|f(q) - q\big\|\big) \\
&\le \max\left\{\|x_n - q\|, \frac{1}{1-\varepsilon}\big(M_1 + \big\|f(q) + q\big\|\big)\right\},
\end{aligned} \tag{3.12}$$

where $M_1 = \sup_{n\ge1}\{\|v_n - q\|\}$. Putting $S = \gamma_0 I + \sum_{j\in J}\gamma_j T_j$, by Proposition 3.1, S is nonexpansive. On the other hand, for all $n \in N$, we have

$$\begin{aligned}
\|x_{n+1} - q\| &\le \beta_n\|x_n - q\| + (1-\beta_n)\|St_n - q\| \\
&\le \beta_n\|x_n - q\| + (1-\beta_n)\|t_n - q\| \\
&\le \beta_n\|x_n - q\| + (1-\beta_n)\|y_n - q\| \\
&\le \max\left\{\|x_n - q\|, \frac{1}{1-\varepsilon}\big(M_1 + \big\|f(q) + q\big\|\big)\right\}.
\end{aligned} \tag{3.13}$$

By induction, we deduce that

$$\|x_{n+1}-q\| \leq \max\left\{\|x_1-q\|, \frac{1}{1-\varepsilon}\bigl(M_1+\bigl\|f(q)+q\bigr\|\bigr)\right\}, \quad \forall n\in N. \tag{3.14}$$

Therefore, $\{x_n\}$ is bounded, and so are $\{y_n\}$, $\{u_n\}$, $\{u_{n,i}\}$, $\{t_n\}$, and $\{k_n\}$.

Claim 2 $\|x_{n+1}-x_n\| \to 0$ *as* $n\to\infty$.

Let $z_n = \frac{1}{1-\beta_n}x_{n+1} - \frac{\beta_n}{1-\beta_n}x_n$. Hence

$$\begin{aligned}\|z_{n+1}-z_n\| &= \left\|\frac{1}{1-\beta_{n+1}}(x_{n+2}-\beta_{n+1}x_{n+1}) - \frac{1}{1-\beta_n}(x_{n+1}-\beta_n x_n)\right\| \\ &= \|St_{n+1}-St_n\| \\ &\leq \|t_{n+1}-t_n\|.\end{aligned} \tag{3.15}$$

Now, by the definition of t_n we have

$$\begin{aligned}\|t_{n+1}-t_n\| &\leq \bigl\|(y_{n+1}-\lambda_{n+1}Ay_{n+1})-(y_n-\lambda_n Ay_n)\bigr\| \\ &\leq \|y_{n+1}-y_n\| + |\lambda_{n+1}-\lambda_n|\|Ay_n\|.\end{aligned} \tag{3.16}$$

Similarly,

$$\|k_{n+1}-k_n\| \leq \|u_{n+1}-u_n\| + |\lambda_{n+1}-\lambda_n|\|Au_n\|. \tag{3.17}$$

By (3.17) and the definition of y_n we obtain

$$\begin{aligned}\|y_{n+1}-y_n\| &\leq \alpha_{n+1}\mu\|v_{n+1}-v_n\| + |\alpha_{n+1}-\alpha_n|\|v_n\| + |\alpha_{n+1}-\alpha_n|\bigl\|f(k_n)\bigr\| \\ &\quad + |\alpha_{n+1}-\alpha_n|\|k_n\| + \bigl\|\bigl(I-\alpha_{n+1}(I-f)\bigr)(k_{n+1}) - \bigl(I-\alpha_{n+1}(I-f)\bigr)(k_n)\bigr\| \\ &\leq \alpha_{n+1}\|v_{n+1}-v_n\| + |\alpha_{n+1}-\alpha_n|\bigl(\|v_n\|+\bigl\|f(k_n)\bigr\|+\|k_n\|\bigr) \\ &\quad + \bigl(1-\alpha_{n+1}(1-\varepsilon)\bigr)\|k_{n+1}-k_n\| \\ &\leq \alpha_{n+1}\|v_{n+1}-v_n\| + |\alpha_{n+1}-\alpha_n|\bigl(\|v_n\|+\bigl\|f(k_n)\bigr\|+\|k_n\|\bigr) \\ &\quad + \bigl(1-\alpha_{n+1}(1-\varepsilon)\bigr)\bigl(\|u_{n+1}-u_n\|+|\lambda_{n+1}-\lambda_n|\|Au_n\|\bigr).\end{aligned} \tag{3.18}$$

Furthermore, by the definition of u_n,

$$\begin{aligned}\|u_{n+1}-u_n\| &= \left\|\sum_{i\in I}(\delta_{n+1,i}u_{n+1,i}-\delta_{n,i}u_{n,i})\right\| \\ &\leq \left\|\sum_{i\in I}\delta_{n+1,i}(u_{n+1,i}-u_{n,i})\right\| + \left\|\sum_{i\in I}(\delta_{n+1,i}-\delta_{n,i})u_{n,i}\right\| \\ &\leq \sum_{i\in I}\delta_{n+1,i}\|u_{n+1,i}-u_{n,i}\| + \sum_{i\in I}|\delta_{n+1,i}-\delta_{n,i}|\|u_{n,i}\|.\end{aligned} \tag{3.19}$$

From (3.7), since for each $i \in I$, $u_{n,i}, u_{n+1,i} \in K$,

$$G_i(u_{n+1,i}, u_{n,i}) + \frac{1}{r_{n+1,i}}\langle u_{n,i} - u_{n+1,i}, u_{n+1,i} - (I - r_{n+1,i}B_i)x_{n+1}\rangle \geq 0 \tag{3.20}$$

and

$$G_i(u_{n,i}, u_{n+1,i}) + \frac{1}{r_{n,i}}\langle u_{n+1,i} - u_{n,i}, u_{n,i} - (I - r_{n,i}B_i)x_n\rangle \geq 0. \tag{3.21}$$

By adding the two inequalities (3.20), (3.21), and the monotonicity of G_i we have

$$\left\langle u_{n+1,i} - u_{n,i}, \frac{u_{n,i} - (I - r_{n,i}B_i)x_n}{r_{n,i}} - \frac{u_{n+1,i} - (I - r_{n+1,i}B_i)x_{n+1}}{r_{n+1,i}}\right\rangle \geq 0, \quad \forall i \in I.$$

So

$$\left\langle u_{n+1,i} - u_{n,i}, u_{n,i} - (I - r_{n,i}B_i)x_n - r_{n,i}B_ix_{n+1} - \frac{r_{n,i}}{r_{n+1,i}}(u_{n+1,i} - x_{n+1})\right\rangle \geq 0, \quad \forall i \in I.$$

Thus, for each $i \in I$,

$$\left\langle u_{n+1,i} - u_{n,i}, (I - r_{n,i}B_i)x_{n+1} - (I - r_{n,i}B_i)x_n + (u_{n,i} - u_{n+1,i}) + \left(1 - \frac{r_{n,i}}{r_{n+1,i}}\right)(u_{n+1,i} - x_{n+1})\right\rangle \geq 0.$$

This yields

$$\begin{aligned}
\|u_{n+1,i} - u_{n,i}\|^2 &\leq \Big\langle u_{n+1,i} - u_{n,i}, (I - r_{n,i}B_i)x_{n+1} - (I - r_{n,i}B_i)x_n \\
&\quad + \left(1 - \frac{r_{n,i}}{r_{n+1,i}}\right)(u_{n+1,i} - x_{n+1})\Big\rangle \\
&\leq \|u_{n+1,i} - u_{n,i}\|\bigg[\big\|(I - r_{n,i}B_i)x_{n+1} - (I - r_{n,i}B_i)x_n\big\| \\
&\quad + \left|1 - \frac{r_{n,i}}{r_{n+1,i}}\right|\|u_{n+1,i} - x_{n+1}\|\bigg] \\
&\leq \|u_{n+1,i} - u_{n,i}\|\left[\|x_{n+1} - x_n\| + \left|1 - \frac{r_{n,i}}{r_{n+1,i}}\right|\|u_{n+1,i} - x_{n+1}\|\right], \quad \forall i \in I,
\end{aligned}$$

or

$$\begin{aligned}
\|u_{n+1,i} - u_{n,i}\| &\leq \|x_{n+1} - x_n\| + \frac{1}{r_{n+1,i}}|r_{n+1,i} - r_{n,i}|\|u_{n+1,i} - x_{n+1}\| \\
&\leq \|x_{n+1} - x_n\| + \frac{1}{\tau}|r_{n+1,i} - r_{n,i}|M_2, \quad \forall i \in I,
\end{aligned} \tag{3.22}$$

where $\tau = \inf_{n\geq 1}\{r_{n,i}\}$ and $M_2 = \sup_{n\geq 1}\{\|u_{n,i} - x_n\|\}$. Thus, from (3.15), (3.16), (3.18), (3.19), and (3.22) we obtain

$$\begin{aligned}
\|z_{n+1} - z_n\| &\leq \|x_{n+1} - x_n\| + \alpha_{n+1}\|v_{n+1} - v_n\| \\
&\quad + |\alpha_{n+1} - \alpha_n|\big(\|v_n\| + \|k_n\| + \big\|f(k_n)\big\|\big)
\end{aligned}$$

$$+ |\lambda_{n+1} - \lambda_n| \|Ay_n\| + \big(1 - \alpha_{n+1}(1-\varepsilon)\big)\Bigg[\sum_{i\in I} \delta_{n+1,i}\frac{1}{\tau}|r_{n+1,i} - r_{n,i}|M_2$$
$$+ |\lambda_{n+1} - \lambda_n| \|Au_n\| + \sum_{i\in I} |\delta_{n+1,i} - \delta_{n,i}| \|u_{n,i}\|\Bigg].$$

So, by assumptions 1-6 of the theorem

$$\limsup_{n\to\infty}\big\{\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|\big\} \le 0,$$

and by Lemma 2.3, we have

$$\lim_{n\to\infty} \|z_n - x_n\| = 0.$$

But, since $x_{n+1} - x_n = (1-\beta_n)(z_n - x_n)$, we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.23}$$

Claim 3 $\lim_{n\to\infty} \|x_n - Sx_n\| = 0$.

Note that

$$\begin{aligned}\|x_n - Sx_n\| &\le \|x_{n+1} - x_n\| + \|x_{n+1} - St_n\| + \|St_n - Sx_n\| \\ &\le \|x_{n+1} - x_n\| + \|x_{n+1} - St_n\| + \|t_n - x_n\|. \end{aligned} \tag{3.24}$$

First we show that $\lim_{n\to\infty} \|x_{n+1} - St_n\| = 0$. From (3.5)

$$\begin{aligned}\|x_{n+1} - St_n\| &\le \beta_n \|x_n - St_n\| \\ &\le \beta_n \|x_n - x_{n+1}\| + \beta_n \|x_{n+1} - St_n\|.\end{aligned}$$

Hence

$$\|x_{n+1} - St_n\| \le \frac{\beta_n}{1-\beta_n}\|x_n - x_{n+1}\|.$$

This implies that

$$\lim_{n\to\infty} \|x_{n+1} - St_n\| = 0. \tag{3.25}$$

Now, we prove that $\lim_{n\to\infty} \|t_n - x_n\| = 0$. To do this, it suffices to show that $\lim_{n\to\infty} \|x_n - u_n\| = 0$ and $\lim_{n\to\infty} \|u_n - t_n\| = 0$. By the definition of t_n we have

$$\begin{aligned}\|t_n - q\|^2 &\le \big\|(y_n - \lambda_n Ay_n) - (q - \lambda_n Aq)\big\|^2 \\ &\le \|y_n - q\|^2 + \lambda_n(\lambda_n - 2\sigma)\|Ay_n - Aq\|^2 \\ &\le \|x_n - q\|^2 + \lambda_n(\lambda_n - 2\sigma)\|Ay_n - Aq\|^2. \end{aligned} \tag{3.26}$$

So, by (3.26) and the convexity of $\|\cdot\|^2$, we have

$$\begin{aligned}\|x_{n+1}-q\|^2 &\le \beta_n\|x_n-q\|^2+(1-\beta_n)\|St_n-q\|^2\\ &\le \beta_n\|x_n-q\|^2+(1-\beta_n)\|t_n-q\|^2\\ &\le \beta_n\|x_n-q\|^2+(1-\beta_n)\bigl(\|x_n-q\|^2+\lambda_n(\lambda_n-2\sigma)\|Ay_n-Aq\|^2\bigr)\\ &= \|x_n-q\|^2+(1-\beta_n)\lambda_n(\lambda_n-2\sigma)\|Ay_n-Aq\|^2.\end{aligned}$$

Hence

$$\begin{aligned}(1-\beta_n)\lambda_n(2\sigma-\lambda_n)\|Ay_n-Aq\|^2 &\le \|x_n-q\|^2-\|x_{n+1}-q\|^2\\ &\le \|x_n-x_{n+1}\|\bigl(\|x_n-q\|+\|x_{n+1}-q\|\bigr),\end{aligned}$$

and then

$$\lim_{n\to\infty}\|Ay_n-Aq\|=0. \tag{3.27}$$

Using the projection properties gives us

$$\begin{aligned}\|t_n-q\|^2 &= \bigl\|P_K(y_n-\lambda_nAy_n)-P_K(q-\lambda_nAq)\bigr\|^2\\ &\le \bigl\langle (y_n-\lambda_nAy_n)-(q-\lambda_nAq),t_n-q\bigr\rangle\\ &= \frac{1}{2}\bigl[\bigl\|(y_n-\lambda_nAy_n)-(q-\lambda_nAq)\bigr\|^2+\|t_n-q\|^2\\ &\quad -\bigl\|(y_n-\lambda_nAy_n)-(q-\lambda_nAq)-(t_n-q)\bigr\|^2\bigr]\\ &\le \frac{1}{2}\bigl[\|y_n-q\|^2+\|t_n-q\|^2-\bigl\|y_n-t_n-\lambda_n(Ay_n-Aq)\bigr\|^2\bigr]\\ &\le \frac{1}{2}\bigl[\|y_n-q\|^2+\|t_n-q\|^2-\|y_n-t_n\|^2-\lambda_n^2\|Ay_n-Aq\|^2\\ &\quad +2\lambda_n\langle y_n-t_n,Ay_n-Aq\rangle\bigr].\end{aligned}$$

This implies that

$$\begin{aligned}\|t_n-q\|^2 &\le \|y_n-q\|^2-\|y_n-t_n\|^2-\lambda_n^2\|Ay_n-Aq\|^2\\ &\quad +2\lambda_n\langle y_n-t_n,Ay_n-Aq\rangle\\ &\le \|y_n-q\|^2-\|y_n-t_n\|^2+2\lambda_n\langle y_n-t_n,Ay_n-Aq\rangle.\end{aligned} \tag{3.28}$$

From (3.28) and the convexity of $\|\cdot\|^2$, one can see that, for $q\in\Omega$,

$$\begin{aligned}\|x_{n+1}-q\|^2 &\le \beta_n\|x_n-q\|^2+(1-\beta_n)\|St_n-q\|^2\\ &\le \beta_n\|x_n-q\|^2+(1-\beta_n)\|t_n-q\|^2\\ &\le \beta_n\|x_n-q\|^2+(1-\beta_n)\bigl[\|y_n-q\|^2-\|y_n-t_n\|^2\\ &\quad +2\lambda_n\langle y_n-t_n,Ay_n-Aq\rangle\bigr]\end{aligned}$$

$$\begin{aligned}
&\leq \beta_n\|x_n-q\|^2+(1-\beta_n)\big[\alpha_n\|v_n-q\|^2+\alpha_n\big\|f(k_n)-q\big\|^2 \\
&\quad+(1-\alpha_n)\|k_n-q\|^2-\|y_n-t_n\|^2+2\lambda_n\langle y_n-t_n,Ay_n-Aq\rangle\big] \\
&\leq \beta_n\|x_n-q\|^2+(1-\beta_n)\big[\alpha_n\|v_n-q\|^2+\alpha_n\big\|f(k_n)-q\big\|^2 \\
&\quad+(1-\alpha_n)\|x_n-q\|^2-\|y_n-t_n\|^2+2\lambda_n\langle y_n-t_n,Ay_n-Aq\rangle\big].
\end{aligned}$$

Hence

$$\begin{aligned}
(1-\beta_n)\|y_n-t_n\|^2 &\leq (1-\beta_n)\big[\alpha_n\|v_n-q\|^2+\alpha_n\big\|f(k_n)-q\big\|^2 \\
&\quad+2\lambda_n\langle y_n-t_n,Ay_n-Aq\rangle\big] \\
&\quad+\|x_n-q\|^2-\|x_{n+1}-q\|^2 \\
&\leq (1-\beta_n)\big[\alpha_n\|v_n-q\|^2+\alpha_n\big\|f(k_n)-q\big\|^2 \\
&\quad+2\lambda_n\|y_n-t_n\|\|Ay_n-Aq\|\big] \\
&\quad+\|x_n-x_{n+1}\|\big(\|x_n-q\|^2+\|x_{n+1}-q\|^2\big),
\end{aligned}$$

and so by (3.27)

$$\lim_{n\to\infty}\|y_n-t_n\|=0. \tag{3.29}$$

Next, we show that $\lim_{n\to\infty}\|y_n-u_n\|=0$. The definition of k_n and a similar argument to (3.26) give us

$$\|k_n-q\|^2\leq\|x_n-q\|^2+\lambda_n(\lambda_n-2\sigma)\|Au_n-Aq\|^2. \tag{3.30}$$

Then

$$\begin{aligned}
\|x_{n+1}-q\|^2 &\leq \beta_n\|x_n-q\|^2+(1-\beta_n)\|St_n-q\|^2 \\
&\leq \beta_n\|x_n-q\|^2+(1-\beta_n)\|t_n-q\|^2 \\
&\leq \beta_n\|x_n-q\|^2+(1-\beta_n)\|y_n-q\|^2 \\
&\leq \beta_n\|x_n-q\|^2+(1-\beta_n)\big(\alpha_n\|v_n-q\|^2 \\
&\quad+\alpha_n\big\|f(k_n)-q\big\|^2+(1-\alpha_n)\|k_n-q\|^2\big) \\
&\leq \|x_n-q\|^2+(1-\beta_n)\big(\alpha_n\|v_n-q\|^2+\alpha_n\big\|f(k_n)-q\big\|^2\big) \\
&\quad+(1-\beta_n)(1-\alpha_n)\lambda_n(\lambda_n-2\sigma)\|Au_n-Aq\|^2.
\end{aligned}$$

Hence

$$\begin{aligned}
(1-\beta_n)(1-\alpha_n)\lambda_n(2\sigma-\lambda_n)\|Au_n-Aq\|^2 &\leq \|x_n-q\|^2-\|x_{n+1}-q\|^2 \\
&\leq \|x_n-x_{n+1}\|\big(\|x_n-q\|+\|x_{n+1}-q\|\big),
\end{aligned}$$

and therefore

$$\lim_{n\to\infty}\|Au_n-Aq\|=0. \tag{3.31}$$

Similar to (3.28) we can see that

$$\begin{aligned}\|k_n - q\|^2 &\le \|u_n - q\|^2 - \|u_n - k_n\|^2 + 2\lambda_n\langle u_n - k_n, Au_n - Aq\rangle \\ &\le \|x_n - q\|^2 - \|u_n - k_n\|^2 + 2\lambda_n\langle u_n - k_n, Au_n - Aq\rangle.\end{aligned} \tag{3.32}$$

From (3.32) and the convexity of $\|\cdot\|^2$, we have

$$\begin{aligned}\|x_{n+1} - q\|^2 &\le \beta_n\|x_n - q\|^2 + (1-\beta_n)\|St_n - q\|^2 \\ &\le \beta_n\|x_n - q\|^2 + (1-\beta_n)\|t_n - q\|^2 \\ &\le \beta_n\|x_n - q\|^2 + (1-\beta_n)\|y_n - q\|^2 \\ &\le \beta_n\|x_n - q\|^2 + (1-\beta_n)\big[\alpha_n\|v_n - q\|^2 \\ &\quad + \alpha_n\big\|f(k_n) - q\big\|^2 + (1-\alpha_n)\|k_n - q\|^2\big] \\ &\le \beta_n\|x_n - q\|^2 + (1-\beta_n)\big[\alpha_n\|v_n - q\|^2 + \alpha_n\big\|f(k_n) - q\big\|^2 \\ &\quad + (1-\alpha_n)\big(\|x_n - q\|^2 - \|u_n - k_n\|^2 + 2\lambda_n\langle u_n - k_n, Au_n - Aq\rangle\big)\big] \\ &\le \|x_n - q\|^2 + (1-\beta_n)\big[\alpha_n\|v_n - q\|^2 + \alpha_n\big\|f(k_n) - q\big\|^2 \\ &\quad + (1-\alpha_n)2\lambda_n\langle u_n - k_n, Au_n - Aq\rangle\big] - (1-\beta_n)(1-\alpha_n)\|u_n - k_n\|^2.\end{aligned}$$

So

$$\begin{aligned}(1-\beta_n)(1-\alpha_n)\|u_n - k_n\|^2 &\le (1-\beta_n)\big[\alpha_n\|v_n - q\|^2 + \alpha_n\big\|f(k_n) - q\big\|^2\big] \\ &\quad + (1-\beta_n)(1-\alpha_n)2\lambda_n\langle u_n - k_n, Au_n - Aq\rangle \\ &\quad + \|x_n - q\|^2 - \|x_{n+1} - q\|^2 \\ &\le (1-\beta_n)\big[\alpha_n\|v_n - q\|^2 + \alpha_n\big\|f(k_n) - q\big\|^2\big] \\ &\quad + (1-\beta_n)(1-\alpha_n)2\lambda_n\|u_n - k_n\|\|Au_n - Aq\| \\ &\quad + \|x_n - x_{n+1}\|\big(\|x_n - q\| + \|x_{n+1} - q\|\big).\end{aligned}$$

Then the above inequality and (3.31) imply that

$$\lim_{n\to\infty}\|u_n - k_n\| = 0. \tag{3.33}$$

But from (3.5),

$$\|y_n - u_n\| \le \alpha_n\|v_n - u_n\| + \alpha_n\big\|f(k_n) - u_n\big\| + (1-\alpha_n)\|k_n - u_n\|.$$

So, from (3.33) we have

$$\lim_{n\to\infty}\|y_n - u_n\| = 0. \tag{3.34}$$

Then by (3.29) and (3.34) we have

$$\lim_{n\to\infty}\|u_n - t_n\| = 0. \tag{3.35}$$

Now, we show that $\lim_{n\to\infty}\|x_n - u_n\| = 0$. To do this, note that, for any $i \in I$,

$$\begin{aligned}
\|u_{n,i} - q\|^2 &= \bigl\|T_{r_{n,i}}(x_n - r_{n,i}B_ix_n) - T_{r_{n,i}}(q - r_{n,i}B_iq)\bigr\|^2 \\
&\le \bigl\langle T_{r_{n,i}}(x_n - r_{n,i}B_ix_n) - T_{r_{n,i}}(q - r_{n,i}B_iq), (x_n - q) - r_{n,i}(B_ix_n - B_iq)\bigr\rangle \\
&= \langle u_{n,i} - q, x_n - q\rangle - r_{n,i}\langle u_{n,i} - q, B_ix_n - B_iq\rangle \\
&\le \langle u_{n,i} - q, x_n - q\rangle - r_{n,i}\theta_i\|B_ix_n - B_iq\|^2 \\
&\le \langle u_{n,i} - q, x_n - q\rangle.
\end{aligned} \tag{3.36}$$

So, from (3.36) and the definition of u_n, we obtain

$$\begin{aligned}
\|u_n - q\|^2 &\le \sum_{i\in I}\delta_{n,i}\|u_{n,i} - q\|^2 \\
&\le \sum_{i\in I}\delta_{n,i}\langle u_{n,i} - q, x_n - q\rangle \\
&= \Bigl\langle \sum_{i\in I}\delta_{n,i}u_{n,i} - q, x_n - q\Bigr\rangle \\
&= \langle u_n - q, x_n - q\rangle \\
&\le \frac{1}{2}\bigl[\|u_n - q\|^2 + \|x_n - q\|^2 - \|u_n - x_n\|^2\bigr].
\end{aligned}$$

Thus,

$$\|u_n - q\|^2 \le \|x_n - q\|^2 - \|u_n - x_n\|^2. \tag{3.37}$$

Since S is nonexpansive, we have

$$\begin{aligned}
\|x_{n+1} - q\|^2 &\le \beta_n\|x_n - q\|^2 + (1-\beta_n)\|St_n - q\|^2 \\
&\le \beta_n\|x_n - q\|^2 + (1-\beta_n)\|t_n - q\|^2 \\
&\le \beta_n\|x_n - q\|^2 + (1-\beta_n)\|y_n - q\|^2 \\
&\le \beta_n\|x_n - q\|^2 + (1-\beta_n)\bigl(\alpha_n\|v_n - q\|^2 + \alpha_n\bigl\|f(k_n) - q\bigr\|^2\bigr) \\
&\quad + (1-\beta_n)(1-\alpha_n)\|k_n - q\|^2 \\
&\le \beta_n\|x_n - q\|^2 + (1-\beta_n)\bigl(\alpha_n\|v_n - q\|^2 + \alpha_n\bigl\|f(k_n) - q\bigr\|^2\bigr) \\
&\quad + (1-\beta_n)(1-\alpha_n)\|u_n - q\|^2 \\
&\le \beta_n\|x_n - q\|^2 + (1-\beta_n)\bigl(\alpha_n\|v_n - q\|^2 + \alpha_n\bigl\|f(k_n) - q\bigr\|^2\bigr) \\
&\quad + (1-\beta_n)(1-\alpha_n)\bigl(\|x_n - q\|^2 - \|x_n - u_n\|^2\bigr).
\end{aligned}$$

Hence

$$\begin{aligned}
(1-\beta_n)(1-\alpha_n)\|x_n - u_n\|^2 &\le (1-\beta_n)\bigl(\alpha_n\|v_n - q\|^2 + \alpha_n\bigl\|f(k_n) - q\bigr\|^2\bigr) \\
&\quad + \|x_n - q\|^2 - \|x_{n+1} - q\|^2
\end{aligned}$$

$$\begin{aligned}
&\leq (1-\beta_n)\big(\alpha_n\|v_n-q\|^2+\alpha_n\big\|f(k_n)-q\big\|^2\big) \\
&\quad+\big(\|x_n-q\|-\|x_{n+1}-q\|\big)\big(\|x_n-q\|+\|x_{n+1}-q\|\big) \\
&\leq (1-\beta_n)\big(\alpha_n\|v_n-q\|^2+\alpha_n\big\|f(k_n)-q\big\|^2\big) \\
&\quad+\big(\|x_n-q\|+\|x_{n+1}-q\|\big)\|x_n-x_{n+1}\|,
\end{aligned}$$

which yields

$$\lim_{n\to\infty}\|x_n-u_n\|=0. \tag{3.38}$$

Since $\|t_n-x_n\|\leq\|t_n-u_n\|+\|u_n-x_n\|$, from (3.35) and (3.38) we obtain

$$\lim_{n\to\infty}\|t_n-x_n\|=0. \tag{3.39}$$

Inequality (3.24) and equations (3.25), (3.39), and $\|x_n-x_{n+1}\|\to 0$ imply that

$$\lim_{n\to\infty}\|x_n-Sx_n\|=0. \tag{3.40}$$

Claim 4 $\limsup_{n\to\infty}\langle v+f(z)-z,y_n-z\rangle\leq 0$, *where* $z=P_\Omega(v+f(z))$.

To prove the claim, let $\{y_{n_k}\}$ be a subsequence of $\{y_n\}$ such that

$$\limsup_{n\to\infty}\big\langle v+f(z)-z,y_n-z\big\rangle=\limsup_{n\to\infty}\big\langle v+f(z)-z,y_{n_k}-z\big\rangle. \tag{3.41}$$

By boundedness of $\{y_{n_k}\}$, there exists a subsequence of $\{y_{n_k}\}$ which is weakly convergent to $z_0\in K$. Without loss of generality, we can assume that $y_{n_k}\rightharpoonup z_0$. So, (3.41) reduces to

$$\limsup_{n\to\infty}\big\langle v+f(z)-z,y_n-z\big\rangle=\big\langle v+f(z)-z,z_0-z\big\rangle. \tag{3.42}$$

Therefore, by projection properties, to prove $\langle v+f(z)-z,z_0-z\rangle\geq 0$, it suffices to show that $z_0\in\Omega$.

(a) First we prove that $z_0\in\bigcap_{j\in J}^m F(T_j)$. From (3.40) and the demiclosedness property of S we obtain $z_0\in F(S)$. So, by Proposition 3.1, $z_0\in\bigcap_{j\in J}^m F(T_j)$.

(b) Next we show that $z_0\in\mathrm{VI}(A,K)$. Note that from boundedneess of $\{x_n\}$, $\{u_n\}$, and equation (3.33), there exist subsequences $\{x_{n_k}\}$ and $\{u_{n_k}\}$ of $\{x_n\}$ and $\{u_n\}$, respectively, which converge weakly to z_0. Suppose that N_Kx is a normal cone to K at x and Q is a mapping defined by

$$Q(x)=\begin{cases}Ax+N_Kx, & x\in K,\\ \emptyset, & x\notin K.\end{cases} \tag{3.43}$$

It is well known that Q is a maximal monotone mapping and $0\in Q(x)$ if and only if $x\in\mathrm{VI}(A,K)$. For details see [2]. If $(x,u)\in G(Q)$, then $u-Ax\in N_Kx$. Since $k_n=P_K(u_n-\lambda_nAu_n)\in K$, we have

$$\langle x-k_n,u-Ax\rangle\geq 0. \tag{3.44}$$

In addition, from projection properties we have $\langle x - k_n, k_n - (u_n - \lambda_n A u_n)\rangle \geq 0$. Then $\langle x - k_n, \frac{k_n - u_n}{\lambda_n} + Au_n\rangle \geq 0$. Hence, from (3.44) we have

$$\begin{aligned}\langle x - k_{n_k}, u\rangle &\geq \langle x - k_{n_k}, Ax\rangle \\ &\geq \langle x - k_{n_k}, Ax\rangle - \left\langle x - k_{n_k}, \frac{k_{n_k} - u_{n_k}}{\lambda_{n_k}} + Au_{n_k}\right\rangle \\ &= \langle x - k_{n_k}, Ax - Ak_{n_k}\rangle + \langle x - k_{n_k}, Ak_{n_k} - Au_{n_k}\rangle \\ &\quad - \left\langle x - k_{n_k}, \frac{k_{n_k} - u_{n_k}}{\lambda_{n_k}}\right\rangle \\ &\geq \langle x - k_{n_k}, Ak_{n_k} - Au_{n_k}\rangle - \left\langle x - k_{n_k}, \frac{k_{n_k} - u_{n_k}}{\lambda_{n_k}}\right\rangle. \end{aligned} \tag{3.45}$$

Since A is a continuous mapping, from (3.34) and (3.45) we deduce that

$$\langle x - z_0, u\rangle \geq 0, \quad \text{as } k \to \infty.$$

Therefore, from maximal monotonicity of Q, we obtain $0 \in Q(z_0)$ and hence $z_0 \in \mathrm{VI}(A, K)$.

(c) Now we prove that $z_0 \in \bigcap_{i\in I} \mathrm{GEP}(G_i, B_i)$. For all $i \in I$, by (3.36),

$$\begin{aligned}\|u_{n,i} - q\|^2 &\leq \langle u_{n,i} - q, x_n - q\rangle \\ &\leq \frac{1}{2}\left[\|u_{n,i} - q\|^2 + \|x_n - q\|^2 - \|u_{n,i} - x_n\|^2\right]\end{aligned}$$

and then

$$\|u_{n,i} - q\|^2 \leq \|x_n - q\|^2 - \|u_{n,i} - x_n\|^2.$$

This implies that

$$\begin{aligned}\|u_n - q\|^2 &\leq \sum_{i\in I} \delta_{n,i}\|u_{n,i} - q\|^2 \\ &\leq \|x_n - q\|^2 - \sum_{i\in I} \delta_{n,i}\|u_{n,i} - x_n\|^2.\end{aligned}$$

Therefore, for any $i \in I$,

$$\begin{aligned}\|u_{n,i} - x_n\|^2 &\leq \sum_{i\in I} \delta_{n,i}\|u_{n,i} - x_n\|^2 \leq \|x_n - q\|^2 - \|u_n - q\|^2 \\ &\leq \|x_n - u_n\|\big(\|x_n - q\| + \|u_n - q\|\big).\end{aligned}$$

So by (3.38),

$$\lim_{n\to\infty} \|u_{n,i} - x_n\| = 0, \quad \forall i \in I. \tag{3.46}$$

Since $\{u_{n,i}\}_{i\in I}$ is bounded, by (3.46), there exists a weakly convergent subsequence $\{u_{n_k,i}\}$ of $\{u_{n,i}\}$ to z_0. Now, we will show that, for any $i \in I$, z_0 is a member of $\mathrm{GEP}(G_i, B_i)$. Since

$u_{n,i} = T_{r_{n,i}}(x_n - r_{n,i}B_i x_n)$, for all $y \in K$ we have

$$G_i(u_{n,i}, y) + \langle B_i x_n, y - u_{n,i}\rangle + \frac{1}{r_{n,i}}\langle y - u_{n,i}, u_{n,i} - x_n\rangle \geq 0, \quad \forall i \in I.$$

From (A2) we obtain

$$\langle B_i x_n, y - u_{n,i}\rangle + \frac{1}{r_{n,i}}\langle y - u_{n,i}, u_{n,i} - x_n\rangle \geq G_i(y, u_{n,i}), \quad \forall y \in K, \forall i \in I.$$

Hence, for all $y \in K$,

$$\langle B_i x_{n_k}, y - u_{n_k,i}\rangle + \left\langle y - u_{n_k,i}, \frac{u_{n_k,i} - x_{n_k}}{r_{n_k,i}}\right\rangle \geq G_i(y, u_{n_k,i}), \quad \forall i \in I. \tag{3.47}$$

Let $y_t = ty + (1-t)z_0$, where $t \in (0,1]$ and $y \in K$. Then $y_t \in K$ and by (3.47),

$$\begin{aligned}
\langle y_t - u_{n_k,i}, B_i y_t\rangle &\geq \langle y_t - u_{n_k,i}, B_i y_t\rangle - \langle y_t - u_{n_k,i}, B_i x_{n_k}\rangle \\
&\quad - \left\langle y_t - u_{n_k,i}, \frac{u_{n_k,i} - x_{n_k}}{r_{n_k,i}}\right\rangle + G_i(y_t, u_{n_k,i}) \\
&= \langle y_t - u_{n_k,i}, B_i y_t - B_i u_{n_k,i}\rangle + \langle y_t - u_{n_k,i}, B_i u_{n_k,i} - B_i x_{n_k}\rangle \\
&\quad - \left\langle y_t - u_{n_k,i}, \frac{u_{n_k,i} - x_{n_k}}{r_{n_k,i}}\right\rangle + G_i(y_t, u_{n_k,i}), \quad \forall i \in I.
\end{aligned}$$

But B_i is a θ_i-inverse strongly monotone mapping and $\|u_{n_k,i} - x_{n_k}\| \to 0$, so $\|B_i u_{n_k,i} - B_i x_{n_k}\| \to 0$ and $\langle y_t - u_{n_k,i}, B_i y_t - B_i u_{n_k,i}\rangle \geq 0$, for all $i \in I$. As $k \to \infty$, the relations $\frac{u_{n_k,i} - x_{n_k}}{r_{n_k,i}} \to 0$, $u_{n_k,i} \rightharpoonup 0$, and condition (A4) imply that

$$\langle y_t - z_0, B_i y_t\rangle \geq G_i(y_t, z_0), \quad \forall i \in I. \tag{3.48}$$

From (A1), (A4), and (3.48) we have

$$\begin{aligned}
0 = G_i(y_t, y_t) &\leq tG_i(y_t, y) + (1-t)G_i(y_t, z_0) \\
&\leq tG_i(y_t, y) + (1-t)\langle y_t - z_0, B_i y_t\rangle \\
&= tG_i(y_t, y) + (1-t)t\langle y - z_0, B_i y_t\rangle \\
&\leq G_i(y_t, y) + (1-t)\langle y - z_0, B_i y_t\rangle, \quad \forall i \in I.
\end{aligned}$$

Letting $t \to 0$, so for each $y \in K$,

$$G_i(z_0, y) + \langle y - z_0, B_i z_0\rangle \geq 0, \quad \forall i \in I.$$

That is, $z_0 \in \mathrm{GEP}(G_i, B_i)$, for all $i \in I$. Now by parts (a), (b) and (c), $z_0 \in \Omega$. Therefore, from (3.42) we obtain

$$\limsup_{n\to\infty}\langle \nu + f(z) - z, y_n - z\rangle = \langle \nu + f(z) - z, z_0 - z\rangle \leq 0. \tag{3.49}$$

Claim 5 *The sequence* $\{x_n\}$ *converges to* z, *where* $z = P_\Omega(\nu + f(z))$.

From the convexity of $\|\cdot\|^2$ and (2.1) we deduce that

$$\begin{aligned}
\|x_{n+1} - z\|^2 &\le \beta_n \|x_n - z\|^2 + (1-\beta_n)\|St_n - z\|^2 \\
&\le \beta_n \|x_n - z\|^2 + (1-\beta_n)\|t_n - z\|^2 \\
&\le \beta_n \|x_n - z\|^2 + (1-\beta_n)\|y_n - z\|^2 \\
&\le \beta_n \|x_n - z\|^2 + (1-\beta_n)\big\|\alpha_n\big[\nu_n + f(k_n) - z\big] \\
&\quad + (1-\alpha_n)(k_n - z)\big\|^2 \\
&\le \beta_n \|x_n - z\|^2 + (1-\beta_n)(1-\alpha_n)^2\|k_n - z\|^2 \\
&\quad + 2\alpha_n(1-\beta_n)\big\langle \nu_n + f(k_n) - z, y_n - z\big\rangle \\
&\le \beta_n \|x_n - z\|^2 + (1-\beta_n)(1-\alpha_n)\|x_n - z\|^2 \qquad (3.50) \\
&\quad + 2\alpha_n(1-\beta_n)\big\langle \nu_n + f(k_n) - z, y_n - z\big\rangle \\
&= \big(1 - \alpha_n(1-\beta_n)\big)\|x_n - z\|^2 + \gamma_n, \qquad (3.51)
\end{aligned}$$

where $\gamma_n = 2\alpha_n(1-\beta_n)\langle \nu_n + f(x_n) - z, y_n - z\rangle$. On the other hand

$$\begin{aligned}
\gamma_n &= 2\alpha_n(1-\beta_n)\big\langle \nu_n + f(k_n) - z, y_n - z\big\rangle \\
&= 2\alpha_n(1-\beta_n)\big\langle (\nu_n - \nu) + \big(f(k_n) - f(z)\big), y_n - z\big\rangle \\
&\quad + 2\alpha_n(1-\beta_n)\big\langle \nu + f(z) - z, y_n - z\big\rangle \\
&\le 2\alpha_n(1-\beta_n)\big\{\|\nu_n - \nu\| + \varepsilon\|k_n - z\|\big\}\|y_n - z\| \\
&\quad + 2\alpha_n(1-\beta_n)\big\langle \nu + f(z) - z, y_n - z\big\rangle \\
&\le \alpha_n(1-\beta_n)\big(\|\nu_n - \nu\|^2 + \|y_n - z\|^2\big) \\
&\quad + \alpha_n(1-\beta_n)\varepsilon\big(\|k_n - z\|^2 + \|y_n - z\|^2\big) \\
&\quad + 2\alpha_n(1-\beta_n)\big\langle \nu + f(z) - z, y_n - z\big\rangle.
\end{aligned}$$

Suppose that $M_0 = \sup_{n\in N}\{\|y_n - z\|\}$. So

$$\begin{aligned}
\gamma_n &\le \alpha_n(1-\beta_n)\varepsilon\|x_n - z\|^2 + \alpha_n(1-\beta_n)\big[M_1^2 + (1+\varepsilon)M_0^2\big] \\
&\quad + 2\alpha_n(1-\beta_n)\big\langle \nu + f(z) - z, y_n - z\big\rangle. \qquad (3.52)
\end{aligned}$$

Substitute (3.52) in (3.50), then

$$\begin{aligned}
\|x_{n+1} - z\|^2 &\le \big(1 - \alpha_n(1-\beta_n)\big)\|x_n - z\|^2 + \alpha_n(1-\beta_n)\varepsilon\|x_n - z\|^2 \\
&\quad + \alpha_n(1-\beta_n)\big[(1+\varepsilon)M_1^2 + M_0^2\big] + 2\alpha_n(1-\beta_n)\big\langle \nu + f(z) - z, y_n - z\big\rangle \\
&\le \big[1 - \alpha_n(1-\beta_n)(1-\varepsilon)\big]\|x_n - z\|^2 + \alpha_n(1-\beta_n)M \\
&\quad + 2\alpha_n(1-\beta_n)\big\langle \nu + f(z) - z, y_n - z\big\rangle,
\end{aligned}$$

where $M = (1+\varepsilon)M_1^2 + M_0^2$. Therefore from (3.49) and Lemma 2.1, we conclude that $\lim_{n\to\infty}\|x_n - z\| = 0$. Also from (3.34) and (3.38) we can see that $y_n \to z$ and $u_n \to z$. This completes the proof. □

Let $m = 1$ in the index set I and take $\delta_{n,1} = 1$, so (3.5) becomes the following algorithm:

$$\begin{cases} \Theta(u_n, y) + \langle Cu_n + Bx_n, y - u_n\rangle + \varphi(y) - \varphi(u_n) \\ \quad + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in K, \\ y_n = \alpha_n v_n + (I - \alpha_n(I - f))P_K(u_n - \lambda_n A u_n), \\ x_{n+1} = \beta_n x_n + (1-\beta_n)SP_K(y_n - \lambda_n A y_n). \end{cases} \tag{3.53}$$

Put $\varphi = 0$, $C = 0$, and $\{v_n\} = \{0\}$ in (3.53). If $A = 0$, then by the projection properties, $k_n = P_\Omega u_n$. Since $u_n \in C$, we have $k_n = u_n$. So, we get the following corollary which is the so-called viscosity approximation method.

Corollary 3.3 *Let $\Theta : K \times K \to R$ be a bifunction satisfying* (A1)-(A4) *and B a θ-inverse strongly monotone. Let $S : K \to K$ be a nonexpansive and $f : K \to K$ be an ε-contraction mapping. Suppose that $\Omega = \mathrm{GEP}(\Theta, B) \cap F(S)$ is nonempty. For any initial guess $x_1 \in K$, define the sequence $\{x_n\}$ by*

$$\begin{cases} \Theta(u_n, y) + \langle Bx_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in K, \\ y_n = \alpha_n f(x_n) + (1-\alpha_n)u_n, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)Sy_n, \end{cases} \tag{3.54}$$

where $\{r_n\}$ is a positive real sequence, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ *satisfying the following conditions*:

1. $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$;
2. $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$;
3. *for some $\tau, \rho \in (0, 2\theta)$, $r_n \in [\tau, \rho]$ and $\lim_{n\to\infty}(r_{n+1} - r_n) = 0$.*

Then the sequence $\{x_n\}$ converges strongly to $z \in \Omega$, where $z = P_\Omega fz$.

4 Numerical example

In this section, we present a numerical example which supports our algorithm.

Example 1 Suppose $H = R$ and $K = [-200, 200]$. A system of generalized mixed equilibrium problem is to find a point $x \in K$ such that, for each $i \in I$,

$$\Theta_i(x, y) + \langle A_i x, y - x\rangle + \varphi_i(y) - \varphi_i(x) \geq 0, \quad \forall y \in K. \tag{4.1}$$

For any $i \in I$, define $\varphi_i = 0$, $\Theta_i(x, y) = (y + ix)(y - x)$ and $A_i x = ix$. It is easy to see that, for each $i \in I$, $\Theta_i(x, y)$ satisfies the conditions (A1)-(A4) and A_i is $\frac{1}{i+1}$-inverse strongly monotone mapping. We know that, for each $i \in I$, T_{r_i} is single valued. Thus for any $y \in k$ and $r_i > 0$, we have

$$\begin{aligned} &\Theta_i(u_i, y) + \langle A_i x, y - u_i\rangle + \frac{1}{r_i}\langle y - u_i, u_i - x\rangle \geq 0 \\ &\iff \quad \Theta_i(u_i, y) + \frac{1}{r_i}\bigl\langle y - u_i, u_i - (I - A_i)x\bigr\rangle \geq 0 \\ &\iff \quad r_i y^2 + \bigl[\bigl(1 + r_i(i-1)\bigr)u_i - (1 - ir_i)x\bigr]y + \bigl[(1 - ir_i)u_i x - (1 + ir_i)u_i^2\bigr] \geq 0. \end{aligned}$$

Let $Q_i(y) = r_i y^2 + [(1 + r_i(i-1))u_i - (1 - ir_i)x]y + [(1 - ir_i)u_i x - (1 + ir_i)u_i^2]$. Since Q_i is a quadratic function relative to y, $Q_i(y) \geq 0$ for all $y \in K$, if and only if the coefficient of y^2 is positive and the discriminant $\Delta_i \leq 0$. But

$$\begin{aligned}\Delta_i &= \big[\big(1 + r_i(i-1)\big)u_i - (1 - ir_i)x\big]^2 \\ &\quad - 4r_i\big[(1 - ir_i)u_i x - (1 + ir_i)u_i^2\big] \\ &= \big[\big(1 + r_i(i+1)\big)u_i - (1 - ir_i)x\big]^2,\end{aligned}$$

so we obtain

$$u_i = \frac{(1 - ir_i)}{1 + r_i(i+1)}x$$

and then

$$T_{r_i}(x) = \frac{(1 - ir_i)}{1 + r_i(i+1)}x.$$

Table 1 The behavior of x_n with $x_0 = 10$ and $x_0 = -10$

Iterative number	Initial point	
	$x_0 = 10$	$x_0 = -10$
1	3.3333	–3.5157
2	0.9411	–1.5857
3	0.2374	–0.6642
4	0.0556	–0.2619
5	0.0124	–0.0989
6	0.0027	–0.0362
7	0.0006	–0.0129
8	0.0001	–0.0045
9	0.0000	–0.0016
10	0.0000	–0.0005
11	0.0000	–0.0002
12	0.0000	–0.0001
13	0.0000	–0.0000

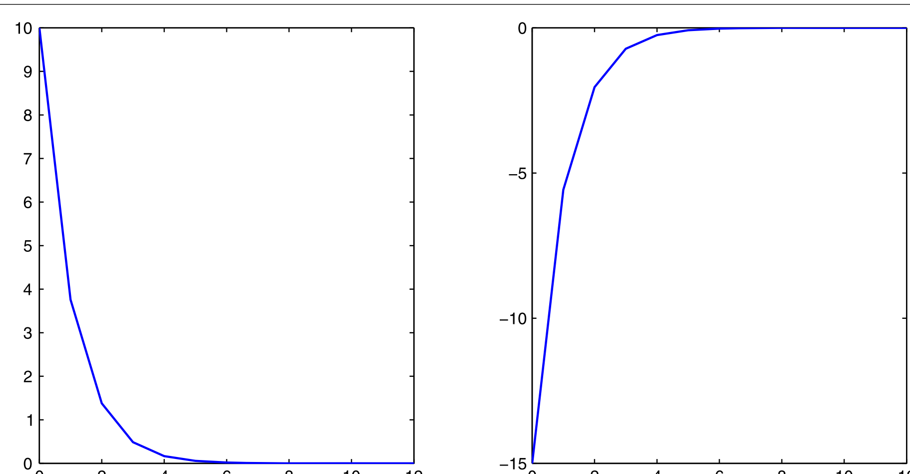

Figure 1 Convergence of the algorithm with initial values $x_0 = 10$ and $x_0 = -10$.

From Lemma 2.5, we have $F(T_{r_i}) = \mathrm{GEP}(\Theta, A_i) = 0$. Define $S : K \to K$ by $S(x) = \sin(x)$. Then S is nonexpansive and $F(\sin(x)) = \{0\}$. So, $\Omega = \{0\}$. Assume that $I = \{1, 2\}$, $A = 0$, $\{\nu_n\} = \{0\}$, $f(x) = \frac{x}{2}$, $r_{n,i} = \frac{2n}{(n+1)(i+1)}$, $\alpha_n = \frac{1}{n}$, $\beta_n = \frac{1}{3}$ and $\delta_{n,i} = \frac{1}{2}$, $C_i = 0$, $i \in I$. Hence,

$$\begin{cases} u_{n,1} = \frac{1}{3n+1}x_n, \\ u_{n,2} = \frac{-n+3}{9n+3}x_n, \\ y_n = \frac{-6n^3+37n^2-5n-6}{108n^3+72n^2+12n}x_n, \\ x_{n+1} = \frac{1}{3}x_n + \frac{2}{3}\sin(\frac{-6n^3+37n^2-5n-6}{108n^3+72n^2+12n}x_n). \end{cases}$$

Then, by Theorem 3.2, the sequence $\{x_n\}$ converges strongly to $0 \in \Omega$. Table 1 and Figure 1 indicate the behavior of x_n for algorithm (3.5) with $x_0 = 10$ and $x_0 = -10$. We have used MATLAB with $\varepsilon = 10^{-4}$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally, and they also read and finalized manuscript.

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