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Fixed point theorems for generalized F-Suzuki-contraction mappings in complete b-metric spaces

Hossein Piri[1] and Poom Kumam[2,3]*

*Correspondence:
poom.kumam@mail.kmutt.ac.th
[2]KMUTT Fixed Point Research Laboratory, Department of Mathematics, Room SCL 802 Fixed Point Laboratory, Science Laboratory Building, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), 126 Pracha Uthit Road, Bang Mod, Thung Khru, Bangkok, 10140, Thailand
[3]KMUTT-Fixed Point Theory and Applications Research Group (KMUTT-FPTA), Theoretical and Computational Science Center (TaCS), Science Laboratory Building, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), 126 Pracha Uthit Road, Bang Mod, Thung Khru, Bangkok, 10140, Thailand
Full list of author information is available at the end of the article

Abstract
The aim of this work is to establish some new fixed point theorems for generalized F-Suzuki-contraction mappings in complete b-metric spaces.

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1 Introduction and mathematical preliminaries

Recently, Wardowski [1] established a new contraction, the so-called F-contraction, and obtained a fixed point result as a generalization of the Banach contraction principle. After that Altun *et al.* [2] introduced the new concept of multivalued F-contraction mappings and gave some fixed point results. Wardowski and Dung [3] further generalized the concept of an F-contraction to an F-weak contraction and also obtained certain fixed point results. Dung and Hang [4] studied the notion of a generalized F-contraction and extended a fixed point theorem for such mappings. Recently Piri and Kumam [5] described a large class of functions by replacing condition (F3′) instead of the condition (F3) in the definition of F-contraction.

Following this direction of research, in this paper, we extend the fixed point results of Wardowski [1], Wardowski and Dung [3], Dung and Hang [4], and Piri and Kumam [5] by introducing a generalized F-Suzuki-contraction in b-metric spaces. We begin with some basic well-known definitions and results which will be used in the rest of this paper.

Throughout this paper, $\mathbb{N}_0$, $\mathbb{N}$, $\mathbb{R}_+$, $\mathbb{R}$ denote the set of nonnegative integer numbers, the set of natural numbers, the set of positive real numbers, and the set of real numbers, respectively.

Definition 1.1 Let $\mathcal{F}$ be the family of all functions $F:\mathbb{R}_+\to\mathbb{R}$ such that:

(F1) F is strictly increasing, *i.e.* for all $x,y\in\mathbb{R}_+$ such that $x<y$, $F(x)<F(y)$;
(F2) for each sequence $\{\alpha_n\}_{n=1}^{\infty}$ of positive numbers, $\lim_{n\to\infty}\alpha_n=0$ if and only if $\lim_{n\to\infty}F(\alpha_n)=-\infty$;
(F3) there exists $k\in(0,1)$ such that $\lim_{\alpha\to 0^+}\alpha^k F(\alpha)=0$.

Definition 1.2 [1] Let (X,d) be a metric space. A mapping $T: X \to X$ is said to be an F-contraction on (X,d) if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that, for all $x, y \in X$,

$$d(Tx, Ty) > 0 \quad \Rightarrow \quad \tau + F\bigl(d(Tx, Ty)\bigr) \leq F\bigl(d(x,y)\bigr). \tag{1}$$

A new generalization of Banach contraction principle has been given by Wardowski [1] as follows.

Theorem 1.3 [1] *Let (X,d) be a complete metric space and let $T: X \to X$ be an F-contraction. Then T has a unique fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T^n x\}_{n=1}^{\infty}$ converges to x^*.*

In 2014, Wardowski and Dung [3] introduced the notion of an F-weak contraction and proved a related fixed point theorem as follows.

Definition 1.4 [3] Let (X,d) be a metric space. A mapping $T: X \to X$ is said to be an F-weak contraction on (X,d) if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that, for all $x, y \in X$,

$$d(Tx, Ty) > 0 \quad \Rightarrow \quad \tau + F\bigl(d(Tx, Ty)\bigr) \leq F\bigl(M(x,y)\bigr), \tag{2}$$

in which

$$M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty) + d(y,Tx)}{2}\right\}.$$

Theorem 1.5 [3] *Let (X,d) be a complete metric space and let $T: X \to X$ be an F-weak contraction. If T or F is continuous, then T has a unique fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T^n x\}_{n=1}^{\infty}$ converges to x^*.*

Recall that a contraction conditions for a self-mapping T on a metric space (X,d), usually contained at most five values $d(x,y)$, $d(x,Tx)$, $d(y,Ty)$, $d(x,Ty)$, $d(y,Tx)$ (for example see [6, 7]). Recently, by adding four new values $d(T^2x,x)$, $d(T^2x,Tx)$, $d(T^2x,y)$, $d(T^2x,Ty)$ to a contraction condition, Kumam *et al.* [8] stated a new generalization of the Ćirić fixed point theorem in [9]. Motivated and inspired by the idea of Kumam *et al.* [8], Dung and Hang [4] generalized the notion of a generalized F-contraction and proved some fixed point theorems for such maps. They gave examples to show that their result is a real generalization of Theorem 1.5 and some others in the literature.

Definition 1.6 [4] Let (X,d) be a metric space. A mapping $T: X \to X$ is said to be a generalized F-contraction on (X,d) if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that, for all $x, y \in X$,

$$d(Tx, Ty) > 0 \quad \Rightarrow \quad \tau + F\bigl(d(Tx, Ty)\bigr) \leq F\bigl(N(x,y)\bigr),$$

in which

$$\begin{aligned} N(x,y) = \max\biggl\{& d(x,y), d(x,Tx), d(y,Ty), \\ & \frac{d(x,Ty) + d(y,Tx)}{2}, \frac{d(T^2x,x) + d(T^2x,Ty)}{2}, \\ & d\bigl(T^2x, Tx\bigr), d\bigl(T^2x, y\bigr), d\bigl(T^2x, Ty\bigr)\biggr\}. \end{aligned} \tag{3}$$

Theorem 1.7 [4] *Let (X,d) be a complete metric space and let $T:X\to X$ be a generalized F-contraction mapping. If T or F is continuous, then T has a unique fixed point $x^*\in X$ and for every $x\in X$ the sequence $\{T^nx\}_{n=1}^\infty$ converges to x^*.*

In 2014, Piri and Kumam [5] described a large class of functions by replacing the condition (F3) in the definition of an F-contraction introduced by Wardowski [1] with the following one:

(F3′) F is continuous on $(0,\infty)$.

They denote by $\mathfrak{F}$ the family of all functions $F:\mathbb{R}_+\to\mathbb{R}$ which satisfy conditions (F1), (F2), and (F3′). Under this new set-up, Piri and Kumam proved some Wardowski and Suzuki type fixed point results in metric spaces as follows.

Theorem 1.8 [5] *Let T be a self-mapping of a complete metric space X into itself. Suppose that there exist $F\in\mathfrak{F}$ and $\tau>0$ such that, for all $x,y\in X$,*

$$d(Tx,Ty)>0 \quad\Rightarrow\quad \tau+F\bigl(d(Tx,Ty)\bigr)\le F\bigl(d(x,y)\bigr).$$

Then T has a unique fixed point $x^*\in X$ and for every $x\in X$ the sequence $\{T^nx\}_{n=1}^\infty$ converges to x^*.

Theorem 1.9 [5] *Let T be a self-mapping of a complete metric space X into itself. Suppose that there exist $F\in\mathfrak{F}$ and $\tau>0$ such that*

$$\forall x,y\in X,\quad \frac{1}{2}d(x,Tx)<d(x,y) \quad\Rightarrow\quad \tau+F\bigl(d(Tx,Ty)\bigr)\le F\bigl(d(x,y)\bigr).$$

Then T has a unique fixed point $x^*\in X$ and for every $x_0\in X$ the sequence $\{T^nx_0\}_{n=1}^\infty$ converges to x^*.

Definition 1.10 [10] Let X be a nonempty set and $s\ge 1$ be a given real number. A mapping $d:X\times X\to\mathbb{R}^+$ is said to be a b-metric if for all $x,y,z\in X$ the following conditions are satisfied:

(bm_1) $d(x,y)=0$ if and only if $x=y$;
(bm_2) $d(x,y)=d(y,x)$;
(bm_3) $d(x,z)\le s[d(x,y)+d(y,z)]$.

In this case, the pair (X,d) is called a b-metric space (with constant s).

Definition 1.11 [11] Let (X,d) be a b-metric space. A sequence $\{x_n\}_{n=1}^\infty$ in X is called:

(A) Convergent if and only if there exists $x\in X$ such that $\lim_{n\to\infty}d(x_n,x)=0$. In this case, we write $\lim_{n\to}x_n=x$.
(B) Cauchy if and only if $\lim_{n,m\to\infty}d(x_n,x_m)=0$.

Remark 1.12 [11] Notice that in a b-metric space (X,d) the following assertions hold:

(A) a convergent sequence has a unique limit;
(B) each convergent sequence is Cauchy;
(C) in general, a b-metric is not continuous;
(D) in general, a b-metric does not induce a topology on X.

Definition 1.13 [11] The b-metric space (X,d) is complete if every Cauchy sequence in X converges in X.

Definition 1.14 [12] Let (X,d_X) and (Y,d_Y) be b-metric spaces; a mapping $f:X\to Y$ is called:

(A) continuous at a point $x\in X$, if for every sequence $\{x_n\}_{n=1}^{\infty}$ in X such that $\lim_{n\to} x_n = x$, then $\lim_{n\to} f(x_n) = f(x)$;

(B) continuous on X, if it is continuous at each point $x\in X$.

2 Main results

We use $\mathfrak{F}_G$ to denote the set of all functions $F:\mathbb{R}_+\to\mathbb{R}$ which satisfy conditions (F1) and (F3′) and Ψ to denote the set of all functions $\psi:[0,\infty)\to[0,\infty)$ such that ψ is continuous and $\psi(t)=0$ if and only $t=0$.

Definition 2.1 Let (X,d) be a b-metric space. A self-mapping $T:X\to X$ is said to be a generalized F-Suzuki-contraction if there exists $F\in\mathfrak{F}_G$ such that, for all $x,y\in X$ with $x\neq y$,

$$\frac{1}{2s}d(x,Tx)<d(x,y) \quad\Rightarrow\quad F\bigl(s^5 d(Tx,Ty)\bigr)\le F\bigl(M_T(x,y)\bigr)-\psi\bigl(M_T(x,y)\bigr),$$

in which $\psi\in\Psi$ and

$$\begin{aligned} M_T(x,y)=\max\biggl\{&d(x,y),d\bigl(T^2x,y\bigr),\frac{d(Tx,y)+d(x,Ty)}{2s},\\ &\frac{d(T^2x,x)+d(T^2x,Ty)}{2s},d\bigl(T^2x,Ty\bigr)+d\bigl(T^2x,Tx\bigr),\\ &d\bigl(T^2x,Ty\bigr)+d(Tx,x),d(Tx,y)+d(y,Ty)\biggr\}. \end{aligned} \tag{4}$$

Theorem 2.2 *Let (X,d) be a complete b-metric space and $T:X\to X$ be a generalized F-Suzuki-contraction. Then T has a unique fixed point $x^*\in X$ and for every $x\in X$ the sequence $\{T^nx\}_{n=1}^{\infty}$ converges to x^*.*

Proof Take $x_0=x\in X$. Let $x_n=Tx_{n-1}$ for all $n\in\mathbb{N}$. If there exists $n\in\mathbb{N}$ such that $d(x_n,Tx_n)=0$ then $x=x_n$ becomes a fixed point of T, which completes the proof. So, in the rest of the proof, we assume that

$$0<d(x_n,Tx_n),\quad \forall n\in\mathbb{N}. \tag{5}$$

Hence, we have

$$\frac{1}{2s}d(x_n,Tx_n)<d(x_n,Tx_n)=d(x_n,x_{n+1}),\quad \forall n\in\mathbb{N}. \tag{6}$$

So by the assumption of the theorem, we have

$$F\bigl(d(Tx_n,Tx_{n+1})\bigr)\le F\bigl(M_T(x_n,x_{n+1})\bigr)-\psi\bigl(M_T(x_n,x_{n+1})\bigr).$$

Since

$$\begin{aligned}
&\max\{d(x_n,x_{n+1}),d(T^2x_n,x_{n+1})\} \\
&\quad\le M_T(x_n,x_{n+1}) \\
&\quad= \max\biggl\{d(x_n,x_{n+1}),d(x_{n+2},x_{n+1}),\frac{d(x_n,x_{n+2})}{2s},\frac{d(x_{n+2},x_n)}{2s}, \\
&\qquad d(x_{n+2},x_{n+1}),d(x_{n+1},x_{n+2}),d(x_n,x_{n+1})\biggr\} \\
&\quad\le \max\biggl\{d(x_n,x_{n+1}),d(x_{n+2},x_{n+1}),\frac{s[d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})]}{2s}, \\
&\qquad \frac{s[d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})]}{2s},d(x_{n+2},x_{n+1}),d(x_{n+1},x_{n+2}),d(x_n,x_{n+1})\biggr\} \\
&\quad\le \max\{d(x_n,x_{n+1}),d(x_{n+2},x_{n+1})\},
\end{aligned}$$

we get

$$\begin{aligned}
F(d(x_{n+1},x_{n+2})) \le{}& F(\max\{d(x_n,x_{n+1}),d(x_{n+1},x_{n+2})\}) \\
&- \psi(\max\{d(x_n,x_{n+1}),d(x_{n+1},x_{n+2})\}).
\end{aligned} \tag{7}$$

If $d(x_{n+1},x_{n+2}) > d(x_n,x_{n+1})$, then

$$\max\{d(x_n,x_{n+1}),d(x_{n+1},x_{n+2})\} = d(x_{n+1},x_{n+2}),$$

so (7) becomes

$$F(d(x_{n+1},x_{n+2})) \le F(d(x_{n+1},x_{n+2})) - \psi(d(x_{n+1},x_{n+2})),$$

which is a contradiction (from (5) and the property of φ, we have $\psi(d(x_{n+1},x_{n+2})) > 0$). Thus, we conclude that

$$\begin{aligned}
F(d(x_{n+1},x_{n+2})) &\le F(d(x_n,x_{n+1})) - \psi(d(x_n,x_{n+1})) \\
&< F(d(x_n,x_{n+1})).
\end{aligned} \tag{8}$$

It follows from (8) and (F1) that

$$d(x_n,x_{n+1}) < d(x_{n-1},x_n), \quad \forall n \in \mathbb{N}. \tag{9}$$

Therefore $\{d(x_{n+1},x_n)\}_{n\in\mathbb{N}}$ is a nonnegative decreasing sequence of real numbers. Thus, there exists $\gamma \ge 0$ such that $\lim_{n\to\infty} d(x_{n+1},x_n) = \gamma$. Letting $n \to \infty$ in (8), we have

$$F(\gamma) \le F(\gamma) - \psi(\gamma).$$

This implies that $\psi(\gamma) = 0$ and thus $\gamma = 0$. Consequently, we have

$$\lim_{n\to\infty} d(x_n, Tx_n) = \lim_{n\to\infty} d(x_n,x_{n+1}) = 0. \tag{10}$$

Now, we claim that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Arguing by contradiction, we assume that there exist $\epsilon > 0$, and the sequences $\{p(n)\}_{n=1}^{\infty}$ and $\{q(n)\}_{n=1}^{\infty}$ of natural numbers such that, for all $n \in \mathbb{N}$,

$$p(n) > q(n) > n, \quad d(x_{p(n)}, x_{q(n)}) \geq \epsilon, \qquad d(x_{p(n)-1}, x_{q(n)}) < \epsilon. \tag{11}$$

Observe that

$$\begin{aligned}\epsilon \leq d(x_{p(n)}, x_{q(n)}) &\leq s\big[d(x_{p(n)}, x_{p(n)-1}) + d(x_{p(n)-1}, x_{q(n)})\big] \\ &\leq sd(x_{p(n)}, x_{p(n)-1}) + s\epsilon.\end{aligned}$$

So from (10), we get

$$\epsilon \leq \limsup_{n\to\infty} d(x_{p(n)}, x_{q(n)}) \leq s\epsilon. \tag{12}$$

From the triangle inequality, we have

$$\epsilon \leq d(x_{p(n)}, x_{q(n)}) \leq s\big[d(x_{p(n)}, x_{q(n)+1}) + d(x_{q(n)+1}, x_{q(n)})\big] \tag{13}$$

and

$$d(x_{p(n)}, x_{q(n)+1}) \leq s\big[d(x_{p(n)}, x_{q(n)}) + d(x_{q(n)}, x_{q(n)+1})\big]. \tag{14}$$

It follows from (10), (12), (13), and (14) that

$$\frac{\epsilon}{s} \leq \limsup_{n\to\infty} d(x_{p(n)}, x_{q(n)+1}) \leq s^2\epsilon. \tag{15}$$

Again, using above process, we get

$$\frac{\epsilon}{s} \leq \limsup_{n\to\infty} d(x_{p(n)+1}, x_{q(n)}) \leq s^2\epsilon. \tag{16}$$

From (15) and the inequality

$$d(x_{p(n)}, x_{q(n)+1}) \leq s\big[d(x_{p(n)}, x_{p(n)+1}) + d(x_{p(n)+1}, x_{q(n)+1})\big],$$

we have

$$\frac{\epsilon}{s^2} \leq \limsup_{n\to\infty} d(x_{p(n)+1}, x_{q(n)+1}). \tag{17}$$

From (12) and the inequality

$$\begin{aligned}d(x_{p(n)+1}, x_{q(n)+1}) &\leq s\big[d(x_{p(n)+1}, x_{q(n)}) + d(x_{q(n)}, x_{q(n)+1})\big] \\ &\leq s^2\big[d(x_{p(n)+1}, x_{p(n)}) + d(x_{p(n)}, x_{q(n)})\big] + sd(x_{q(n)}, x_{q(n)+1}),\end{aligned}$$

we have

$$\limsup_{n\to\infty} d(x_{p(n)+1}, x_{q(n)+1}) \leq s^3\epsilon. \tag{18}$$

It follows from (17) and (18) that

$$\frac{\epsilon}{s^2} \leq \limsup_{n\to\infty} d(x_{p(n)+1}, x_{q(n)+1}) \leq s^3\epsilon. \tag{19}$$

From (10) and (11), we can choose a positive integer $n_1 \in \mathbb{N}$ such that

$$\frac{1}{2s} d(x_{p(n)}, Tx_{p(n)}) < \frac{1}{2s}\epsilon < d(x_{p(n)}, x_{q(n)}), \quad \forall n \geq n_1.$$

Therefore by assumption of theorem for every $n \geq n_1$, we have

$$F\big(d(x_{p(n)+1}, x_{q(n)+1})\big) \leq F\big(M_T(x_{p(n)}, x_{q(n)})\big) - \psi\big(M_T(x_{p(n)}, x_{q(n)})\big). \tag{20}$$

Since

$$\begin{aligned}
&d(x_{p(n)}, x_{q(n)}) \\
&\quad \leq M_T(x_{p(n)}, x_{q(n)}) \\
&\quad = \max\bigg\{ d(x_{p(n)}, x_{q(n)}), d(x_{p(n)+2}, x_{q(n)}), \frac{d(x_{p(n)+1}, x_{q(n)}) + d(x_{p(n)}, x_{q(n)+1})}{2s}, \\
&\qquad \frac{d(x_{p(n)+2}, x_{p(n)}) + d(x_{p(n)+2}, x_{q(n)+1})}{2s}, d(x_{p(n)+2}, x_{q(n)+1}) + d(x_{p(n)+2}, x_{p(n)+1}), \\
&\qquad d(x_{p(n)+2}, x_{q(n)+1}) + d(x_{p(n)+1}, x_{p(n)}), d(x_{p(n)+1}, x_{q(n)}) + d(x_{q(n)}, x_{q(n)+1}) \bigg\} \\
&\quad \leq \max\bigg\{ d(x_{p(n)}, x_{q(n)}), s\big[d(x_{p(n)+2}, x_{p(n)+1}) + d(x_{p(n)+1}, x_{q(n)})\big], \\
&\qquad \frac{d(x_{p(n)+1}, x_{q(n)}) + d(x_{p(n)}, x_{q(n)+1})}{2s}, \\
&\qquad \frac{s[d(x_{p(n)+2}, x_{p(n)+1}) + d(x_{p(n)+1}, x_{p(n)})] + s[d(x_{p(n)+2}, x_{p(n)+1}) + d(x_{p(n)+1}, x_{q(n)+1})]}{2s}, \\
&\qquad s\big[d(x_{p(n)+2}, x_{p(n)+1}) + d(x_{p(n)+1}, x_{q(n)+1})\big] + d(x_{p(n)+2}, x_{p(n)+1}), \\
&\qquad s\big[d(x_{p(n)+2}, x_{p(n)+1}) + d(x_{p(n)+1}, x_{q(n)+1})\big] + d(x_{p(n)+1}, x_{p(n)}), \\
&\qquad d(x_{p(n)+1}, x_{q(n)}) + d(x_{q(n)}, x_{q(n)+1}) \bigg\},
\end{aligned}$$

taking the limit supremum as $n \to \infty$ on each side of the above inequality and using (12), (15), (16), and (19) we have

$$\epsilon \leq \limsup_{n\to\infty} M_T(x_{p(n)}, x_{q(n)}) \leq s^3\epsilon. \tag{21}$$

Also, we can show that

$$\epsilon \leq \liminf_{n\to\infty} M_T(x_{p(n)}, x_{q(n)}) \leq s^3\epsilon. \tag{22}$$

Taking the limit supremum as $n \to \infty$ in (20) and using (21) and (22), we get

$$\begin{aligned} F\left(s^3\epsilon\right) = F\left(s^5\frac{\epsilon}{s^2}\right) &\le F\left(\limsup_{n\to\infty} d(x_{p(n)+1}, x_{q(n)+1})\right) \\ &\le F\left(\limsup_{n\to\infty} M_T(x_{p(n)}, x_{q(n)})\right) - \psi\left(\limsup_{n\to\infty} M_T(x_{p(n)}, x_{q(n)})\right) \\ &\le F\left(s^3\epsilon\right) - \psi(\epsilon), \end{aligned}$$

which is a contradiction with $\epsilon > 0$, and it follows that $\{x_n\}$ is a Cauchy sequence in X. By completeness of (X,d), $\{x_n\}_{n=1}^{\infty}$ converges to some point x^* in X. Therefore,

$$\lim_{n\to\infty} d\left(x_n, x^*\right) = 0. \tag{23}$$

We claim that, for every $n \in \mathbb{N}$,

$$\frac{1}{2s} d(x_n, Tx_n) < d\left(x_n, x^*\right) \quad \text{or} \quad \frac{1}{2s} d\left(Tx_n, T^2x_n\right) < d\left(Tx_n, x^*\right). \tag{24}$$

Suppose, on the contrary, that there exists $m \in \mathbb{N}$ such that

$$\frac{1}{2s} d(x_m, Tx_m) \ge d\left(x_m, x^*\right) \quad \text{and} \quad \frac{1}{2s} d\left(Tx_m, T^2x_m\right) \ge d\left(Tx_m, x^*\right). \tag{25}$$

Therefore,

$$2sd\left(x_m, x^*\right) \le d(x_m, Tx_m) \le s\left[d\left(x_m, x^*\right) + d\left(x^*, Tx_m\right)\right],$$

which implies that

$$d\left(x_m, x^*\right) \le d\left(x^*, Tx_m\right). \tag{26}$$

From (9) and (26), we have

$$\begin{aligned} d\left(Tx_m, T^2x_m\right) < d(x_m, Tx_m) &\le sd\left(x_m, x^*\right) + sd\left(x^*, Tx_m\right) \\ &\le 2sd\left(x^*, Tx_m\right). \end{aligned} \tag{27}$$

It follows from (25) and (27) that $d(Tx_m, T^2x_m) < d(Tx_m, T^2x_m)$. This is a contradiction. Hence, (24) holds. If part (I) of (24) is true, then we have

$$\begin{aligned} F\left(d\left(x_{n+1}, Tx^*\right)\right) &= F\left(d\left(Tx_n, Tx^*\right)\right) \\ &\le F\left(M_T\left(x_n, x^*\right)\right) - \psi\left(M_T\left(x_n, x^*\right)\right). \end{aligned} \tag{28}$$

Since

$$\begin{aligned} d\left(x^*, Tx^*\right) &\le M_T\left(x_n, x^*\right) \\ &= \max\left\{d\left(x_n, x^*\right), d\left(x_{n+2}, x^*\right), \frac{d(x_{n+1}, x^*) + d(x_n, Tx^*)}{2s}, \right. \end{aligned}$$

$$\begin{aligned}
&\frac{d(x_{n+2},x_n)+d(x_{n+2},Tx^*)}{2s}, d\bigl(x_{n+2},Tx^*\bigr)+d(x_{n+2},x_{n+1}),\\
&d\bigl(x_{n+2},Tx^*\bigr)+d(x_{n+1},x_n), d\bigl(x_{n+1},x^*\bigr)+d\bigl(x^*,Tx^*\bigr)\Bigr\}\\
&\leq \max\Bigl\{d\bigl(x_n,x^*\bigr), d\bigl(x_{n+2},x^*\bigr), \frac{d(x_{n+1},x^*)+d(x_n,Tx^*)}{2s},\\
&\frac{s[d(x_{n+2},x_{n+1})+d(x_{n+1},x_n)]+d(x_{n+2},Tx^*)}{2s},\\
&d\bigl(x_{n+2},Tx^*\bigr)+d(x_{n+2},x_{n+1}),\\
&d\bigl(x_{n+2},Tx^*\bigr)+d(x_{n+1},x_n), d\bigl(x_{n+1},x^*\bigr)+d\bigl(x^*,Tx^*\bigr)\Bigr\},
\end{aligned}$$

letting $n \to \infty$ and using (23), we get

$$\lim_{n\to\infty} M_T\bigl(x_n,x^*\bigr) = d\bigl(x^*,Tx^*\bigr).$$

It follows from (28), (F3′), and the continuity of φ that

$$F\bigl(d\bigl(x^*,Tx^*\bigr)\bigr) \leq F\bigl(d\bigl(x^*,Tx^*\bigr)\bigr) - \psi\bigl(d\bigl(x^*,Tx^*\bigr)\bigr).$$

This yields $x^* = Tx^*$. If part (II) of (24) is true, using a similar method to the above, we get $x^* = Tx^*$. Hence, x^* is a fixed point of T. Now we show that T has at most one fixed point. Indeed, if $x^*, y^* \in X$ are two fixed points of T, such that $x^* \neq y^*$, then we have $0 = \frac{1}{2s} d(x^*, Tx^*) < d(x^*, y^*)$ and from the assumption of the theorem, we obtain

$$\begin{aligned}
F\bigl(d\bigl(x^*,y^*\bigr)\bigr) &= F\bigl(d\bigl(Tx^*,Ty^*\bigr)\bigr)\\
&\leq F\bigl(M_T\bigl(x^*,y^*\bigr)\bigr) - \psi\bigl(M_T\bigl(x^*,y^*\bigr)\bigr)\\
&= F\bigl(d\bigl(y^*,x^*\bigr)\bigr) - \psi\bigl(d\bigl(y^*,x^*\bigr)\bigr).
\end{aligned}$$

This gives $\psi(d(y^*,x^*)) \leq 0$. Hence $y^* = x^*$. This completes the proof. □

The following two theorems can be obtained easily by repeating the steps in the proof of Theorem 2.2.

Theorem 2.3 *Let (X,d) be a complete b-metric space and $T : X \to X$ be a self-mapping such that, for every $x, y \in X$,*

$$\frac{1}{2s} d(x,Tx) < d(x,y) \quad \Rightarrow \quad F\bigl(d(Tx,Ty)\bigr) \leq F\bigl(M_T(x,y)\bigr) - \psi\bigl(N(x,y)\bigr),$$

where $N(x,y)$ is defined by (3) and ψ is defined as in Theorem 2.2. Then T has a unique fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T^n x\}_{n=1}^{\infty}$ converges to x^*.

Theorem 2.4 *Let (X,d) be a complete b-metric space and $T : X \to X$ be a self-mapping such that, for every $x, y \in X$,*

$$\frac{1}{2s} d(x,Tx) < d(x,y) \quad \Rightarrow \quad F\bigl(d(Tx,Ty)\bigr) \leq F\bigl(M_T(x,y)\bigr) - \psi\bigl(d(x,y)\bigr),$$

where $M_T(x,y)$ is defined by (4) *and ψ is defined as in Theorem* 2.2. *Then T has a unique fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T^n x\}_{n=1}^{\infty}$ converges to x^*.*

Since a b-metric space is a metric space when $s = 1$, so we obtain the following theorems.

Theorem 2.5 *Let (X,d) be a complete metric space and $T : X \to X$ be a generalized F-Suzuki-contraction. Then T has a unique fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T^n x\}_{n=1}^{\infty}$ converges to x^*.*

Theorem 2.6 *Let (X,d) be a complete metric space and $T : X \to X$ be a self-mapping such that, for every $x, y \in X$,*

$$\frac{1}{2s} d(x, Tx) < d(x,y) \quad \Rightarrow \quad F\bigl(d(Tx, Ty)\bigr) \le F\bigl(M_T(x,y)\bigr) - \psi\bigl(d(x,y)\bigr),$$

where $M_T(x,y)$ is defined by (4) *and ψ is defined as in Theorem* 2.2. *Then T has a unique fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T^n x\}_{n=1}^{\infty}$ converges to x^*.*

Theorem 2.7 [4] *Let (X,d) be a complete metric space and let $T : X \to X$ be a generalized F-contraction. If F is continuous, then T has a unique fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T^n x\}_{n=1}^{\infty}$ converges to x^*.*

Proof Since $N(x,y) \le M_T(x,y)$, so from (F1) and by taking $\psi = \tau$ in Theorem 2.5 the proof is complete. □

Theorem 2.8 [5] *Let T be a self-mapping of a complete metric space X into itself. Suppose that there exist $F \in \mathfrak{F}$ and $\tau > 0$ such that*

$$\forall x, y \in X, \quad \frac{1}{2} d(x, Tx) < d(x,y) \quad \Rightarrow \quad \tau + F\bigl(d(Tx, Ty)\bigr) \le F\bigl(d(x,y)\bigr).$$

Then T has a unique fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T^n x\}_{n=1}^{\infty}$ converges to x^*.

Proof Since $d(x,y) \le M_T(x,y)$, from (F1) and by taking $\psi = \tau$ and $s = 1$ in Theorem 2.2 the proof is complete. □

Example 2.9 Let $X = \{-2, -1, 0, 1, 2\}$ and define a metric d on X by

$$d(x,y) = \begin{cases} 0, & \text{if } x = y, \\ 2, & \text{if } (x,y) \in \{(1,-1), (-1,1)\}, \\ 1, & \text{otherwise.} \end{cases}$$

Then (X,d) is a b-metric space with coefficient $s = 2$. But it is not a metric space since the triangle inequality is not satisfied. Let $T : X \to X$ be defined by

$$T(-2) = T(0) = T(2) = 0, \qquad T(-1) = 1, \qquad T(1) = -2.$$

First observe that

$$\begin{aligned} & d(Tx,Ty)>0 \\ & \quad\Leftrightarrow\quad \bigl[\bigl(x\in\{-2,0,2\}\wedge y=1\bigr)\vee\bigl(x\in\{-2,0,2\}\wedge y=-1\bigr)\vee(x=1\wedge y=-1)\bigr]. \end{aligned}$$

Now we consider the following cases:

Case 1. Let $x\in\{-2,0,2\}\wedge y=1$, then

$$\begin{aligned} & d(Tx,Ty)=d(0,-2)=1, \qquad d(x,y)=d(x,1)=1, \qquad d(x,Tx)=d(x,0)=0\vee 1, \\ & d(y,Ty)=d(1,-2)=1, \qquad \frac{d(x,Ty)+d(Tx,y)}{2}=\frac{d(x,-2)+d(0,1)}{2}=\frac{1}{2}\vee 1, \\ & \frac{d(T^2x,x)+d(T^2x,Ty)}{2}=\frac{d(0,x)+d(0,-2)}{2}=\frac{1}{2}\vee 1, \\ & d(x,Ty)=d(x,-2)=0\vee 1, \qquad d(Tx,y)=d(0,-2)=1, \\ & d\bigl(T^2x,Tx\bigr)=d(0,0)=0, \qquad d\bigl(T^2x,y\bigr)=d(0,1)=1, \\ & d\bigl(T^2x,Ty\bigr)+d(x,Tx)=d(0,-2)+d(x,0)=1\vee 2, \\ & d(Tx,y)+d(y,Ty)=d(0,1)+d(1,-2)=2. \end{aligned}$$

Case 2. Let $x\in\{-2,0,2\}\wedge y=-1$, then

$$\begin{aligned} & d(Tx,Ty)=d(2,1)=1, \qquad d(x,y)=d(x,-1)=1, \qquad d(x,Tx)=d(x,0)=0\vee 1, \\ & d(y,Ty)=d(-1,1)=4, \qquad \frac{d(x,Ty)+d(Tx,y)}{2}=\frac{d(x,1)+d(0,-1)}{2}=\frac{1}{2}\vee 1, \\ & \frac{d(T^2x,x)+d(T^2x,Ty)}{2}=\frac{d(0,x)+d(0,1)}{2}=\frac{1}{2}\vee 1, \\ & d(x,Ty)=d(x,1)=1, \qquad d(Tx,y)=d(0,-1)=1, \\ & d\bigl(T^2x,Tx\bigr)=d(0,0)=0, \qquad d\bigl(T^2x,y\bigr)=d(0,-1)=1, \\ & d\bigl(T^2x,Ty\bigr)+d(x,Tx)=d(0,1)+d(x,0)=1\vee 2 \\ & d(Tx,y)+d(y,Ty)=d(0,-1)+d(-1,1)=5. \end{aligned}$$

Case 3. Let $x=1\wedge y=-1$, then

$$\begin{aligned} & d(Tx,Ty)=d(-2,1)=1, \qquad d(x,y)=d(1,-1)=4, \qquad d(x,Tx)=d(1,-2)=1, \\ & d(y,Ty)=d(-1,1)=4, \qquad \frac{d(x,Ty)+d(Tx,y)}{2}=\frac{d(1,1)+d(-2,-1)}{2}=\frac{1}{2}, \\ & \frac{d(T^2x,x)+d(T^2x,Ty)}{2}=\frac{d(0,1)+d(0,1)}{2}=1, \\ & d(x,Ty)=d(1,1)=0, \qquad d(Tx,y)=d(-2,-1)=1, \\ & d\bigl(T^2x,Tx\bigr)=d(0,-2)=1, \qquad d\bigl(T^2x,y\bigr)=d(0,-1)=1, \\ & d\bigl(T^2x,Ty\bigr)+d(x,Tx)=d(0,1)+d(1,-2)=2, \\ & d(Tx,y)+d(y,Ty)=d(-2,-1)+d(-1,1)=5. \end{aligned}$$

In Case 1, we have

$$\begin{aligned} d(Tx,Ty) &= \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\} \\ &= \max\left\{\frac{d(T^2x,x)+d(T^2x,Ty)}{2}, d\big(T^2x,Tx\big), d\big(T^2x,y\big), d\big(T^2x,Ty\big)\right\} = 1. \end{aligned}$$

This proves that for all $F \in \mathcal{F}$, T is not an F-weak contraction, generalized F-contraction, and F-contraction. Hence Theorem 1.3, Theorem 1.5, and Theorem 2.7 are not applicable for this example. However, we see that, for all $x, y \in X$,

$$\frac{1}{2}d(x,Tx) < d(x,y), \qquad d(Tx,Ty) = 1, \quad \text{and} \quad M_T(x,y) \geq 2.$$

Since

$$\begin{aligned} \ln\big(d(Tx,Ty)\big) &\leq \ln\big(M_T(x,y)\big) + \ln\left(\frac{1}{2}\right) \\ &\leq \ln\big(M_T(x,y)\big) - \frac{68}{100}. \end{aligned}$$

So by taking $F(t) = \ln(t)$ and $\varphi(t) = \frac{1}{20}t$, we have

$$F\big(d(Tx,Ty)\big) \leq F\big(M_T(x,y)\big) - \varphi\big(M_T(x,y)\big).$$

Hence T satisfies the assumption of Theorem 2.2.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing the article. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Basic Science Faculty, University of Bonab, Bonab, 5551761167, Iran. [2]KMUTT Fixed Point Research Laboratory, Department of Mathematics, Room SCL 802 Fixed Point Laboratory, Science Laboratory Building, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), 126 Pracha Uthit Road, Bang Mod, Thung Khru, Bangkok, 10140, Thailand. [3]KMUTT-Fixed Point Theory and Applications Research Group (KMUTT-FPTA), Theoretical and Computational Science Center (TaCS), Science Laboratory Building, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), 126 Pracha Uthit Road, Bang Mod, Thung Khru, Bangkok, 10140, Thailand.

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