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Schwarz lemma involving the boundary fixed point

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Abstract

Let f be an holomorphic function which maps the unit disk into itself. In this paper, consider the zero of order k (*i.e.*, $f(z) - f(0)$ (or $f(z)$) has a zero of order k at $z = 0$), we obtain the sharp estimates of the classical boundary Schwarz lemma involving the boundary fixed point. The results presented here would generalize the corresponding result obtained by Frolova *et al.* (Complex Anal. Oper. Theory 8:1129-1149, 2004).

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1 Introduction and preliminaries

It is well known that the Schwarz lemma serves as a very powerful tool to study several research fields in complex analysis. For example, almost all results in the geometric function theory have the Schwarz lemma lurking in the background [2–6].

On the other hand, Schwarz lemma at the boundary is also an active topic in complex analysis, various interesting results have been obtained [7–14]. Before summarizing these results, it is necessary to give some elementary contents on the boundary fixed points [15].

Let $\mathbb{D}$ denote the unit disk in $\mathbb{C}$, $H(\mathbb{D}, \mathbb{D})$ denote the class of holomorphic self-mappings of $\mathbb{D}$, $\mathbb{N}$ denote the set of all positive integers. The boundary point $\xi \in \partial\mathbb{D}$ is called a fixed point of $f \in H(\mathbb{D}, \mathbb{D})$ if

$$f(\xi) = \lim_{r\to 1^-} f(r\xi) = \xi.$$

The classification of the boundary fixed points of $f \in H(\mathbb{D}, \mathbb{D})$ can be performed via the value of the angular derivative

$$f'(\xi) = \angle\lim_{z\to\xi} \frac{f(z) - \xi}{z - \xi},$$

which belongs to $(0, \infty]$ due to the celebrated Julia-Carathédory theorem [13]. This theorem also asserts that the finite angular derivative at the boundary fixed point ξ exists if and only if the holomorphic function $f'(z)$ has the finite angular limit $\angle\lim_{z\to\xi} f'(z)$. For a boundary fixed point ξ of f, if

$$f'(\xi) \in (0, \infty),$$

then ξ is called a regular boundary fixed point. The regular fixed points can be attractive if $f'(\xi) \in (0,1)$, neutral if $f'(\xi) = 1$, or repulsive if $f'(\xi) \in (1,\infty)$.

By the Julia-Carathédory theorem [13] (see also [7]) and the Wolff lemma [11], if $f \in H(\mathbb{D},\mathbb{D})$ with no interior fixed point, then there exists a unique regular boundary fixed point ξ such that $f'(\xi) \in (0,1]$; and if $f \in H(\mathbb{D},\mathbb{D})$ with an interior fixed point, then $f'(\xi) > 1$ for any boundary fixed point $\xi \in \partial\mathbb{D}$.

In particular, Unkelbach [16] (see also [17]) obtain the following boundary Schwarz lemma.

Theorem A *If $f \in H(\mathbb{D},\mathbb{D})$ has a regular boundary fixed point* 1, *and* $f(0) = 0$, *then*

$$f'(1) \geq \frac{2}{1 + |f'(0)|}. \tag{1}$$

Moreover, equality in (1) *holds if and only if f is of the form*

$$f(z) = -z\frac{a - z}{1 - az}, \quad \forall z \in \mathbb{D},$$

for some constant $a \in (-1, 0]$.

Theorem A was improved 60 years later by Osserman [18] by removing the assumption $f(0) = 0$.

Theorem B ([18]) *If $f \in H(\mathbb{D},\mathbb{D})$ with $\xi = 1$ as its regular boundary fixed point. Then*

$$f'(1) \geq \frac{2(1 - |f(0)|)^2}{1 - |f(0)|^2 + |f'(0)|}. \tag{2}$$

In [1], Frolova *et al.* proved the following theorem, which is an improvement of Theorem B.

Theorem C ([1]) *If $f \in H(\mathbb{D},\mathbb{D})$ with $\xi = 1$ as its regular boundary fixed point. Then*

$$f'(1) \geq \frac{2}{\Re\mathrm{e}(\frac{1 - f(0)^2 + f'(0)}{(1 - f(0))^2})}. \tag{3}$$

Recently, Ren and Wang [15] offered an alternative and elementary proof of Theorem C and studied the extremal functions of the inequality (3). Their method of proof is quite different from that which Frolova *et al.* have used in [1].

In this paper, stimulated by the above-cited work (especially [15]), considering the zero of order, we obtain a version of boundary Schwarz lemma. This result is a generalization of the boundary Schwarz-Pick lemma obtained by Frolova *et al.* [1].

In order to prove the desired results, we first recall the classical Julia lemma [3] and the Julia-Carathéodory theorem [19].

Lemma 1 ([3]) *Let $f \in H(\mathbb{D},\mathbb{D})$ and let $\xi \in \partial\mathbb{D}$. Suppose that there exists a sequence $\{z_n\}_{n\in\mathbb{N}} \subset \mathbb{D}$ converging to ξ as n tends to ∞, such that the limits*

$$\alpha = \lim_{n\to\infty} \frac{1 - |f(z_n)|}{1 - |z_n|}$$

and

$$\eta = \lim_{n\to\infty} f(z_n)$$

exist (*finitely*). *Then* $\alpha > 0$ *and the inequality*

$$\frac{|f(z)-\eta|^2}{1-|f(z)|^2} \leq \alpha \frac{|z-\xi|^2}{1-|z|^2} \tag{4}$$

holds throughout the open unit disk $\mathbb{D}$ *and is strict except for Möbius transformations of* $\mathbb{D}$.

Lemma 2 ([19]) *Let* $f \in H(\mathbb{D},\mathbb{D})$ *and let* $\xi \in \partial\mathbb{D}$. *Then the following conditions are equivalent*:

(i) *The lower limit*

$$\alpha = \liminf_{z\to\xi} \frac{1-|f(z)|}{1-|z|} \tag{5}$$

is finite, where the limit is taken as z *approaches* ξ *unrestrictedly in* $\mathbb{D}$;

(ii) f *has a non-tangential limit, say* $f(\xi)$, *at the point* ξ, *and the difference quotient*

$$\frac{f(z)-f(\xi)}{z-\xi}$$

has a non-tangential limit, say $f(\xi)$, *at the point* ξ;

(iii) *the derivative* f' *has a non-tangential limit, say* $f'(\xi)$, *at the point* ξ. *Moreover, under the above conditions we have*:

(a) α *in* (i);

(b) *the derivatives* $f'(\xi)$ *in* (ii) *and* (iii) *are the same*;

(c) $f'(\xi) = \alpha\overline{\xi}f(\xi)$;

(d) *the quotient* $\frac{1-|f(z)|}{1-|z|}$ *has the non-tangential limit* α, *at the point* ξ.

Lemma 3 ([17], p.35) *Let* $\varphi \in H(\mathbb{D},\mathbb{D})$, *and* $\varphi(z) = \sum_{n=0}^{\infty} b_n z^{\nu}$. *Then*

$$|b_n| \leq 1 - |b_0|^2, \quad n \geq 1.$$

2 Main results and their proofs

We now state and prove each of our main results given by Theorems 1 and 2 below.

Theorem 1 *Let* $f \in H(\mathbb{D},\mathbb{D})$ *with* $\xi = 1$ *as its regular boundary fixed point and suppose* $f(0) = f'(0) = \cdots = f^{(k-1)}(0) = 0, a_k = \frac{f^{(k)}(0)}{k!} \neq 0, k \in \mathbb{N}$, *we can obtain*:

(I) *if* $0 < |a_k| < 1$, *then*

$$f'(1) \geq k + \frac{|1-a_k|^2}{1-|a_k|^2} \frac{2}{1 + \Re\mathrm{e}\, \frac{1-\overline{a_k}}{1-a_k} \frac{a_{k+1}}{1-|a_k|^2}}, \tag{6}$$

where $a_{k+1} = \frac{f^{(k+1)}(0)}{(k+1)!}$. *Equality holds in the inequality if and only if f is of the form*

$$f(z) = z^k \frac{a_k - z\frac{z-a}{1-az}\frac{a_k-1}{1-\overline{a_k}}}{1 - z\frac{z-a}{1-az}\frac{a_k-1}{1-\overline{a_k}}\overline{a_k}}, \quad \forall z \in \mathbb{D}, \tag{7}$$

for some constant $a \in [-1, 1)$.

(II) *If* $|a_k| = 1$, *then* $f(z) = z^k$.

Proof In view of Lemma 3, we consider the following two cases.

Case I If $0 < |a_k| < 1$, let

$$g(z) = \begin{cases} \frac{1-\overline{a_k}}{a_k-1}\frac{a_k-\frac{f(z)}{z^k}}{1-\overline{a_k}\frac{f(z)}{z^k}}, & 0 < |z| < 1, \\ 0, & z = 0. \end{cases}$$

It is elementary to see that $g \in H(\mathbb{D}, \mathbb{D})$, and $\xi = 1$ is its regular boundary fixed point. A straightforward computation shows that

$$f'(1) = k + \frac{|1-a_k|^2}{1-|a_k|^2} g'(1) \tag{8}$$

and

$$g'(0) = \frac{1-\overline{a_k}}{1-a_k} \cdot \frac{a_{k+1}}{1-|a_k|^2}, \tag{9}$$

which is no larger than 1 in modulus. Applying Lemmas 1 and 2 to the holomorphic function $h : \mathbb{D} \to \overline{\mathbb{D}}$ defined by

$$h(z) = \frac{g(z)}{z}, \quad \forall z \in \mathbb{D},$$

we obtain

$$g'(1) = 1 + h'(1) \geq 1 + \frac{|1-g'(0)|^2}{1-|g'(0)|^2} = \frac{2(1-\Re\mathrm{e}\, g'(0))}{1-|g'(0)|^2}. \tag{10}$$

In particular,

$$g'(1) \geq \frac{2}{1+\Re\mathrm{e}\, g'(0)}. \tag{11}$$

By combining (8), (9), and (11), we get the estimate in (6).

Furthermore, this bound in (6) is sharp. Indeed, if equality holds in (6) for $z \in \mathbb{D}$, then we must have equalities in the corresponding inequalities in (4) and (11). Thus, we can obtain

$$g(z) = z\frac{z-a}{1-\overline{a}z}\frac{1-\overline{a}}{1-a} \tag{12}$$

for some constant $a \in \overline{\mathbb{D}}$, and $g'(0) \in (-1, 1]$, which is possible only if $a \in [-1, 1)$.

Consequently, f must be of the form

$$f(z) = z^k \frac{a_k - z\frac{z-a}{1-az}\frac{a_k-1}{1-\overline{a_k}}}{1 - z\frac{z-a}{1-az}\frac{a_k-1}{1-\overline{a_k}}\overline{a_k}}, \quad \forall z \in \mathbb{D}, \tag{13}$$

for some constant $a \in [-1, 1)$.

Case II If $|a_k| = 1$, set

$$g(z) = \begin{cases} \frac{f(z)}{z^k}, & 0 < |z| < 1, \\ a_k, & z = 0. \end{cases}$$

It is clear that $g \in H(\mathbb{D}, \mathbb{D})$, $|g(0)| = |a_k| = 1$. Thus by the principle of the maximum modulus, g is a constant function, and $g(z) = a_k = g(1) = 1$, and hence $f(z) \equiv z^k$. This completes the proof. □

Taking into account the relation $|\frac{1-\overline{a_k}}{1-a_k} \cdot \frac{a_{k+1}}{1-|a_k|^2}| \leq 1$ and using (6) in Theorem 1, we can readily deduce the following corollary (the proof is omitted here).

Corollary 1 *Let $f \in H(\mathbb{D}, \mathbb{D})$ with $\xi = 1$ as its regular boundary fixed point and suppose $f(0) = f'(0) = \cdots = f^{(k-1)}(0) = 0, a_k = \frac{f^{(k)}(0)}{k!} \neq 0, k \in \mathbb{N}$; we have the following.*

If $0 < |a_k| < 1$, then

$$f'(1) \geq k + \frac{|1 - a_k|^2}{1 - |a_k|^2}. \tag{14}$$

In particular,

$$f'(1) \geq k - 1 + \frac{2}{1 + \Re\mathrm{e}\, a_k}. \tag{15}$$

Remark 1 When $n = 1$, it follows from (15) that

$$f'(1) \geq \frac{2}{1 + \Re\mathrm{e}\, a_1} = \frac{2}{1 + \Re\mathrm{e} f'(0)}.$$

Note that

$$\frac{2}{1 + \Re\mathrm{e} f'(0)} \geq \frac{2}{1 + |f'(0)|}.$$

Therefore, Theorem 1 (or Corollary 1) generalizes and improves Theorem A.

Theorem 2 *Let $f \in H(\mathbb{D}, \mathbb{D})$ with $\xi = 1$ as its regular boundary fixed point and suppose $f'(0) = \cdots = f^{(k-1)}(0) = 0, a_k = \frac{f^{(k)}(0)}{k!} \neq 0, k \in \mathbb{N}$, we can obtain:*

(I) *If $0 < |a_k| < 1 - |f(0)|^2$, then*

$$f'(1) \geq (k - 1)\frac{|1 - f(0)|^2}{1 - |f(0)|^2} + \frac{2}{\Re\mathrm{e}(\frac{1-f^2(0)+a_k}{(1-f(0))^2})}. \tag{16}$$

Equality holds in the inequality if and only if f is of the form

$$f(z) = \frac{f(0) - z^k \frac{a-z}{1-az} \frac{1-f(0)}{1-\overline{f(0)}}}{1 - z^k \frac{a-z}{1-az} \frac{1-f(0)}{1-\overline{f(0)}} \overline{f(0)}}.$$

(II) *If* $|a_k| = 1 - |f(0)|^2$, *then*

$$f(z) = \frac{\frac{1-f(0)}{1-\overline{f(0)}} z^k + f(0)}{1 + \overline{f(0)} \frac{1-f(0)}{1-\overline{f(0)}} z^k}. \tag{17}$$

Proof Set

$$g(z) = \frac{f(z) - f(0)}{1 - \overline{f(0)} f(z)} \frac{1 - \overline{f(0)}}{1 - f(0)}.$$

It is not difficult to verify that $g \in H(\mathbb{D}, \mathbb{D})$, and $\xi = 1$ is its regular boundary fixed point. Elementary computations yield

$$f'(1) = \frac{|1 - f(0)|^2}{1 - |f(0)|^2} g'(1) \tag{18}$$

and

$$\frac{g^{(k)}(0)}{k!} = \frac{a_k}{1 - |f(0)|^2} \frac{1 - \overline{f(0)}}{1 - f(0)}. \tag{19}$$

On the other hand, let

$$h(z) = \begin{cases} \frac{g(z)}{z^k}, & 0 < |z| < 1, \\ \frac{g^{(k)}(0)}{k!}, & z = 0, \end{cases} \tag{20}$$

which is in $H(\mathbb{D}, \mathbb{D})$. By Lemma 3, we obtain the following results:

(I) If $0 < |a_k| < 1 - |f(0)|^2$, then it follows from (19) that $|\frac{g^{(k)}(0)}{k!}| < 1$. By using Lemmas 1 and 2, we have

$$g'(1) = k + h'(1) \geq k + \frac{|1 - \frac{g^{(k)}(0)}{k!}|^2}{1 - |\frac{g^{(k)}(0)}{k!}|^2} = k - 1 + \frac{2(1 - \Re\mathrm{e}\, \frac{g^{(k)}(0)}{k!})}{1 - |\frac{g^{(k)}(0)}{k!}|^2}.$$

In particular,

$$g'(1) \geq k - 1 + \frac{2}{1 + \Re\mathrm{e}\, \frac{g^{(k)}(0)}{k!}}.$$

From the above relation and (18), we deduce that

$$f'(1) \geq \frac{|1 - f(0)|^2}{1 - |f(0)|^2} \left(k - 1 + \frac{2}{1 + \Re\mathrm{e}\, \frac{g^{(k)}(0)}{k!}} \right)$$

$$= \frac{|1-f(0)|^2}{1-|f(0)|^2}\left(k-1+\frac{2}{1+\Re\mathrm{e}(\frac{a_k}{1-|f(0)|^2}\frac{1-\overline{f(0)}}{1-f(0)})}\right)$$
$$= (k-1)\frac{|1-f(0)|^2}{1-|f(0)|^2}+\frac{2}{\Re\mathrm{e}(\frac{1-f^2(0)+a_k}{(1-f(0))^2})}.$$

Applying a similar argument to Theorem 1, we deduce that equality holds in inequality (16) if and only if f is of the form

$$f(z) = \frac{f(0)-z^k\frac{a-z}{1-az}\frac{1-f(0)}{1-\overline{f(0)}}}{1-z^k\frac{a-z}{1-az}\frac{1-f(0)}{1-\overline{f(0)}}\overline{f(0)}}.$$

(II) If $|a_k| = 1-|f(0)|^2$, then we find from (19) and (20) that $|h(0)| = |\frac{g^{(k)}(0)}{k!}| = 1$. By the principle of the maximum modulus, h is a constant function, and $h(z) = g(1) = 1$, and hence $g(z) \equiv z^k$, which yields the assertion (17). This completes the proof. □

Remark 2 By setting $k = 1$ in (16) of Theorem 2, we get the following estimate:

$$f'(1) \geq \frac{2}{\Re\mathrm{e}(\frac{1-f^2(0)+a_1}{(1-f(0))^2})} = \frac{2}{\Re\mathrm{e}(\frac{1-f(0)^2+f'(0)}{(1-f(0))^2})},$$

which is Theorem C obtained by Frolova *et al.* [1]. Thus, Theorem 2 is a generalization of Theorem C.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally to this work.

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