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A fixed point theorem for weakly inward A-proper maps and application to a Picard boundary value problem

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Abstract

A fixed point theorem for weakly inward A-proper maps defined on cones in Banach spaces is established using a fixed point index for such maps. The result generalizes a theorem in Deimling (Nonlinear Functional Analysis, 1985) for weakly inward maps defined on a cone in $\mathbb{R}^n$. We then apply the theorem to a Picard boundary value problem and obtain the existence of a positive solution.

MSC: Primary 34B18; secondary 34B15

Keywords: fixed point index; cone; positive solutions; boundary value problem

1 Introduction

The purpose of this paper is to establish a fixed point theorem for weakly inward A-proper maps defined on cones in Banach spaces that generalizes a result in Deimling [1], p.254, for weakly inward maps defined on a cone in $\mathbb{R}^n$. We use the fixed point index for weakly inward A-proper maps introduced by Lan and Webb [2] to obtain our new result. As an application, we obtain a positive solution to the Picard boundary value problem

$$-x''(t) = f\bigl(t, x(t), x'(t), x''(t)\bigr), \quad \text{where } x(0) = x(1) = 0,$$

under suitable conditions on f. This problem has been extensively studied; in particular, we refer to [3], where the concept of P_γ-compact maps and quasinormal cones is used, [4], where the problem is formulated as a semilinear equation, [5], where f is allowed to take negative values, and [6], where positive solutions for three-point boundary value problems are obtained. As mentioned in [5], in [3] and [4], examples were provided with conflicting hypotheses; our theorem will allow a different approach, which corrects the hypotheses of the analogous examples.

2 Preliminaries

Let X be a Banach space, $X_n \subset X$ a sequence of oriented finite-dimensional subspaces, and $P_n : X \to X_n$ a sequence of continuous linear projections such that $P_n x \to x$ for each $x \in X$. Then X is called a Banach space with projection scheme $\Gamma = \{X_n, P_n\}$.

A map $f : \mathrm{dom} f \subset X \to X$ is said to be *A-proper* with respect to Γ if $P_n f : X_n \to X_n$ is continuous for each n and for any bounded sequence $\{x_{n_j} | x_{n_j} \in X_{n_j}\}$ such that $f_{n_j}(x_{n_j}) \to y$, there exists a subsequence $\{x_{n_{j_k}}\}$ such that $x_{n_{j_k}} \to x$ and $f(x) = y$.

A closed convex set K in a Banach space X is called a *cone* if $\lambda K \subset K$ for all $\lambda \geq 0$ and $K \cap \{-K\} = 0$.

Let $K \subset X$ be a closed convex set. For each $x \in K$, the set $I_K(x) = \{x + c(z - x) : z \in K, c \geq 0\}$ is called the *inward set* of x with respect to K. A map $f : K \to X$ is called *inward* (respectively, *weakly inward*) if for all $x \in K, f(x) \in I_K(x)$ $(f(x) \in \bar{I}_K(x))$.

A map $f : \overline{\Omega}_K \to X$ is said to be *inward* (respectively, *weakly inward*) on $\overline{\Omega}_K$ relative to K if $f(x) \in I_K(x)$ (respectively, $f(x) \in \bar{I}_K(x)$) for $x \in \overline{\Omega}_K$, where $\Omega \subset X$ is open and bounded with $\Omega_K = \Omega \cap K \neq \emptyset$.

For the definition and properties of the Lan-Webb fixed point index, see [2].

3 An existence theorem for weakly inward *A*-proper maps

Theorem 3.1 *Let K be a closed convex set, and $f : K \to X$ be weakly inward on K, where $I - f$ is A-proper. Suppose that*

(a) *$f(x) \not\leq x$ on $\|x\| = r$, and*
(b) *there exists $\rho \in (0, r)$ such that $\lambda x \not\leq f(x)$ for $\|x\| = \rho$ and $\lambda > 1$.*

Then f has a fixed point in $\{x \in K : \rho < \|x\| < r\}$.

Proof Let $B_r = \{x \in X : \|x\| < r\}$, $B_{r_K} = B_r \cap K$, $B_\rho = \{x \in X : \|x\| < \rho\}$, and $B_{\rho_K} = B_\rho \cap K$. We show that $i_K(f, B_{r_K}) = \{0\}$ and $i_K(f, B_{\rho_K}) = \{1\}$, so that by the additivity property of the index $i_K(f, B_{r_K} \backslash B_{\rho_K}) = i_K(f, B_{r_K}) - i_K(f, B_{\rho_K}) = \{0\} - \{1\} = \{-1\} \neq \{0\}$, which implies the existence of a fixed point $x \in K$ such that $\rho < \|x\| < r$.

To show that $i_K(f, B_{r_K}) = \{0\}$, suppose instead that $i_K(f, B_{r_K}) \neq \{0\}$. Then we choose an a with $\|f(x)\| \leq a$ on $\overline{B}_{r_K}$ and an $e \in K$ with $\|e\| > r + a$. Define the weakly inward A-proper homotopy $H(x, t) = f(x) + te$. Now if $H(x, t) = x$ for some $(x, t) \in \partial B_{r_K} \times [0, 1]$, then $f(x) + te = x$, so that $x \in K$ and $x - f(x) = te \in K$ so $f(x) \leq x$, which contradicts (a). Thus, H is an admissible homotopy, and $i_K(H(x, 1), B_{r_K}) = i_K(f, B_{r_K}) \neq \{0\}$. Then there exists $x \in B_{r_K}$ with $f(x) + e = x$, so that $\|e\| = \|x - f(x)\| \leq \|x\| + \|f(x)\| \leq r + a$, which contradicts $\|e\| > r + a$. Hence, $i_K(f, B_{r_K}) = \{0\}$.

Now we show that $i_K(f, B_{\rho_K}) = \{1\}$. Define the weakly inward A-proper homotopy $H(x, t) = tf(x)$.

If $H(x, t) = x$ for some $(x, t) \in \partial B_{\rho_K} \times [0, 1]$, then $t \neq 0$ (this would give $0 = x$ on ∂B_{r_K}) and $tf(x) = x$ and $x \in K$, so that $f(x) = \frac{1}{t} x \geq x$, which contradicts (b).

Thus $H(x, t) \neq x$ on $\partial B_{\rho_K} \times [0, 1]$.

By the homotopy property of the index, $i_K(H(x, 0), B_{\rho_K}) = i_K(H(x, 1), B_{\rho_K}) = i_K(f, B_{\rho_K}) = \{1\}$.

Consequently, $i_K(f, B_{r_K} \backslash B_{\rho_K}) = i_K(f, B_{r_K}) - i_K(f, B_{\rho_K}) = \{0\} - \{1\} = \{-1\}$.

Since the index is not 0, the existence property implies that there exists a fixed point $x \in K$ such that

$$\rho < \|x\| < r. \qquad \square$$

Remark 3.1 The conclusion of Theorem 3.1 is valid if condition (a) holds for $\|x\| = \rho$ and condition (b) holds for $\|x\| = r$, that is,

(a) $f(x) \not\leq x$ on $\|x\| = \rho$, and
(b) $\lambda x \not\leq f(x)$ for $\|x\| = r$ and $\lambda > 1$.

We shall use these conditions in the following application.

4 Application

We formulate the Picard boundary value problem

$$-x''(t) = f\bigl(t, x(t), x'(t), x''(t)\bigr), \quad \text{where } x(0) = x(1) = 0 \tag{1}$$

as a fixed point equation of the operator $T : \overline{K}_r \to K$, $K_r = \{x \in K : \|x\| < r\}$,

$$Ty(t) = f\left(t, L^{-1}y, \frac{d}{dt}\bigl(L^{-1}y\bigr), -y\right),$$

where $L : X \to Y$ is defined by $Lx = -x''(t)$. Observe that (1) is equivalent to $y = Ty$.

Let $X = \{x \in C^2[0,1] : x(0) = x(1) = 0\}$, $Y = C[0,1]$, and $K = \{y \in C[1,0] : y(t) \geq 0\}$ with norms $\|x\|_X = \max\{\|x\|_Y, \|x'\|_Y, \|x''\|_Y\}$ and $\|x\|_Y = \max_{t\in[0,1]}\{|x(t)|\}$. Then L is a linear bounded isometric homeomorphism.

Theorem 4.1 *Under the above assumptions, suppose also that*

(a′) *there exist* $r > 0$ *and* $k \in (0,1)$ *such that* $f : [0,1] \times [0,r] \times [-r,r] \times R^- \to R^+$ *is continuous with* $|f(t,p,q,s_1) - f(t,p,q,s_2)| \leq k|s_1 - s_2|$ *for* $t \in [0,1]$, $p \in [0,r]$, $q \in [-r,r]$, $s_1, s_2 \in R^-$;
(b′) $f(t,p,q,s) < r$ *for every* $t \in [0,1]$, $p \in [0,r]$, $q \in [-r,r]$, $s = -r$;
(c′) *there are* $\rho \in (0,r)$, $t_0 \in [0,1]$ *such that* $f(t_0,p,q,s) > \rho$ *for* $p \in [0,\rho]$, $q \in [-\rho,\rho]$, $s = -\rho$.

Then there exists a positive solution $x \in K$ *to equation* (1) *with* $\rho < \|x\|_X < r$.

Proof Since T maps K to K, T is weakly inward. Condition (a′) implies that T is $(\beta_K)k$-ball contractive, where β_K is the ball measure of noncompactness associated with K, and thus $\lambda I - T$ is A-proper with respect to the projection scheme $\Gamma = \{X_n, P_n\}$ for every $\lambda \geq \gamma$, $\gamma \in (k,1)$ (*cf.* [3]). To verify the remaining hypotheses of Remark 3.1, we first show that (b′) implies (b). Let r be as in (b′) and $y \in K$ such that $\|y\|_Y = r$. Then there exists $x \in L^{-1}(K)$ such that $Lx = y$ and $\|x\|_X = \|y\|_Y = \|x''\|_Y$, so that $r = \|x''\|_Y = \|x\|_X$ and there exists $t_0 \in [0,1]$ such that $y(t_0) = r$. Now since $y = Lx$ for some $x \in L^{-1}(K)$, we have that $x(t) \in [0,r]$, $x'(t) \in [-r,r]$ for all $t \in [0,1]$ and $r = -x''(t_0)$. Then if $Ty = \lambda y$ for some $\lambda > 1$ and $y \in K$ with $\|y\|_Y = r$, we would have $f(t,x(t),x'(t),x''(t)) = \lambda y(t)$ for all $t \in [0,1]$, including t_0, but then this implies $\lambda r < r$, a contradiction. So (b) holds.

To show that (c′) implies (a) of Remark 3.1, let $x \in K$ with $\|x\|_X = \rho$. Then $\|Lx\|_Y = \|-x''\|_Y = \rho$, and there exists $t_1 \in [0,1]$ such that $-x''(t_1) = \rho$ or $x''(t_1) = -\rho$. So we have for $t \in [0,1]$ that $x(t) \in [0,\rho]$, $x'(t_1) \in [-\rho,\rho]$, and $x''(t_1) = -\rho$. By (c′) we have $Ty(t_1) = f(t_1,x(t_1),x'(t_1),x''(t_1)) > \rho$, and so (a) is satisfied.

Thus, there exists a solution to equation (1) with $x \in K$ and $\rho < \|x\| < r$. □

Example 4.1 The function $f(t,x,x',x'') = 1 + \frac{3}{4}\sin x''$ with $r = \frac{3\pi}{2}$ and $\rho = \frac{\pi}{2}$ shows that the class of maps that satisfy the conditions of Theorem 4.1 is nonempty.

Competing interests
The author declares that he has no competing interests.

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