Saddeek *Fixed Point Theory and Applications* (2016) 2016:60
DOI 10.1186/s13663-016-0549-9

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

On the convergence of a generalized modified Krasnoselskii iterative process for generalized strictly pseudocontractive mappings in uniformly convex Banach spaces

Ali Mohamed Saddeek*

*Correspondence:
dr.ali.saddeek@gmail.com
Department of Mathematics, Faculty of Science, Assiut University, Assiut, Egypt

Abstract
This paper aims to study the strong convergence of generalized modified Krasnoselskii iterative process for finding the minimum norm solutions of certain nonlinear equations with generalized strictly pseudocontractive, demiclosed, coercive, bounded, and potential mappings in uniformly convex Banach spaces. An application to nonlinear pseudomonotone equations is provided. The results extend and improve recent work in this direction.

MSC: 47H05; 47H10

Keywords: generalized modified Krasnoselskii iterative; generalized strictly pseudocontractive mappings; minimum norm solutions; uniformly convex Banach spaces

1 Introduction and preliminaries

Let H be a real Hilbert space with norm $\|\cdot\|_H$ and inner product $(\cdot,\cdot)$. Let C be a nonempty closed and convex subset of H. Let T be a nonlinear mapping of H into itself. Let I denote the identity mapping on H. Denote by $\mathfrak{F}(T)$ the set of fixed points of T.

Moreover, the symbols $\rightharpoonup$ and $\rightarrow$ stand for weak and strong convergence, respectively.

We say that T is generalized Lipschitzian iff there exists a nonnegative real valued function $r(x,y)$ satisfying $\sup_{x,y\in H}\{r(x,y)\} = \lambda < \infty$ such that

$$\|Tx - Ty\|_H \le r(x,y)\|x-y\|_H, \quad \forall x,y \in H. \tag{1.1}$$

Recently, this class of mappings has been studied by Saddeek and Ahmed [1], and Saddeek [2].

For $r(x,y) = \lambda \in (0,1)$ (resp., $r(x,y) = 1$) such mappings are said to be λ-contractive (resp., nonexpansive) mappings.

If $r(x,y) = \lambda > 0$, then the class of generalized Lipschitzian mappings coincide with the class of λ-Lipschitzian mappings.

We say that T is generalized strictly pseudocontractive iff for each pair of points x, y in H there exist nonnegative real valued functions $r_i(x,y)$, $i = 1,2$, satisfying

$$\sup_{x,y\in H}\left\{\sum_{i=1}^{2} r_i(x,y)\right\} = \lambda' < \infty$$

such that

$$\|Tx - Ty\|_H^2 \leq r_1(x,y)\|x-y\|_H^2 + r_2(x,y)\bigl\|(I-T)(x) - (I-T)(y)\bigr\|_H^2. \tag{1.2}$$

By letting $r_1(x,y) = 1$ and $r_2(x,y) = \lambda \in [0,1)$ (resp., $r_i(x,y) = 1$, $i = 1,2$) in (1.2), we may derive the class of λ-strictly pseudocontractive (resp., pseudocontractive) mappings, which is due to Browder and Petryshyn [3].

The class of λ-strictly pseudocontractive mappings has been studied recently by various authors (see, for example, [4–9]).

It worth noting that the class of generalized strictly pseudocontractive mappings includes generalized Lipschizian mappings, λ-strictly pseudocontractive mappings, λ-Lipschitzian mappings, pseudocontractive mappings, nonexpansive (or 0-strictly pseudocontractive) mappings.

These mappings appear in nonlinear analysis and its applications.

Definition 1.1 For any $x,y,z \in H$ the mapping T is said to be

(i) demiclosed at 0 (see, for example, [10]) if $Tx = 0$ whenever $\{x_n\} \subset H$ with $x_n \rightharpoonup x$ and $Tx_n \to 0$, as $n \to \infty$;

(ii) pseudomonotone (see, for example, [11]) if it is bounded and $x_n \rightharpoonup x \in H$ and

$$\limsup_{n\to\infty}(Tx_n, x_n - x) \leq 0 \quad \Longrightarrow \quad \liminf_{n\to\infty}(Tx_n, x_n - y) \geq (Tx, x-y);$$

(iii) coercive (see, for example, [12]) if

$$(Tx,x) \geq \rho\bigl(\|x\|_H\bigr)\|x\|_H, \qquad \lim_{\xi\to+\infty}\rho(\xi) = +\infty;$$

(iv) potential (see, for example, [13]) if

$$\int_0^1 \Bigl(\bigl(T\bigl(t(x+y)\bigr), x+y\bigr)\Bigr) - \bigl(T(tx),x\bigr)\Bigr)\,dt = \int_0^1 \bigl(T(x+ty),y\bigr)\,dt;$$

(v) hemicontinuous (see, for example, [12]) if

$$\lim_{t\to 0}\bigl(T(x+ty),z\bigr) = (Tx,z);$$

(vi) demicontinuous (see, for example, [12]) if

$$\lim_{\|x_n - x\|_H\to 0}(Tx_n, y) = (Tx,y);$$

(vii) uniformly monotone (see, for example, [11]) if there exist $p \geq 2$, $\alpha > 0$ such that

$$(Tx - Ty, x-y) \geq \alpha\|x-y\|_H^p;$$

(viii) bounded Lipschitz continuous (see, for example, [13]) if there exist $p \geq 2$, $M > 0$ such that

$$\|Tx - Ty\|_H \leq M\big(\|x\|_H + \|y\|_H\big)^{p-2}\|x - y\|_H.$$

It should be noted that any demicontinuous mapping is hemicontinuous and every uniformly monotone is monotone (*i.e.*, $(Tx - Ty, x - y) \geq 0$, $\forall x, y \in H$) and every monotone hemicontinuous is pseudomonotone.

If T is uniformly monotone (resp. bounded Lipschitz continuous) with $p = 2$, then T is called strongly monotone (resp. M-Lipschitzian).

For $x_0 \in C$ the Krasnoselskii iterative process (see, for example, [14]) starting at x_0 is defined by

$$x_{n+1} = (1 - \tau)x_n + \tau Tx_n, \quad n \geq 0, \tag{1.3}$$

where $\tau \in (0,1)$.

Recently, in a real Hilbert space setting, Saddeek and Ahmed [1] proved that the Krasnoselskii iterative sequence given by (1.3) converges weakly to a fixed point of T under the basic assumptions that $I - T$ is generalized Lipschitzian, demiclosed at 0, coercive, bounded, and potential. Moreover, they also applied their result to the stationary filtration problem with a discontinuous law.

However, the convergence in [1] is in general not strong. Very recently, motivated and inspired by the work in He and Zhu [15], Saddeek [2] introduced the following modified Krasnoselskii iterative algorithm by the boundary method:

$$x_{n+1} = \big(1 - \tau h(x_n)\big)x_n + \tau T_\tau x_n, \quad n \geq 0, \tag{1.4}$$

where $x_0 = x \in C$, $\tau \in (0,1)$, $T_\tau = (1 - \tau)I + \tau T$ and $h :\to [0,1]$ is a function defined by

$$h(x) = \inf\big\{\alpha \in [0,1] : \alpha x \in C\big\}, \quad \forall x \in C.$$

By replacing T_τ by T and taking $h(x_n) = 1$, $\forall n \geq 0$ in (1.4), we can obtain (1.3).

Saddeek [2] obtained some strong convergence theorems of the iterative algorithm (1.4) for finding the minimum norm solutions of certain nonlinear operator equations.

The class of uniformly convex Banach spaces play an important role in both the geometry of Banach spaces and relative topics in nonlinear functional analysis (see, for example, [16, 17]).

Let X be a real Banach space with its dual X^*. Denote by $\langle \cdot, \cdot \rangle$ the duality pairing between X^* and X. Let $\|\cdot\|_X$ be a norm in X, and $\|\cdot\|_{X^*}$ be a norm in X^*.

A Banach space X is said to be strictly convex if $\|x + y\|_X < 2$ for every $x, y \in X$ with $\|x\|_X \leq 1$, $\|y\|_X \leq 1$ and $x \neq y$.

A Banach space X is said to be uniformly convex if for every $\varepsilon > 0$, there exists an increasing positive function $\delta(\varepsilon)$ with $\delta(0) = 0$ such that $\|x\|_X \leq 1$, $\|y\|_X \leq 1$ with $\|x - y\|_X \geq \varepsilon$ imply $\|x + y\|_X \leq 2(1 - \delta(\varepsilon))$ for every $x, y \in X$.

It is well known that every Hilbert space is uniformly convex and every uniformly convex Banach space is reflexive and strictly convex.

A Banach space X is said to have a Gateaux differentiable norm (see, for example, [10], p.69) if for every $x, y \in X$ with $\|x\|_X = 1$, $\|y\|_X = 1$ the following limit exists:

$$\lim_{t\to 0^+} \frac{[\|x+ty\|_X - \|x\|_X]}{t}.$$

X is said to have a uniformly Gateaux differentiable norm if for all $y \in X$ with $\|y\|_X = 1$, the limit is attained uniformly for $\|x\|_X = 1$.

Hilbert spaces, L^p (or l_p) spaces, and Sobolev spaces W_p^1 $(1 < p < \infty)$ are uniformly convex and have a uniformly Gateaux differentiable norm.

The generalized duality mapping J_p, $p > 1$ from X to 2^{X^*} is defined by

$$J_p(x) = \{x^* \in X^* : \langle x^*, x\rangle = \|x\|_X^p, \|x^*\|_{X^*} = \|x\|_X^{p-1}\}, \quad \forall x \in X.$$

It is well known that (see, for example, [18, 19]) if the uniformly convex Banach space X is a uniformly Gateaux differentiable norm, then J_p is single valued (we denote it by j_p), one to one and onto. In this case the inverse of j_p will be denoted by j_p^{-1}.

Definition 1.1 above can easily be stated for mappings T from C to X^*. The only change here is that one replaces the inner product $(\cdot,\cdot)$ by the bilinear form $\langle\cdot,\cdot\rangle$.

Given a nonlinear mapping A of C into X^*. The variational inequality problem associated with C and A is to find

$$x \in C : \langle Ax - f, y - x\rangle \geq 0, \quad \forall y \in C, f \in X^*. \tag{1.5}$$

The set of solutions of the variational inequality (1.5) is denoted by $\mathrm{VI}(C, A)$.

It is well known (see, for example, [12, 20, 21]) that if A is pseudomonotone and coercive, then $\mathrm{VI}(C, A)$ is a nonempty, closed, and convex subset of X. Further, if $A = j_p - T$, then $\tilde{\mathfrak{F}}(j_p, T) = \{x \in C : j_p x = Tx\} = A^{-1}0$. In addition, there exists also a unique element $z = \mathrm{proj}_{A^{-1}0}(0) \in \mathrm{VI}(A^{-1}0, j_p)$, called the minimum norm solution of variational inequality (1.5) (or the metric projection of the origin onto $A^{-1}0$). If $X = H$, then $j_p = I$ and hence $\tilde{\mathfrak{F}} = \mathfrak{F}$.

Example 1.1 Let Ω be a bounded domain in $\mathbb{R}^n$ with Lipschitz continuous boundary. Let us consider $p \geq 2$, $\frac{1}{p} + \frac{1}{q} = 1$, and $X = \mathring{W}_p^{(1)}(\Omega)$, $X^* = W_q^{(-1)}(\Omega)$. The p-Laplacian is the mapping $-\Delta_p : \mathring{W}_p^{(1)}(\Omega) \to W_q^{(-1)}(\Omega)$, $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ for $u \in \mathring{W}_p^{(1)}(\Omega)$.

It is well known that the p-Laplacian is in fact the generalized duality mapping j_p (more specifically, $j_p = -\Delta_p$), *i.e.*, $\langle j_p u, v\rangle = \int_\Omega |\nabla u|^{p-2}(\nabla u, \nabla u)\,dx$, $\forall u, v \in \mathring{W}_p^{(1)}(\Omega)$.

From [22], p.312, we have

$$\begin{aligned}\langle j_p u - j_p v, u - v\rangle &= \int_\Omega \bigl(\bigl(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\bigr), \nabla(u-v)\bigr)\,dx \\ &\geq M \int_\Omega |\nabla u - \nabla v|^p\,dx \quad \text{for some } M > 0,\end{aligned}$$

which implies that j_p is uniformly monotone.

By [22], p.314, we have

$$\bigl|\langle j_p u - j_p v, w\rangle\bigr| \leq M\|u - v\|_{\mathring{W}_p^{(1)}(\Omega)} \bigl(\|u\|_{\mathring{W}_p^{(1)}(\Omega)} + \|v\|_{\mathring{W}_p^{(1)}(\Omega)}\bigr)^{p-2} \|w\|_{\mathring{W}_p^{(1)}(\Omega)},$$

or

$$\begin{aligned}\|j_p u - j_p v\|_{W_q^{(-1)}(\Omega)} &= \sup_{w\in \mathring{W}_p^{(1)}(\Omega)} \frac{|\langle j_p u - j_p v, w\rangle|}{\|w\|_{\mathring{W}_p^{(1)}(\Omega)}} \\ &\le M\|u-v\|_{\mathring{W}_p^{(1)}(\Omega)}\bigl(\|u\|_{\mathring{W}_p^{(1)}(\Omega)} + \|v\|_{\mathring{W}_p^{(1)}(\Omega)}\bigr)^{p-2},\end{aligned}$$

this shows that j_p is bounded Lipschitz continuous.

The generalized duality mapping $j_p = -\Delta_p$ is bounded, demicontinuous (and hence hemicontinuous) and monotone, and hence j_p is pseudomonotone.

From the definition of j_p, it follows that j_p is coercive.

Since $j_p u \in W_q^{(-1)}(\Omega)$, $\forall u \in \mathring{W}_p^{(1)}(\Omega)$ is the subgradient of $\frac{1}{p}\|u\|^p_{\mathring{W}_p^{(1)}(\Omega)}$, it follows that j_p is potential.

Since j_p is pseudomonotone and coercive (it is surjective), then j_p is demiclosed at 0 (see Saddeek [2] for an explanation).

The mapping j_p is generalized strictly pseudocontractive with $r_1(x,y) = 1$.

The following two lemmas play an important role in the sequel.

Lemma 1.1 ([23]) *Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be nonnegative real sequences satisfying*

$$a_{n+1} \le (1-\gamma_n)a_n + b_n + c_n, \quad \forall n \ge 0,$$

where $\gamma_n \subset (0,1)$, $\sum_{n=0}^{\infty}\gamma_n = \infty$, $\limsup_{n\to\infty}\frac{b_n}{\gamma_n} \le 0$, and $\sum_{n=0}^{\infty} c_n < \infty$. Then $\lim_{n\to\infty} a_n = 0$.

Lemma 1.2 ([24]) *Let X be a real uniformly convex Banach space with a uniformly Gateaux differentiable norm, and let X^* be its dual. Then, for all $x^*, y^* \in X^*$, the following inequality holds*:

$$\|x^* + y^*\|^2_{X^*} \le \|x^*\|^2_{X^*} + 2\langle y^*, j_p^{-1}x^* - y\rangle, \quad y \in X,$$

where j_p^{-1} is the inverse of the duality mapping j_p.

Let us now generalize the algorithm (1.4) for a pair of mappings as follows:

$$j_p x_{n+1} = \bigl(1 - \tau h(x_n)\bigr) j_p x_n + \tau T_\tau^{j_p} x_n, \quad n \ge 0, \tag{1.6}$$

where $x_0 = x \in C$, $\tau \in (0,1)$, $T_\tau^{j_p} = (1-\tau)j_p + \tau T$, $T : C \to X^*$ is a suitable mapping, and $j_p : X \to X^*$ is the generalized duality mapping.

This algorithm can also be regarded as a modification of algorithm (3) in [1]. We shall call this algorithm the generalized modified Krasnoselskii iterative algorithm.

In the case when X is uniformly convex Banach space, the generalized strictly pseudocontractive mapping (1.2) can be written as follows:

$$\begin{aligned}\|Tx - Ty\|^p_{X^*} &\le r_1(x,y)\|j_p x - j_p y\|^p_{X^*} \\ &\quad + r_2(x,y)\bigl\|(j_p - T)(x) - (j_p - T)(y)\bigr\|^p_{X^*}, \quad p \in [2,\infty),\end{aligned} \tag{1.7}$$

where $r_1(x,y)$ and $r_2(x,y)$ satisfy the same conditions as above.

Obviously, (1.6) and (1.7) reduce to (1.4) and (1.2), respectively, when X is a Hilbert space.

The main purpose of this paper is to extend the results in [2] to uniformly convex Banach spaces and to generalized modified iterative processes with generalized strictly pseudocontractive mappings.

2 Main results

Now we are ready to state and prove the results of this paper.

Theorem 2.1 *Let X be a real uniformly convex Banach space with a uniformly Gateaux differentiable norm and X^* be its dual. Let C be a nonempty closed convex subset of X. Let $j_p : X \to X^*$ be the generalized duality mapping and let $T : C \to X^*$ be a bounded Lipschitz continuous nonlinear mapping. Define $S_{h(x)} : C \to X^*$ by*

$$S_{h(x)}x = \bigl(h(x) + \tau - 1\bigr)j_p x - \tau Tx, \quad \forall x \in C,$$

where the function $h(x)$ is defined as above and $\tau \in (0,1)$.

Assume that $S_{h(x)}$ is demiclosed at 0, *coercive, potential, bounded, and generalized strictly pseudocontractive in the sense of* (1.7), *here $r_i = r_i(x,y)$, $i = 1,2$, satisfy the following condition*:

$$\sup_{x,y\in C}\bigl[r_1 + \bigl(2 - h(x)\bigr)^p r_2\bigr] = \bigl(\lambda'\bigr)^p < \infty, \quad p \geq 2.$$

Suppose that the constant α appearing in (1.5) *is as follows*:

$$\alpha = \sup_{x,y\in C}\Bigl[\|x-y\|_X + 2\sup_{x\in C}\|x\|_X\Bigr]^{p-2}\|x-y\|_X^{2-p}, \quad p \geq 2.$$

Then the iterative sequence $\{x_n\}$ generated by algorithm (1.6) *with $\sum_{n=0}^{\infty} h(x_n) = \infty$ and $0 < \tau = \min\{1, \frac{1}{\lambda' M}\}$, converges strongly to $\bar{x} \in \mathrm{VI}(S_{h(\bar{x})}^{-1}0, j_p)$, $\bar{x} = \mathrm{proj}_{S_{h(\bar{x})}^{-1}0}(0)$, where $S_{h(\bar{x})}^{-1}0 = \tilde{\mathfrak{F}}(h(\bar{x})j_p, T_\tau^{j_p})$.*

Proof First observe that $\{x_n\}$ is well defined because $S_{h(x)}$ is bounded and $\lambda' < \infty$. Next, we show that the sequence $\{x_n\}$ is bounded. Since $S_{h(x)}$ is coercive, it is sufficient (see proof of Theorem 4.1 in [2]) to show that

$$\{x_n\} \subset S_0, \qquad \|x_n\|_X \leq R_0, \quad n \geq 0, \tag{2.1}$$

where $S_0 = \{x \in C : F(x) \leq F(x_0)\}$, $R_0 = \sup_{x\in S_0}\|x\|_X$, and $F : X \to (-\infty, \infty]$ is a real function defined as follows:

$$F(x) = \int_0^1 \bigl\langle S_{h(x)}(tx), x\bigr\rangle\,dt, \quad \forall x \in X. \tag{2.2}$$

From the definition of S_0, it follows immediately that $x_0 \in S_0$. Suppose, for $n \geq 1$, that $x_n \in S_0$. We now claim that $x_{n+} \in S_0$. Indeed, from (1.7), the bounded Lipschitz continuity of j_p, T, and the definition of $S_{h(x)}$, we obtain

$$\begin{aligned}&\bigl\|S_{h(x_n)}\bigl(x_{n+1} + t(x_n - x_{n+1})\bigr) - S_{h(x_n)}(x_n)\bigr\|_{X^*}^p\\ &\quad\leq r_1\bigl\|j_p\bigl(x_{n+1} + t(x_n - x_{n+1})\bigr) - j_p x_n\bigr\|_{X^*}^p\end{aligned}$$

$$
\begin{aligned}
&\quad + r_2 \big\| (j_p - S_{h(x_n)})\big(x_{n+1} + t(x_n - x_{n+1})\big) - (j_p - S_{h(x_n)})(x_n) \big\|_{X^*}^p \\
&\leq r_1 \big\| j_p\big(x_{n+1} + t(x_n - x_{n+1})\big) - j_p(x_n) \big\|_{X^*}^p \\
&\quad + r_2 \big[\big(2 - \tau - h(x_n)\big) \big\| j_p\big(x_{n+1} + t(x_n - x_{n+1})\big) - j_p(x_n) \big\|_{X^*} \\
&\quad + \tau \big\| T\big(x_{n+1} + t(x_n - x_{n+1})\big) - T(x_n) \big\|_{X^*} \big]^p \\
&\leq (1-t)^p M^p \big[r_1 + \big(2 - h(x_n)\big)^p r_2 \big] \\
&\quad \times \big(\big\| x_{n+1} + t(x_n - x_{n+1}) \big\|_X + \|x_n\|_X \big)^{p(p-2)} \|x_n - x_{n+1}\|_X^p \\
&= (1-t)^p M^p \big[r_1 + \big(2 - h(x_n)\big)^p r_2 \big] \\
&\quad \times \big(\big\| x_{n+1} + t(x_n - x_{n+1}) \big\|_X - \|x_n\|_X + 2\|x_n\|_X \big)^{p(p-2)} \|x_n - x_{n+1}\|_X^p \\
&\leq (1-t)^p M^p \big[r_1 + \big(2 - h(x_n)\big)^p r_2 \big] \\
&\quad \times \big[(1-t)\|x_n - x_{n+1}\|_X + 2\|x_n\|_X \big]^{p(p-2)} \|x_n - x_{n+1}\|_X^p \\
&\leq M^p \big[r_1 + \big(2 - h(x_n)\big)^p r_2 \big] \\
&\quad \times \big[\|x_n - x_{n+1}\|_X + 2R_0 \big]^{p(p-2)} \|x_n - x_{n+1}\|_X^p \quad \text{for } t \in [0,1].
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\big\| S_{h(x_n)}\big(x_{n+1} + t(x_n - x_{n+1})\big) - S_{h(x_n)}(x_n) \big\|_{X^*}^p \\
&\quad \leq M\lambda' \big[\|x_n - x_{n+1}\|_X + 2R_0 \big]^{p-2} \|x_n - x_{n+1}\|_X.
\end{aligned} \tag{2.3}
$$

This implies that

$$
\begin{aligned}
&\big| \big\langle S_{h(x_n)}\big(x_{n+1} + t(x_n - x_{n+1})\big) - S_{h(x_n)}(x_n), x_n - x_{n+1} \big\rangle \big| \\
&\quad \leq M\lambda' \big[\|x_n - x_{n+1}\|_X + 2R_0 \big]^{p-2} \|x_n - x_{n+1}\|_X^2.
\end{aligned} \tag{2.4}
$$

Since $S_{h(x)}$ is potential and j_p is uniformly monotone, by (2.2), (1.6), and (2.4) it follows that

$$
\begin{aligned}
F(x_n) - F(x_{n+1}) &= \int_0^1 \big(\big\langle S_{h(x_n)}(tx_n), x_n \big\rangle - \big\langle S_{h(x_n)}(tx_{n+1}), x_{n+1} \big\rangle \big)\, dt \\
&= \int_0^1 \big(\big\langle S_{h(x_n)}\big(x_{n+1} + t(x_n - x_{n+1})\big), x_n - x_{n+1} \big\rangle \big)\, dt \\
&= \int_0^1 \big(\big\langle S_{h(x_n)}\big(x_{n+1} + t(x_n - x_{n+1})\big), x_n - x_{n+1} \big\rangle \big)\, dt \\
&\quad - \int_0^1 \big\langle S_{h(x_n)}(x_n), x_n - x_{n+1} \big\rangle\, dt + \big\langle S_{h(x_n)}(x_n), x_n - x_{n+1} \big\rangle \\
&\geq - \int_0^1 \big| \big\langle S_{h(x_n)}\big(x_{n+1} + t(x_n - x_{n+1})\big) - S_{h(x_n)}(x_n), x_n - x_{n+1} \big\rangle \big|\, dt \\
&\quad + \big\langle S_{h(x_n)}(x_n), x_n - x_{n+1} \big\rangle \\
&\geq -M\lambda' \big[\|x_n - x_{n+1}\|_X + 2R_0 \big]^{p-2} \|x_n - x_{n+1}\|_X^2 \\
&\quad + \frac{1}{\tau} \langle j_p x_n - j_p x_{n+1}, x_n - x_{n+1} \rangle \\
&\geq -M\lambda' \big[\|x_n - x_{n+1}\|_X + 2R_0 \big]^{p-2} \|x_n - x_{n+1}\|_X^2 + \frac{\alpha}{\tau} \|x_n - x_{n+1}\|_X^p,
\end{aligned}
$$

which together with the restriction on α implies that

$$F(x_n) - F(x_{n+1}) \geq \mu\bigl[\|x_n - x_{n+1}\|_X + 2R_0\bigr]^{p-2}\|x_n - x_{n+1}\|_X^2, \quad \mu = \frac{1}{\tau} - M\lambda' > 0. \tag{2.5}$$

Therefore, $F(x_{n+1}) \leq F(x_n) \leq F(x_0)$, which implies that $x_{n+1} \in S_0$. Thus, by mathematical induction we get $x_n \in S_0$ for all $n \geq 0$. This shows that x_n is bounded. This, together with the definition of j_p, the boundedness of $S_{h(x_n)}$, and (1.6), (2.2), implies that the sequences $\{S_{h(x_n)}(x_n)\}$, $\{j_p(x_n)\}$, $\{T_\tau^{j_p}(x_n)\}$, and $\{F(x_n)\}$ are also bounded.

Further, it follows from (2.5) that the sequence $\{F(x_n)\}$ is monotonically decreasing and therefore convergent. Consequently, from (2.5), we have

$$\lim_{n\to\infty}\|x_n - x_{n+1}\|_X = 0. \tag{2.6}$$

Hence, by the bounded Lipschitz continuity of j_p, we obtain

$$\lim_{n\to\infty}\|j_p x_n - j_p x_{n+1}\|_{X^*} = 0. \tag{2.7}$$

Therefore, by (1.6) and the definition of $S_{h(x)}$, we then have

$$\lim_{n\to\infty}\|S_{h(x_n)}x_n\|_{X^*} = 0. \tag{2.8}$$

Let $\bar{x}$ be a weak limit point of $\{x_n\}$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\lim_{k\to\infty}\|x_{n_k} - \bar{x}\|_X \to \sigma_{\bar{x}}. \tag{2.9}$$

Since $S_{h(x)}$ is demiclosed at 0, it follows from (2.8) and (2.9) that $S_{h(\bar{x})}\bar{x} = 0$, and hence

$$\bar{x} \in S_{h(\bar{x})}^{-1}0. \tag{2.10}$$

Now, we show that

$$\limsup_{n\to\infty}\langle S_{h(x_n)}x_n, x_{n+1} - \tilde{x}\rangle \leq 0, \quad \forall\tilde{x} \in S_{h(\tilde{x})}^{-1}0. \tag{2.11}$$

Using (1.6) and the definition of $S_{h(x)}$, we get

$$\begin{aligned}\bigl\langle S_{h(x_n)}(x_n), x_{n+1} - \tilde{x}\bigr\rangle &\leq \bigl\langle S_{h(x_n)}(x_n), x_{n+1} - x_n\bigr\rangle + \tau^{-1}\bigl\langle j_p(x_n) - j_p(x_{n+1}), x_n - \tilde{x}\bigr\rangle \\ &\leq \bigl\|S_{h(x_n)}(x_n)\bigr\|_{X^*}\|x_n - x_{n+1}\|_X \\ &\quad + \tau^{-1}\|j_p x_n - j_p x_{n+1}\|_{X^*}\|x_n - \tilde{x}\|_X.\end{aligned} \tag{2.12}$$

Taking the lim sup as $n \to \infty$ in (2.12) and using (2.6), (2.7), and (2.8) yield the desired inequality (2.11).

Now, let us show that

$$\limsup_{n\to\infty}\bigl\langle -j_p(\bar{x}), x_{n+1} - \bar{x}\bigr\rangle \leq 0, \tag{2.13}$$

where $\bar{x}$ is the metric projection of the origin onto $S_{h(\bar{x})}^{-1}0$.

Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that $x_{n_{k+1}} \to \tilde{x} \in S^{-1}_{h(\bar{x})}0$ and

$$\limsup_{n\to\infty}\langle -j_p(\bar{x}), x_{n+1}-\bar{x}\rangle = \limsup_{k\to\infty}\langle -j_p(\bar{x}), x_{n_{k+1}}-\bar{x}\rangle.$$

It follows from Kato [21] that

$$\limsup_{n\to\infty}\langle -j_p(\bar{x}), x_{n+1}-\bar{x}\rangle = \langle -j_p(\bar{x}), \tilde{x}-\bar{x}\rangle \le 0. \tag{2.14}$$

This proves the desired inequality (2.13), and, hence by (2.10) and (2.14), we obtain

$$\bar{x} \in S^{-1}_{h(\bar{x})}0 \cap \mathrm{VI}\big(S^{-1}_{h(\bar{x})}0, j_p\big). \tag{2.15}$$

Now, we prove that $j_p x_n \to j_p\bar{x}$ as $n \to \infty$.

By using (1.6) and Lemma 1.2, we get

$$\begin{aligned}
\|j_p x_{n+1} - j_p\bar{x}\|^2_{X^*} &= \big\|\big(1-\tau h(x_n)\big)(j_p x_n - j_p\bar{x}) + \tau\big(T^{j_p}_\tau x_n - h(x_n)j_p\bar{x}\big)\big\|^2_{X^*} \\
&\le \big(1-\tau h(x_n)\big)^2\|j_p x_n - j_p\bar{x}\|^2_{X^*} + 2\tau\big\langle T^{j_p}_\tau x_n - h(x_n)j_p\bar{x}, x_{n+1}-\bar{x}\big\rangle \\
&\le \big(1-\tau h(x_n)\big)^2\|j_p x_n - j_p\bar{x}\|^2_{X^*} + 2\tau\big[\big\langle T^{j_p}_\tau x_n - h(x_n)j_p x_n, x_{n+1}-\bar{x}\big\rangle \\
&\quad + h(x_n)\big\langle -j_p(\bar{x}), x_{n+1}-\bar{x}\big]\big\rangle + 2\tau h(x_n)\big\|j_p(x_n)\big\|_{X^*}\|x_{n+1}-\bar{x}\|_X.
\end{aligned} \tag{2.16}$$

Set $\gamma_n = \tau h(x_n)(2-\tau h(x_n))$, $a_n = \|j_p x_n - j_p\bar{x}\|^2_{X^*}$, $b_n = 2\tau[\langle T^{j_p}_\tau x_n - h(x_n)j_p x_n, x_{n+1}-\bar{x}\rangle + h(x_n)\langle -j_p(\bar{x}), x_{n+1}-\bar{x}\rangle]$, and $c_n = 2\tau h(x_n)\|j_p(x_n)\|_{X^*}\|x_{n+1}-\bar{x}\|_X$.

Then inequality (2.16) becomes

$$a_{n+1} \le (1-\gamma_n)a_n + b_n + c_n, \quad \forall n \ge 0. \tag{2.17}$$

From $\sum_{n=0}^\infty h(x_n) = \infty$, and (2.13), it follows that $\sum_{n=0}^\infty \gamma_n = \infty$, $\limsup_{n\to\infty}\frac{b_n}{\gamma_n} \le 0$, and $\sum_{n=0}^\infty c_n < \infty$. Consequently, applying Lemma 1.1 to (2.17), we conclude that

$$\lim_{n\to\infty}\|j_p x_n - j_p\bar{x}\|_{X^*} = 0. \tag{2.18}$$

Finally, we show that $x_n \to \bar{x}$ as $n \to \infty$.

From the uniform monotonicity of j_p, we have

$$\begin{aligned}
\|x_n - \bar{x}\|^p_X &\le \frac{1}{\alpha}\langle j_p x_n - j_p\bar{x}, x_n - \bar{x}\rangle \\
&\le \frac{1}{\alpha}\|j_p x_n - j_p\bar{x}\|_{X^*}\big[\|x_n\|_X + \|\bar{x}\|_X\big].
\end{aligned} \tag{2.19}$$

Letting $n \to \infty$ in (2.19) and using (2.18) and the boundedness of $\{x_n\}$, we obtain $x_n \to \bar{x}$, as $n \to \infty$. This completes the proof. □

An immediate consequence of Theorem 2.1 is the following corollary.

Corollary 2.1 *Let $X = H$ be a real Hilbert space, and let C be a nonempty closed convex subset of H. Let $T : C \to H$ be an M-Lipschitzian mapping. Define $\hat{S}_{h(x)} : C \to H$ by*

$$\hat{S}_{h(x)}x = h(x)x - T_\tau x, \quad \forall x \in C,$$

where $T_\tau = (1-\tau)I + \tau T$. *Let* $\hat{S}_{h(x)}$, τ, *and* $h(x)$ *be as in Theorem* 2.1 *and* $p = 2$ (*i.e.*, $j_p = I$, $\alpha = 1$, *and* $\sup_{x,y\in C}[r_1 + (2-h(x))^2 r_2] = (\lambda')^2 < \infty$). *Then the sequence* $\{x_n\}$ *defined by*

$$x_{n+1} = \big(1 - \tau h(x_n)\big)x_n + \tau T_\tau x_n, \quad n \geq 0, \tag{2.20}$$

with $\sum_{n=0}^{\infty} h(x_n) = \infty$ *converges strongly to* $\bar{x} \in \mathrm{VI}(\hat{S}^{-1}_{h(\bar{x})}0, I)$, $\bar{x} = \mathrm{proj}_{S^{-1}_{h(\bar{x})}0}(0)$, *where* $\hat{S}^{-1}_{h(\bar{x})}0 = \tilde{\mathfrak{F}}(h(\bar{x})I, T_\tau)$.

A special case of Corollary 2.1 is the following theorem due to Saddeek [2], who proved it under the condition that T is generalized Lipschitzian, which in turn, is a generalization of Theorem 2 of Saddeek and Ahmed [1].

Corollary 2.2 *Except for the M-Lipschitzian condition for the mapping* T, *let all the other assumptions of Corollary* 2.1 *be satisfied and* $r_2 = 0$. *Then the sequence* $\{x_n\}$ *defined by* (2.20) *with* $\sum_{n=0}^{\infty} h(x_n) = \infty$, $0 < \tau = \min\{1, \frac{1}{\lambda}\}$, *and* $\sup_{x,y\in C}[r_1(x,y)] = (\lambda)^2 < \infty$, *converges strongly to* $\bar{x} = \mathrm{proj}_{S^{-1}_{h(\bar{x})}0}(0)$.

Remark 2.1 All conditions imposed in Theorem 2.1 on the mapping $S_{h(x)}$ are quintessential to prove the main theorem, more precisely for the existence solution of $S_{h(x)}x = 0$, and to ensure the strong convergence of the generalized modified Krasnoselski iterative algorithm.

3 Application to nonlinear pseudomonotone equations

In this section, we study nonlinear equations for pseudomonotone mappings; that is; we seek $x \in C$ such that

$$Ax = f, \quad f \in X^*, \tag{3.1}$$

where $A : C \to X^*$ is a nonlinear pseudomonotone mapping.

To ensure the existence of solutions of (3.1), we shall assume that A is pseudomonotone and coercive on $\mathring{W}_p^{(1)}(\Omega)$ $(1 < p < \infty)$ (see, for example, [12]). Such nonlinear equations occur, in particular, in descriptions of a stabilized filtration and in problems of finding the equilibria of soft shells (see, for example, [25, 26]).

Theorem 3.1 *Besides the assumptions on* A, *let* A *be potential and satisfy the following condition*:

$$\|Ax - Ay\|_{X^*} \leq \|j_p x - j_p y\|_{X^*}, \quad \forall x, y \in C. \tag{3.2}$$

Then the sequence $\{x_n\}$ *generated by* $x_0 = x \in C$,

$$j_p x_{n+1} = j_p x_n - \tau\big(A(x_n) - f\big), \quad n \geq 0, \tag{3.3}$$

where $0 < \tau = \min\{1, \frac{1}{M}\}$, *converges strongly to the minimum norm solution of equation* (3.1), *provided that* $\sum_{n=0}^{\infty} h(x_n) = \infty$.

Proof Define $S_{h(x)} : C \to X^*$ by $S_{h(x)}x = Ax - f$, $\forall x \in C$. Since (3.1) has at least one solution, then $S_{h(x)}^{-1}0 \neq \phi$. On the other hand, condition (3.2) with the bounded Lipschitz continuity of j_p clearly imply that A is bounded Lipschitz continuous and the potentiality of j_p imply that $s_{h(x)}$ is potential.

Now, we show that condition (1.7) is implied by (3.2). Indeed by (3.2) and the definition of $S_{h(x)}$, we get

$$\|S_{h(x)}x - S_{h(x)}y\|_{X^*}^p = \|Ax - Ay\|_{X^*}^p \leq \|j_p x - j_p y\|_{X^*}^p.$$

Hence $S_{h(x)}$ satisfies condition (1.7) with $r_1(x,y) = 1$, $r_2(x,y) = 0$, and $\lambda' = 1$.

Finally, the pseudomonotonicity of A implies that $S_{h(x)}$ is demiclosed at 0 can be proved by proceeding as in the proof of Theorem 5.1 of [2]. Now we apply Theorem 2.1 to yield the desired result. □

Remark 3.1 If we set $X = H$ (*i.e.*, $j_p = I$ and $p = 2$), then the condition (3.2) reduces to the M-Lipschitzian condition of the operator A. Hence from Theorem 3.1 we obtain Theorem 5.1 of [2], which in turn is a generalization of Theorem 3 of [1].

4 Conclusion

In this work, we introduce a generalized modified Krasnoselskii iterative process involving a pair of a generalized strictly pseudocontractive mapping and a generalized duality mapping and prove some strong convergence theorems of the proposed iterative process to the minimum norm solutions of certain nonlinear equations in the framework of uniformly convex Banach spaces. These results improve and generalize recent work in this direction.

Competing interests
The author declares that he has no competing interests.

Author's contributions
The author have read and approved the final manuscript.

Acknowledgements
The author would like to thank the editor and the reviewers for their valuable suggestions and comments.

Received: 25 August 2015 Accepted: 25 April 2016 Published online: 10 May 2016

References

1. Saddeek, AM, Ahmed, SA: Iterative solution of nonlinear equations of the pseudo-monotone type in Banach spaces. Arch. Math. **44**, 273-281 (2008)
2. Saddeek, AM: A strong convergence theorem for a modified Krasnoselskii iteration method and its application to seepage theory in Hilbert spaces. J. Egypt. Math. Soc. **22**, 476-480 (2014)
3. Browder, FE, Petryshyn, WV: Construction of fixed points of nonlinear mappings in Hilbert spaces. J. Math. Anal. Appl. **20**, 197-228 (1967)
4. Marino, G, Xu, HK: Weak and strong convergence theorems for *k*-strict pseudocontractions in Hilbert spaces. J. Math. Anal. Appl. **329**, 336-349 (2007)
5. Zhou, H: Convergence theorems of fixed points for *k*-strict pseudocontractions in Hilbert spaces. Nonlinear Anal. **69**, 456-462 (2008)
6. Enyi, CD, Iyiola, OS: A new iterative scheme for common solution of equilibrium problems, variational inequalities and fixed point of *k*-strictly pseudocontractive mappings in Hilbert spaces. Br. J. Math. Comput. Sci. **4**(4), 512-527 (2014)
7. Hao, Y: A strong convergence theorem on generalized equilibrium problems and strictly pseudocontractive mappings. Proc. Est. Acad. Sci. **60**(1), 12-24 (2011)
8. He, Z: Strong convergence of the new modified composite iterative method for strict pseudocontractions in Hilbert spaces. Note Mat. **31**(2), 67-78 (2011)
9. Li, M, Yao, Y: Strong convergence of an iterative algorithm for λ-strictly pseudocontractive mappings in Hilbert spaces. An. Ştiinţ. Univ. 'Ovidius' Constanţa **18**(1), 219-228 (2010)
10. Goebel, K, Kirk, WA: Topics in Metric Fixed Point Theory. Cambridge University Press, Cambridge (1990)

11. Zeidler, E: Nonlinear Functional Analysis and Its Applications. II/B. Nonlinear Monotone Operators. Springer, Berlin (1990)
12. Lions, JL: Quelques Methodes de Resolution des Problemes aux Limites Nonlineaires. Dunod/Gauthier-Villars, Paris (1969)
13. Gajewski, H, Groger, K, Zacharias, K: Nichtlineare Operatorgleichungen und Operatordifferentialgleichungen. Akademie Verlag, Berlin (1974)
14. Krasnoselskii, MA: Two observations about the method of successive approximations. Usp. Mat. Nauk **10**, 123-127 (1955)
15. He, S, Zhu, W: A modified Mann iteration by boundary point method for finding minimum-norm fixed point of nonexpansive mappings. Abstr. Appl. Anal. **2013**, Article ID 768595 (2013)
16. Benyamini, Y, Lindenstrauss, J: Geometric Nonlinear Functional Analysis, vol. 1. Amer. Math. Soc. Colloq. Publ., vol. 48. Am. Math. Soc., Providence (2000)
17. Diestel, J: Sequences and Series in Banach Spaces. Graduate Texts in Mathematics, vol. 92. Springer, New York (1984)
18. Reich, S: On the asymptotic behavior of nonlinear semigroups and the range of accretive operators. J. Math. Anal. Appl. **79**(1), 123-126 (1981)
19. Takahashi, W: Nonlinear Functional Analysis - Fixed Point Theory and Its Applications. Yokohama Publishers, Yokohama (2000)
20. Alber, YI: Metric and generalized projection operators in Banach spaces. In: Kartsatos, A (ed.) Properties and Applications: Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, pp. 15-50. Dekker, New York (1996)
21. Kato, T: Nonlinear semigroups and evolution equations. J. Math. Soc. Jpn. **19**, 508-520 (1957)
22. Ciarlet, P: The Finite Element Method for Elliptic Problems. North-Holland, New York (1978)
23. Liu, LS: Ishikawa and Mann iterative process with errors for nonlinear strongly accretive mappings in Banach spaces. J. Math. Anal. Appl. **194**, 114-125 (1995)
24. Cioranescu, I: Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems. Mathematics and Its Applications, vol. 62. Kluwer Academic, Dordrecht (1990)
25. Badriev, IB, Shagidullin, RR: Investigating one dimensional equations of the soft shell statistical condition and algorithm of their solution. Izv. Vysš. Učebn. Zaved., Mat. **6**, 8-16 (1992)
26. Lapin, AV: On the research of some problems of nonlinear filtration theory. Ž. Vyčisl. Mat. Mat. Fiz. **19**(3), 689-700 (1979)