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A note on the paper 'Fixed point theorems for cyclic weak contractions in compact metric spaces'

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Abstract

We show that the result on cyclic weak contractions of Harjani *et al.* (*J. Nonlinear Sci. Appl.* 6:279-284, 2013) holds without the assumption of compactness of the underlying space, and also without the assumption of continuity of the given mapping.

MSC: 47H10; 54H25

Keywords: compact metric space; cyclic contraction; weak contraction; well posed problem

1 Introduction

In 2003, Kirk *et al.* introduced in [2] an interesting concept of cyclic contractions in metric spaces, and obtained the corresponding generalizations of Banach's, as well as some other fixed point results. In subsequent articles, several authors obtained various fixed point theorems, adapting some other known results to their cyclic variants (see, *e.g.*, [1, 3–9]).

The first results for cyclic contractions in compact metric spaces were obtained already in [2]. Recently, Harjani *et al.* [1] obtained a cyclic fixed point result in compact spaces for weak contractions (in the sense of [10, 11]), thus modifying a theorem of Karapınar [4] and Karapınar and Sadarangani [5].

In this note, we are going to improve the result from [1] showing that it still holds without the assumption of compactness of the underlying space, and also without the assumption of continuity of the given mapping. An application to well posedness of the corresponding problem is also obtained.

2 Preliminaries

Let X be a nonempty set, and A_1, A_2, \dots, A_p be its nonempty subsets. Recall that $Y = \bigcup_{i=1}^p A_i$ is said to be a cyclic representation of Y with respect to a mapping $T : Y \rightarrow Y$ if

$$T(A_1) \subset A_2, \quad \dots, \quad T(A_{p-1}) \subset A_p, \quad T(A_p) \subset A_1. \quad (2.1)$$

Roughly speaking, there are two kinds of fixed point results for cyclic contractions - those assuming that $Y = X$, and those that do not use this assumption.

In [4, 5], the following class of functions was used:

$$\mathcal{J} = \left\{ \phi : [0, +\infty) \rightarrow [0, +\infty) : \phi \text{ is continuous, nondecreasing, } \phi(0) = 0 \text{ and } \phi(t) > 0 \text{ for } t \in (0, +\infty) \right\}.$$

The main result in [4, 5] is the following.

Theorem 1 ([4], Theorem 6, [5], Theorem 2) *Let (X, d) be a complete metric space, and let $Y = \bigcup_{i=1}^p A_i$ be a cyclic representation of $Y \subseteq X$ with respect to a mapping $T : Y \rightarrow Y$, where the sets A_i are closed. If, for some $\phi \in \mathcal{J}$,*

$$d(Tx, Ty) \leq d(x, y) - \phi(d(x, y)) \tag{2.2}$$

for any $x \in A_i$ and $y \in A_{i+1}$, $i = 1, 2, \dots, p$, where $A_{p+1} = A_1$, then T has a unique fixed point $z \in \bigcap_{i=1}^p A_i$.

We will use the following auxiliary result.

Lemma 1 ([9], Lemma 2.1) *Let (X, d) be a metric space, $T : X \rightarrow X$ be a mapping and let $X = \bigcup_{i=1}^p A_i$ be a cyclic representation of X w.r.t. T . Assume that*

$$\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0,$$

where $x_{n+1} = Tx_n$, $x_1 \in A_1$, $n \in \mathbb{N}$. If $\{x_n\}$ is not a Cauchy sequence, then there exist an $\varepsilon > 0$ and two sequences $\{n(k)\}$ and $\{m(k)\}$ of positive integers such that $n(k) > m(k) > k$ and the following sequences tend to ε^+ as $k \rightarrow \infty$:

$$\begin{aligned} d(x_{m(k)-j(k)}, x_{n(k)}), & \quad d(x_{m(k)-j(k)+1}, x_{n(k)}), \\ d(x_{m(k)-j(k)}, x_{n(k)+1}), & \quad d(x_{m(k)-j(k)+1}, x_{n(k)+1}), \end{aligned} \tag{2.3}$$

where $j(k) \in \{1, 2, \dots, p\}$ is chosen so that $n(k) - (m(k) - j(k)) \equiv 1 \pmod{p}$, for each $k \in \mathbb{N}$.

Remark 1 Using the previous lemma, similar to [9], one can prove that there is an equivalence between Theorem 1 and the corresponding fixed point result for non-cyclic weak contraction. In the case when $Y = X = \bigcup_{i=1}^p A_i$, no assumption on the closedness of sets A_i is necessary, while in the case when $Y \neq X$, it is enough to assume that one of these sets is closed (similarly as in [3], Theorem 2.1). However, in this case the cyclic and non-cyclic versions of this assertion are not equivalent anymore.

In [1], the following class of functions (obviously wider than \mathcal{J}) was used in order to obtain a version of Theorem 1 in compact metric spaces (and assuming that $Y = X$):

$$\mathcal{F} = \left\{ \phi : [0, +\infty) \rightarrow [0, +\infty) : \phi \text{ is nondecreasing, } \phi(0) = 0 \text{ and } \phi(t) > 0 \text{ for } t \in (0, +\infty) \right\}.$$

Theorem 2 ([1], Theorem 2.1) *Let (X, d) be a compact metric space and $T : X \rightarrow X$ a continuous mapping. Suppose that $X = \bigcup_{i=1}^p A_i$ is a cyclic representation of X with respect to T . If, for some $\phi \in \mathcal{F}$, relation (2.2) holds for any $x \in A_i$ and $y \in A_{i+1}$, $i = 1, 2, \dots, p$ where $A_{p+1} = A_1$, then T has a unique fixed point $z \in \bigcap_{i=1}^p A_i$.*

Remark 2 Under the assumptions of Theorem 2, it can easily be seen that $X = \bigcup_{i=1}^p \overline{A_i}$ is also a cyclic representation of X w.r.t. T . Moreover, all sets $\overline{A_i}$, $i \in \{1, \dots, p\}$ are compact in X and (2.2) implies that

$$d(Tx, Ty) < d(x, y)$$

holds for all $x \in A_i, y \in A_{i+1}$ with $x \neq y$. But then the result follows directly from [2], Theorem 2.1. Hence, Theorem 2 is not a new result.

Note also that [1], Remark 2.2, is not correct. Namely, Theorem 2 is not a generalization of Theorem 1, since it uses the assumption $Y = X$, which is not present in Theorem 1.

The following result is a version of [4], Theorem 7, when (X, d) is a compact metric space.

Theorem 3 ([1], Theorem 2.3) *Under the assumptions of Theorem 2, the fixed point problem for T is well posed, that is, if there exists a sequence $\{y_n\}$ in X with $d(y_n, Ty_n) \rightarrow 0$ as $n \rightarrow \infty$, then $y_n \rightarrow z$ as $n \rightarrow \infty$, where z is the unique fixed point of T .*

3 Results

Our main result is the following improvement of Theorem 2, that is, the main result from [1]. Note that compactness of the space is not assumed, nor is the continuity of T . Also, the subsets A_i of X need not be closed. Moreover, the proof is much shorter than the proof of the relevant theorem in [1].

Theorem 4 *Let (X, d) be a complete metric space and $T : X \rightarrow X$ be a mapping. Suppose that $X = \bigcup_{i=1}^p A_i$ is a cyclic representation of X with respect to T . If, for some $\phi \in \mathcal{F}$, relation (2.2) holds for any $x \in A_i$ and $y \in A_{i+1}$, $i = 1, 2, \dots, p$, where $A_{p+1} = A_1$, then T has a unique fixed point $z \in X$ and $z \in \bigcap_{i=1}^p A_i$. Moreover, each Picard sequence $x_n = T^n x_0, x_0 \in X$ converges to z .*

Proof Take arbitrary $x_0 \in X$. It belongs to some of the subsets $A_i, i \in \{1, \dots, p\}$, say $x_0 \in A_1$. Then the Picard sequence $x_n = Tx_{n-1}, n \in \mathbb{N}$, is divided into the following p subsequences, each belonging to some $A_i, i \in \{1, \dots, p\}$:

$$\{x_{np}\} \subset A_1, \quad \{x_{np+1}\} \subset A_2, \quad \dots, \quad \{x_{np+p-1}\} \subset A_p.$$

Suppose that $x_{n+1} \neq x_n$, for each n (otherwise there is nothing to prove). It easily follows that $x_n \neq x_m$ for $n \neq m$. Moreover, it can be proved in a standard way (e.g., as in the proof of [1], Theorem 2.1) that

$$d(x_n, x_{n+1}) \downarrow 0 \quad \text{as } n \rightarrow \infty.$$

We will prove that $\{x_n\}$ is a Cauchy sequence.

Suppose that this is not true. Then, using Lemma 1, we conclude that there exist an $\varepsilon > 0$ and two sequences $\{n(k)\}$ and $\{m(k)\}$ of positive integers, with $n(k) > m(k) > k$, such that the sequences (2.3) tend to ε^+ when $k \rightarrow \infty$. Applying (2.2) with $x = x_{m(k)-j(k)}$, $y = x_{n(k)}$, we get

$$d(x_{m(k)-j(k)+1}, x_{n(k)+1}) \leq d(x_{m(k)-j(k)}, x_{n(k)}) - \phi(d(x_{m(k)-j(k)}, x_{n(k)})),$$

i.e.,

$$0 < \phi(d(x_{m(k)-j(k)}, x_{n(k)})) \leq d(x_{m(k)-j(k)}, x_{n(k)}) - d(x_{m(k)-j(k)+1}, x_{n(k)+1}),$$

since $x_{m(k)-j(k)} \neq x_{n(k)}$ (which follows from $n(k) > m(k)$). Passing to the limit as $k \rightarrow \infty$, we get

$$\phi(d(x_{m(k)-j(k)}, x_{n(k)})) \rightarrow 0, \quad \text{as } k \rightarrow \infty.$$

Since $\varepsilon \leq d(x_{m(k)-j(k)}, x_{n(k)})$ for k sufficiently large, we obtain $0 < \phi(\varepsilon) \leq \phi(d(x_{m(k)-j(k)}, x_{n(k)}))$, a contradiction. Hence, $\{x_n\}$ is a Cauchy sequence.

Since the space X is complete, there is $z \in X = \bigcup_{i=1}^p A_i$ such that $x_n \rightarrow z$ as $n \rightarrow \infty$. This means that $z \in A_m$ for some $m \in \{1, \dots, p\}$ and, hence, $Tz \in A_{m+1}$. Consider the subsequence $\{x_{np+m}\} \subset A_{m+1}$ of $\{x_n\}$. Applying (2.2), we get

$$\begin{aligned} d(x_{np+m+1}, Tz) &= d(Tx_{np+m}, Tz) \\ &\leq d(x_{np+m}, z) - \phi(d(x_{np+m}, z)) \\ &\leq d(x_{np+m}, z) \rightarrow 0, \quad \text{as } n \rightarrow \infty. \end{aligned}$$

It follows that $x_{np+m+1} \rightarrow Tz$ as $n \rightarrow \infty$ and, by the uniqueness of the limit, $Tz = z$. By the cyclic property (2.1) of T , it follows that $z \in \bigcap_{i=1}^p A_i$.

The uniqueness of the fixed point follows from the contractive condition (2.2), since, again by (2.1), each fixed point of T belongs to $\bigcap_{i=1}^p A_i$. □

Taking $\phi(t) = (1 - k)t$, $t \in [0, +\infty)$, $k \in [0, 1)$ in Theorem 4, we obtain the following version of a Banach-type cyclic contraction result.

Corollary 1 *Let (X, d) be a complete metric space and $T : X \rightarrow X$ be a mapping. Suppose that $X = \bigcup_{i=1}^p A_i$ is a cyclic representation of X with respect to T . If, for some $k \in [0, 1)$, the inequality $d(Tx, Ty) \leq kd(x, y)$ holds for any $x \in A_i$ and $y \in A_{i+1}$, $i = 1, 2, \dots, p$, where $A_{p+1} = A_1$, then T has a unique fixed point $z \in X$ and $z \in \bigcap_{i=1}^p A_i$. Moreover, each Picard sequence $x_n = T^n x_0$, $x_0 \in X$, converges to z .*

Example 1 Let $X = \mathbb{R}$ be endowed with the standard metric. Take $A_1 = (-\infty, 2)$, $A_2 = (-2, +\infty)$, and define $T : X \rightarrow X$ and $\phi \in \mathcal{F}$ by

$$Tx = \begin{cases} -\frac{1}{2}x, & |x| < 2, \\ -\frac{1}{3}x, & |x| \geq 2, \end{cases} \quad \phi(t) = \begin{cases} \frac{1}{2}t, & t \in [0, 2), \\ \frac{1}{3}t, & t \in [2, +\infty). \end{cases}$$

Then all the assumptions of Theorem 4 are fulfilled; let us check the condition (2.2). Suppose that $x \in A_1$ and $y \in A_2$ (the other case can be treated symmetrically). Consider the following possibilities:

(1) $x \in (-\infty, -2]$, (2) $x \in (-2, 0)$, (3) $x \in [0, 2)$, as well as: (a) $y \in (-2, 0)$, (b) $y \in [0, 2)$, (c) $y \in [2, +\infty)$.

Case 1(a). $Tx = -\frac{1}{3}x < -\frac{1}{2}x$, $Ty = -\frac{1}{2}y$, and $d(Tx, Ty) < \frac{1}{2}(-x - (-y)) = \frac{1}{2}(y - x) = \frac{1}{2}d(x, y)$.

Case 1(b). $Tx = -\frac{1}{3}x < -\frac{1}{2}x$, $Ty = -\frac{1}{2}y$, $d(Tx, Ty) < \frac{1}{2}(-x - (-y)) = \frac{1}{2}(y - x) = \frac{1}{2}d(x, y)$.

Case 1(c). $Tx = -\frac{1}{3}x$, $Ty = -\frac{1}{3}y$, $d(Tx, Ty) = \frac{1}{3}(-x - (-y)) = \frac{1}{3}(y - x) < \frac{1}{2}d(x, y)$.

Case 2(a). $Tx = -\frac{1}{2}x$, $Ty = -\frac{1}{2}y$, $d(Tx, Ty) = \frac{1}{2}|x - y| = \frac{1}{2}d(x, y)$.

Case 2(b). $Tx = -\frac{1}{2}x$, $Ty = -\frac{1}{2}y$, $d(Tx, Ty) = \frac{1}{2}(x - y) = \frac{1}{2}d(x, y)$.

The cases 2(c), 3(a), 3(b), and 3(c) are symmetric to the cases 1(b), 2(b), 2(a), and 1(a), respectively. On the other hand, $\phi(t) \geq \frac{1}{2}t$ for all $t \geq 0$.

Hence, in all possible cases,

$$d(Tx, Ty) \leq \frac{1}{2}d(x, y) \leq d(x, y) - \phi(d(x, y)),$$

and the contractive condition (2.2) of Theorem 4 is fulfilled. We conclude that T has a unique fixed point (which is $z = 0$). Since:

1. X is not compact;
2. T is not continuous;
3. $A_i, i \in \{1, 2\}$, are not closed,

Theorems 1 and 2 cannot be used to reach this conclusion.

The following modification of Theorem 4 can be proved nearly in the same way.

Theorem 5 *Let X be a nonempty set, and A_1, A_2, \dots, A_p be its nonempty subsets, with at least one of them being closed. Let $Y = \bigcup_{i=1}^p A_i$ be a cyclic representation of Y with respect to a mapping $T : Y \rightarrow Y$. If, for some $\phi \in \mathcal{F}$, relation (2.2) holds for any $x \in A_i$ and $y \in A_{i+1}, i = 1, 2, \dots, p$, where $A_{p+1} = A_1$, then T has a unique fixed point $z \in Y$ and $z \in \bigcap_{i=1}^p A_i$. Moreover, each Picard sequence $x_n = T^n x_0, x_0 \in Y$ converges to z .*

Proof Suppose, without loss of generality, that the subset A_1 is closed (and hence complete) in X . The only difference in the proof, compared with Theorem 4, is that, after proving that $\{x_n\}$ is a Cauchy sequence (in Y), we conclude that there is $z \in A_1 (\subseteq Y)$ such that $x_n \rightarrow z$, as $n \rightarrow \infty$. See also the proof of [3], Theorem 2.1. □

As another consequence of Theorem 4, we obtain the following improvement of Theorem 3.

Corollary 2 *Under the assumptions of Theorem 4, the fixed point problem for T is well posed, that is, if $\{y_n\}$ is a sequence in X satisfying $d(y_n, Ty_n) \rightarrow 0$ as $n \rightarrow \infty$, then $y_n \rightarrow z$ as $n \rightarrow \infty$, where z is the unique fixed point of T (whose existence is guaranteed by Theorem 4).*

Proof Since, according to Theorem 4, it follows that the unique fixed point z of T belongs to $\bigcap_{i=1}^p A_i$, the rest of the proof is the same as in [1], p. 282. Namely, this proof uses neither compactness of X , nor continuity of T . □

Remark 3 As a kind of conclusion, we state once more that there are two kinds of cyclic fixed point results - those treating mappings $T : Y \rightarrow Y$, where $Y = \bigcup_{i=1}^p A_i \subset X$ (possibly with $Y \neq X$) and those where $Y = X$. In results of the first kind, it is enough to assume that one of the sets A_i is closed, while in results of the second kind, no closedness assumption is needed. In both cases, if the considered mapping T is continuous, all the results reduce to the case when all A_i 's are closed.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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