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Approximation of a zero point of monotone operators with nonsummable errors

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Abstract
In this paper, we study an iterative scheme for two different types of resolvents of a monotone operator defined on a Banach space. These resolvents are generalizations of resolvents of a monotone operator in a Hilbert space. We obtain iterative approximations of a zero point of a monotone operator generated by the shrinking projection method with errors in a Banach space. Using our result, we discuss some applications.

MSC: 47H05; 47H09; 47J25

Keywords: resolvent; monotone operator; metric projection

1 Introduction

Let H be a real Hilbert space and let $A \subset H \times H$ be a maximal monotone operator. Then the zero point problem is to find $u \in H$ such that

$$0 \in Au. \tag{1.1}$$

Such a $u \in H$ is called a zero point (or a zero) of A. The set of zero points of A is denoted by $A^{-1}0$. This problem is connected with many problems in Nonlinear Analysis and Optimization, that is, convex minimization problems, variational inequality problems, equilibrium problems and so on. A well-known method for solving (1.1) is the proximal point algorithm: $x_1 \in H$ and

$$x_{n+1} = J_{r_n} x_n, \quad n = 1, 2, \dots, \tag{1.2}$$

where $\{r_n\} \subset]0, \infty[$ and $J_{r_n} = (I + r_n A)^{-1}$. This algorithm was first introduced by Martinet [1]. In 1976, Rockafellar [2] proved that if $\liminf_n r_n > 0$ and $A^{-1}0 \neq \emptyset$, then the sequence $\{x_n\}$ defined by (1.2) converges weakly to a solution of the zero point problem. Later, many researchers have studied this problem; see [3–9] and others.

On the other hand, Kimura [10] introduced the following iterative scheme for finding a fixed point of nonexpansive mappings by the shrinking projection method with error in a Hilbert space:

Theorem 1.1 (Kimura [10]) *Let C be a bounded closed convex subset of a Hilbert space H with $D = \operatorname{diam} C = \sup_{x,y\in C} \|x - y\| < \infty$, and let $T : C \to H$ be a nonexpansive mapping having a fixed point. Let $\{\epsilon_n\}$ be a nonnegative real sequence such that $\epsilon_0 = \limsup_n \epsilon_n < \infty$. For a given point $u \in H$, generate an iterative sequence $\{x_n\}$ as follows: $x_1 \in C$ such that $\|x_1 - u\| < \epsilon_1$, $C_1 = C$,*

$$C_{n+1} = \{z \in C : \|z - Tx_n\| \le \|z - x_n\|\} \cap C_n,$$
$$x_{n+1} \in C_{n+1} \quad \textit{such that} \quad \|u - x_{n+1}\|^2 \le d(u, C_{n+1})^2 + \epsilon_{n+1}^2$$

for all $n \in \mathbb{N}$. Then

$$\limsup_{n\to\infty} \|x_n - Tx_n\| \le 2\epsilon_0.$$

Further, if $\epsilon_0 = 0$, then $\{x_n\}$ converges strongly to $P_{F(T)}u \in F(T)$.

We remark that the original result of the theorem above deals with a family of nonexpansive mappings, and the shrinking projection method was first introduced by Takahashi *et al.* [11]. This result was extended to more general Banach spaces by Kimura [12] (see also Ibaraki and Kimura [13]).

In this paper, we study the shrinking projection method with error introduced by Kimura [10] (see also [12, 14]). We obtain an iterative approximation of a zero point of a monotone operator generated by the shrinking projection method with errors in a Banach space. Using our result, we discuss some applications.

2 Preliminaries

Let E be a real Banach space with its dual E^*. The normalized duality mapping J from E into E^* is defined by

$$Jx = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}$$

for each $x \in E$. We also know the following properties: see [15, 16] for more details.

1. (1) $Jx \neq \emptyset$ for each $x \in E$;
2. (2) if E is reflexive, then J is surjective;
3. (3) if E is smooth, then the duality mapping J is single valued.
4. (4) if E is strictly convex, then J is one-to-one and satisfies that $\langle x - y, x^* - y^*\rangle > 0$ for each $x, y \in E$ with $x \neq y$, $x^* \in Jx$ and $y^* \in Jy$;
5. (5) if E is reflexive, smooth, and strictly convex, then the duality mapping $J_* : E^* \to E$ is the inverse of J, that is, $J_* = J^{-1}$;
6. (6) if E uniformly smooth, then the duality mapping J is uniformly norm to norm continuous on each bounded set of E.

Let E be a reflexive and strictly convex Banach space and let C be a nonempty closed convex subset of E. It is well known that for each $x \in E$ there exists a unique point $z \in C$ such that $\|x - z\| = \min\{\|x - y\| : y \in C\}$. Such a point z is denoted by $P_C x$ and P_C is called the metric projection of E onto C. The following result is well known; see, for instance, [16].

Lemma 2.1 *Let E be a reflexive, smooth, and strictly convex Banach space, let C be a nonempty closed convex subset of E, let P_C be the metric projection of E onto C, let $x \in E$ and let $x_0 \in C$. Then $x_0 = P_C x$ if and only if*

$$\langle x_0 - y, J(x - x_0)\rangle \geq 0$$

for all $y \in C$.

Let C be a nonempty closed convex subset of a smooth Banach space E. A mapping $T : C \to E$ is said to be of type (P) [17] if

$$\langle Tx - Ty, J(x - Tx) - J(y - Ty)\rangle \geq 0$$

for each $x, y \in C$. A mapping $T : C \to E$ is said to be of type (Q) [17, 18] if

$$\langle Tx - Ty, (Jx - JTx) - (Jy - JTy)\rangle \geq 0$$

for each $x, y \in C$. We denote by $F(T)$ the set of fixed points of T. A point p in C is said to be an asymptotic fixed point of T if C contains a sequence $\{x_n\}$ such that $x_n \rightharpoonup p$ and $x_n - Tx_n \to 0$. The set of all asymptotic fixed points of T is denoted by $\hat{F}(T)$. It is clear that if $T : C \to E$ is of type (P) and $F(T)$ is nonempty, then

$$\langle Tx - p, J(x - Tx)\rangle \geq 0 \tag{2.1}$$

for each $x \in C$ and $p \in F(T)$. Let E be a reflexive, smooth, and strictly convex Banach space and let C be a nonempty closed convex subset of E. It is well known that the metric projection P_C of E onto C is a mapping of type (P). We also know that if $T : C \to E$ is of type (Q) and $F(T)$ is nonempty, then

$$\langle Tx - p, Jx - JTx\rangle \geq 0 \tag{2.2}$$

for each $x \in C$ and $p \in F(T)$.

The following results describe the relation between the set of fixed points and that of asymptotic fixed points for each type of mapping.

Lemma 2.2 (Aoyama-Kohsaka-Takahashi [19]) *Let E be a smooth Banach space, let C be a nonempty closed convex subset of E and let $T : C \to E$ be a mapping of type (P). If $F(T)$ is nonempty, then $F(T)$ is closed and convex and $F(T) = \hat{F}(T)$.*

Lemma 2.3 (Kohsaka-Takahashi [18]) *Let E be a strictly convex Banach space whose norm is uniformly Gâteaux differentiable, let C be a nonempty closed convex subset of E and let $T : C \to E$ be a mapping of type (Q). If $F(T)$ is nonempty, then $F(T)$ is closed and convex and $F(T) = \hat{F}(T)$.*

In 1984, Tsukada [20] proved the following theorem for the metric projections in a Banach space. For the exact definition of Mosco limit $\text{M-lim}_n C_n$, see [21].

Theorem 2.4 (Tsukada [20]) *Let E be a reflexive and strictly convex Banach space and let $\{C_n\}$ be a sequence of nonempty closed convex subsets of E. If $C_0 = \text{M-lim}_n C_n$ exists and is nonempty, then for each $x \in E$, $\{P_{C_n}x\}$ converges weakly to $P_{C_0}x$, where P_{C_n} is the metric projection of E onto C_n. Moreover, if E has the Kadec-Klee property, the convergence is in the strong topology.*

One of the simplest example of the sequence $\{C_n\}$ satisfying the condition in this theorem above is a decreasing sequence with respect to inclusion; $C_{n+1} \subset C_n$ for each $n \in \mathbb{N}$. In this case, $\text{M-lim}\, C_n = \bigcap_{n=1}^{\infty} C_n$ (see [7, 12, 21, 22] for more details).

Let E be a smooth Banach space and consider the following function $V : E \times E \to \mathbb{R}$ defined by

$$V(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \tag{2.3}$$

for each $x, y \in E$. We know the following properties:

(1) $(\|x\| - \|y\|)^2 \le V(x,y) \le (\|x\| + \|y\|)^2$ for each $x, y \in E$;
(2) $V(x,y) + V(y,x) = 2\langle x - y, Jx - Jy\rangle$ for each $x, y \in E$;
(3) $V(x,y) = V(x,z) + V(z,y) + 2\langle x - z, Jz - Jy\rangle$ for each $x, y, z \in E$;
(4) if E is additionally assumed to be strictly convex, then $V(x,y) = 0$ if and only if $x = y$.

Lemma 2.5 (Kamimura-Takahashi [23]) *Let E be a smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be sequences in E such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim_n V(x_n, y_n) = 0$, then $\lim_n \|x_n - y_n\| = 0$.*

The following results show the existence of mappings $\underline{g}_r$ and $\overline{g}_r$, related to the convex structures of a Banach space E. These mappings play important roles in our result.

Theorem 2.6 (Xu [24]) *Let E be a Banach space, $r \in]0, \infty[$ and $B_r = \{x \in E : \|x\| \le r\}$. Then*

(i) *if E is uniformly convex, then there exists a continuous, strictly increasing, and convex function $\underline{g}_r : [0, 2r] \to [0, \infty[$ with $\underline{g}_r(0) = 0$ such that*

$$\|\alpha x + (1-\alpha)y\|^2 \le \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\underline{g}_r(\|x - y\|)$$

for all $x, y \in B_r$ and $\alpha \in [0,1]$;

(ii) *if E is uniformly smooth, then there exists a continuous, strictly increasing, and convex function $\overline{g}_r : [0, 2r] \to [0, \infty[$ with $\overline{g}_r(0) = 0$ such that*

$$\|\alpha x + (1-\alpha)y\|^2 \ge \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\overline{g}_r(\|x - y\|)$$

for all $x, y \in B_r$ and $\alpha \in [0,1]$.

Theorem 2.7 (Kimura [12]) *Let E be a uniformly smooth and uniformly convex Banach space and let $r > 0$. Then the function $\underline{g}_r$ and $\overline{g}_r$ in Theorem 2.6 satisfies*

$$\underline{g}_r(\|x - y\|) \le V(x,y) \le \overline{g}_r(\|x - y\|)$$

for all $x, y \in B_r$.

3 Approximation theorem for the resolvents of type (P)

In this section, we discuss an iterative scheme of resolvents of a monotone operator defined on a Banach space. Let E be a reflexive, smooth, and strictly convex Banach space. An operator $A \subset E \times E^*$ with domain $D(A) = \{x \in E : Ax \neq \emptyset\}$ and range $R(A) = \bigcup\{Ax : x \in D(A)\}$ is said to be monotone if $\langle x - y, x^* - y^* \rangle \geq 0$ for any $(x, x^*), (y, y^*) \in A$. A monotone operator A is said to be maximal if $A = B$ whenever $B \subset E \times E^*$ is a monotone operator such that $A \subset B$. We denote by $A^{-1}0$ the set $\{z \in D(A) : 0 \in Az\}$.

Let C be a nonempty closed convex subset of E, let $r > 0$ and let $A \subset E \times E^*$ be a monotone operator satisfying

$$D(A) \subset C \subset R\bigl(I + rJ^{-1}A\bigr) \tag{3.1}$$

for $r > 0$. It is well known that if A is maximal monotone operator, then $R(I + rJ^{-1}A) = E$; see [25–27]. Hence, if A is maximal monotone, then (3.1) holds for $C = \overline{D(A)}$. We also know that $\overline{D(A)}$ is convex; see [28]. If A satisfies (3.1) for $r > 0$, we can define the resolvent (of type (P)) $P_r : C \to D(A)$ of A by

$$P_r x = \bigl\{z \in E : 0 \in J(z - x) + rAz\bigr\} \tag{3.2}$$

for all $x \in C$. In other words, $P_r x = (I + rJ^{-1}A)^{-1}x$ for all $x \in C$. The Yosida approximation $A_r : C \to E^*$ is also defined $A_r x = J(x - P_r x)/r$ for all $x \in C$. We know the following; see, for instance, [15, 17, 19]:

(1) P_r is mapping of type (P) from C into $D(A)$;
(2) $(P_r x, A_r x) \in A$ for all $x \in C$;
(3) $\|A_r x\| \leq |Ax| := \inf\{\|x^*\| : x^* \in Ax\}$ for all $x \in D(A)$;
(4) $F(P_r) = A^{-1}0$.

We obtain an approximation theorem for a zero point of a monotone operator in a smooth and uniformly convex Banach space by using the resolvent of type (P).

Theorem 3.1 *Let E be a smooth and uniformly convex Banach space and let $A \subset E \times E^*$ be a monotone operator with $A^{-1}0 \neq \emptyset$. Let $\{r_n\}$ be a positive real sequence such that $\liminf_n r_n > 0$, let C be a nonempty bounded closed convex subset of E satisfying*

$$D(A) \subset C \subset R\bigl(I + r_n J^{-1}A\bigr)$$

for all $n \in \mathbb{N}$ and let $r \in\,]0, \infty[$ such that $C \subset B_r$. Let $\{\delta_n\}$ be a nonnegative real sequence and let $\delta_0 = \limsup_n \delta_n$. For a given point $u \in E$, generate a sequence $\{x_n\}$ by $x_1 = x \in C$, $C_1 = C$, and

$$\begin{aligned}
&y_n = P_{r_n} x_n,\\
&C_{n+1} = \bigl\{z \in C : \bigl\langle y_n - z, J(x_n - y_n)\bigr\rangle \geq 0\bigr\} \cap C_n,\\
&x_{n+1} \in \bigl\{z \in C : \|u - z\|^2 \leq d(u, C_{n+1})^2 + \delta_{n+1}\bigr\} \cap C_{n+1},
\end{aligned}$$

for all $n \in \mathbb{N}$. Then

$$\limsup_{n\to\infty} \|x_n - y_n\| \leq \underline{g}_r^{-1}(\delta_0).$$

Moreover, if $\delta_0 = 0$, then $\{x_n\}$ converges strongly to $P_{A^{-1}0}u$.

Proof Since C_n includes $A^{-1}0 \neq \emptyset$ for all $n \in \mathbb{N}$, $\{C_n\}$ is a sequence of nonempty closed convex subsets and, by definition, it is decreasing with respect to inclusion. Let $p_n = P_{C_n}u$ for all $n \in \mathbb{N}$. Then, by Theorem 2.4, we see that $\{p_n\}$ converges strongly to $p_0 = P_{C_0}u$, where $C_0 = \bigcap_{n=1}^{\infty} C_n$. Since $x_n \in C_n$ and $d(u, C_n) = \|u - p_n\|$, we see that

$$\|u - x_n\|^2 \leq \|u - p_n\|^2 + \delta_n$$

for every $n \in \mathbb{N} \setminus \{1\}$. From Theorem 2.6(i), we see that for $\alpha \in \,]0,1[$,

$$\begin{aligned}\|p_n - u\|^2 &\leq \big\|\alpha p_n + (1-\alpha)x_n - u\big\|^2 \\ &\leq \alpha\|p_n - u\|^2 + (1-\alpha)\|x_n - u\|^2 - \alpha(1-\alpha)\underline{g}_r\big(\|p_n - x_n\|\big)\end{aligned}$$

and thus

$$\alpha\underline{g}_r\big(\|p_n - x_n\|\big) \leq \|x_n - u\|^2 - \|p_n - u\|^2 \leq \delta_n.$$

As $\alpha \to 1$, we see that $\underline{g}_r(\|p_n - x_n\|) \leq \delta_n$ and thus $\|p_n - x_n\| \leq \underline{g}_r^{-1}(\delta_n)$. Using the definition of p_n, we see that $p_{n+1} \in C_{n+1}$ and thus

$$\big\langle y_n - p_{n+1}, J(x_n - y_n)\big\rangle \geq 0,$$

or equivalently,

$$\big\langle x_n - p_{n+1}, J(x_n - y_n)\big\rangle \geq \|x_n - y_n\|^2.$$

Hence we obtain

$$\|x_n - y_n\| \leq \|x_n - p_{n+1}\| \leq \|x_n - p_n\| + \|p_n - p_{n+1}\| \leq \underline{g}_r^{-1}(\delta_n) + \|p_n - p_{n+1}\|$$

for every $n \in \mathbb{N} \setminus \{1\}$. Since $\lim_n p_n = p_0$ and $\limsup_n \delta_n = \delta_0$, we see that

$$\limsup_{n\to\infty} \|x_n - y_n\| \leq \underline{g}_r^{-1}(\delta_0).$$

For the latter part of the theorem, suppose that $\delta_0 = 0$. Then we see that

$$\limsup_{n\to\infty} \|x_n - y_n\| \leq \underline{g}_r^{-1}(0) = 0$$

and

$$\limsup_{n\to\infty} \underline{g}_r\big(\|x_n - p_n\|\big) \leq \limsup_{n\to\infty} \delta_n = 0.$$

Therefore, we obtain

$$\lim_{n\to\infty} \|x_n - y_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|x_n - p_n\| = 0.$$

Hence, we also obtain

$$\lim_{n\to\infty} x_n = p_0 \quad \text{and} \quad \lim_{n\to\infty} y_n = p_0. \tag{3.3}$$

So, from

$$\|y_n - P_{r_1}y_n\| = r_1\|A_{r_1}y_n\| \le r_1|Ay_n| \le r_1\left\|\frac{J(x_n - y_n)}{r_n}\right\| = r_1\left\|\frac{x_n - y_n}{r_n}\right\|.$$

and $\liminf_n r_n > 0$, we see that $\lim_n \|y_n - P_{r_1}y_n\| = 0$. Then, by Lemma 2.2 and (3.3), we obtain $x_n \to p_0 \in \hat{F}(P_{r_1}) = F(P_{r_1}) = A^{-1}0$. Since $A^{-1}0 \subset C_0$, we get $p_0 = P_{C_0}u = P_{A^{-1}0}u$, which completes the proof. □

4 Approximation theorem for the resolvents of type (Q)

We next consider an iterative scheme of resolvents of a monotone operator which is different type of Section 3, in a Banach space. Let C be a nonempty closed convex subset of a reflexive, smooth, and strictly convex Banach space E, let $r > 0$ and let $A \subset E \times E^*$ be a monotone operator satisfying

$$D(A) \subset C \subset J^{-1}R(J + rA) \tag{4.1}$$

for $r > 0$. It is well known that if A is maximal monotone operator, then $J^{-1}R(J + rA) = E$; see [25–27]. Hence, if A is maximal monotone, then (4.1) holds for $C = \overline{D(A)}$. We also know that $\overline{D(A)}$ is convex; see [28]. If A satisfies (4.1) for $r > 0$, then we can define the resolvent (of type (Q)) $Q_r : C \to D(A)$ of A by

$$Q_rx = \{z \in E : Jx \in Jz + rAz\} \tag{4.2}$$

for all $x \in C$. In other words, $Q_rx = (J + rA)^{-1}Jx$ for all $x \in C$. We know the following; see, for instance, [17, 18]:

(1) Q_r is mapping of type (Q) from C into $D(A)$;
(2) $(Jx - JQ_rx)/r \in AQ_rx$ for all $x \in C$;
(3) $F(Q_r) = A^{-1}0$.

Before our result, we need the following lemma.

Lemma 4.1 *Let E be a reflexive, smooth, and strictly convex Banach space, and let $A \subset E \times E^*$ be a monotone operator. Let $r > 0$ and C be a closed convex subset of E satisfying* (4.1) *for $r > 0$. Then the following holds*:

$$V(x, Q_rx) + V(Q_rx, x) \le 2r\langle x - Q_rx, x^*\rangle$$

for all $(x, x^*) \in A$.

Proof Let $(x, x^*) \in A$. Since $(Jx - JQ_rx)/r \in AQ_rx$, we see that

$$0 \le \left\langle x - Q_rx, x^* - \frac{Jx - JQ_rx}{r}\right\rangle,$$

$$\left\langle x - Q_r x, \frac{Jx - JQ_r x}{r} \right\rangle \leq \langle x - Q_r x, x^* \rangle,$$
$$\langle x - Q_r x, Jx - JQ_r x \rangle \leq r\langle x - Q_r x, x^* \rangle.$$

From the property of V, we see that

$$V(x, Q_r x) + V(Q_r x, x) = 2\langle x - Q_r x, Jx - JQ_r x\rangle \leq 2r\langle x - Q_r x, x^* \rangle$$

for all $(x, x^*) \in A$. □

We obtain an approximation theorem for a zero point of a monotone operator in a smooth and uniformly convex Banach space by using the resolvent of type (Q).

Theorem 4.2 *Let E be a uniformly smooth and uniformly convex Banach space and let $A \subset E \times E^*$ be a monotone operator with $A^{-1}0 \neq \emptyset$. Let $\{r_n\}$ be a positive real numbers such that* $\liminf_n r_n > 0$, *let C be a nonempty bounded closed convex subset of E satisfying*

$$D(A) \subset C \subset J^{-1}R(J + r_n A)$$

for all $n \in \mathbb{N}$ and let $r \in]0, \infty[$ such that $C \subset B_r$. Let $\{\delta_n\}$ be a nonnegative real sequence and let $\delta_0 = \limsup_n \delta_n$. For a given point $u \in E$, generate a sequence $\{x_n\}$ by $x_1 = x \in C$, $C_1 = C$, and

$$y_n = Q_{r_n} x_n,$$
$$C_{n+1} = \{z \in C : \langle y_n - z, Jx_n - Jy_n\rangle \geq 0\} \cap C_n,$$
$$x_{n+1} \in \{z \in C : \|u - z\|^2 \leq d(u, C_{n+1})^2 + \delta_{n+1}\} \cap C_{n+1},$$

for all $n \in \mathbb{N}$. Then

$$\limsup_{n\to\infty} \|x_n - y_n\| \leq \underline{g}_r^{-1}\left(\overline{g}_r\left(\underline{g}_r^{-1}(\delta_0)\right)\right).$$

Moreover, if $\delta_0 = 0$, then $\{x_n\}$ converges strongly to $P_{A^{-1}0} u$.

Proof Since C_n includes $A^{-1}0 \neq \emptyset$ for all $n \in \mathbb{N}$, $\{C_n\}$ is a sequence of nonempty closed convex subsets and, by definition, it is decreasing with respect to inclusion. Let $p_n = P_{C_n} u$ for all $n \in \mathbb{N}$. Then, by Theorem 2.4, we see that $\{p_n\}$ converges strongly to $p_0 = P_{C_0} u$, where $C_0 = \bigcap_{n=1}^{\infty} C_n$. Since $x_n \in C_n$ and $d(u, C_n) = \|u - p_n\|$, we see that

$$\|u - x_n\|^2 \leq \|u - p_n\|^2 + \delta_n$$

for every $n \in \mathbb{N} \setminus \{1\}$. From Theorem 2.6(i), we see that for $\alpha \in]0, 1[$,

$$\begin{aligned}\|p_n - u\|^2 &\leq \|\alpha p_n + (1 - \alpha) x_n - u\|^2 \\ &\leq \alpha\|p_n - u\|^2 + (1 - \alpha)\|x_n - u\|^2 - \alpha(1 - \alpha)\underline{g}_r(\|p_n - x_n\|)\end{aligned}$$

and thus

$$\alpha \underline{g}_r\bigl(\|p_n - x_n\|\bigr) \le \|x_n - u\|^2 - \|p_n - u\|^2 \le \delta_n.$$

As $\alpha \to 1$, we see that $\underline{g}_r(\|p_n - x_n\|) \le \delta_n$ and thus $\|p_n - x_n\| \le \underline{g}_r^{-1}(\delta_n)$. Using the definition of p_n, we see that $p_{n+1} \in C_{n+1}$ and thus

$$\langle y_n - p_{n+1}, Jx_n - Jy_n\rangle \ge 0.$$

From the property of the function V, we see that

$$\begin{aligned} 0 &\le 2\langle y_n - p_{n+1}, Jx_n - Jy_n\rangle \\ &= 2\langle p_{n+1} - y_n, Jy_n - Jx_n\rangle \\ &= V(p_{n+1}, x_n) - V(p_{n+1}, y_n) - V(y_n, x_n) \\ &\le V(p_{n+1}, x_n) - V(y_n, x_n). \end{aligned}$$

By Theorem 2.7, we obtain

$$\begin{aligned} V(y_n, x_n) &\le V(p_{n+1}, x_n) \\ &= V(p_{n+1}, p_n) + V(p_n, x_n) + 2\langle p_{n+1} - p_n, Jp_n - Jx_n\rangle \\ &\le V(p_{n+1}, p_n) + \overline{g}_r\bigl(\|p_n - x_n\|\bigr) + 2\langle p_{n+1} - p_n, Jp_n - Jx_n\rangle \\ &\le V(p_{n+1}, p_n) + \overline{g}_r\bigl(\underline{g}_r^{-1}(\delta_n)\bigr) + 2\langle p_{n+1} - p_n, Jp_n - Jx_n\rangle. \end{aligned}$$

Since $\limsup_n \delta_n = \delta_0$ and $p_n \to p_0$, we see that

$$\limsup_{n\to\infty} V(y_n, x_n) \le \overline{g}_r\bigl(\underline{g}_r^{-1}(\delta_0)\bigr).$$

Therefore, by Theorem 2.7, we see that

$$\limsup_{n\to\infty} \|x_n - y_n\| \le \limsup_{n\to\infty} \underline{g}_r^{-1}\bigl(V(y_n, x_n)\bigr) \le \underline{g}_r^{-1}\bigl(\overline{g}_r\bigl(\underline{g}_r^{-1}(\delta_0)\bigr)\bigr).$$

For the latter part of the theorem, suppose that $\delta_0 = 0$. Then we see that

$$\limsup_{n\to\infty} \|x_n - y_n\| \le \underline{g}_r^{-1}\bigl(\overline{g}_r\bigl(\underline{g}_r^{-1}(0)\bigr)\bigr) = 0$$

and

$$\limsup_{n\to\infty} \underline{g}_r\bigl(\|x_n - p_n\|\bigr) \le \limsup_{n\to\infty} \delta_n = 0.$$

Therefore, we obtain

$$\lim_{n\to\infty} \|x_n - y_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|x_n - p_n\| = 0.$$

Hence, we also obtain

$$\lim_{n\to\infty} x_n = p_0 \quad \text{and} \quad \lim_{n\to\infty} y_n = p_0. \tag{4.3}$$

Since E is uniformly smooth, the duality mapping J is uniformly norm-to-norm continuous on each bounded subset on E. Therefore, we obtain

$$\lim_{n\to\infty} \|Jx_n - Jy_n\| = 0. \tag{4.4}$$

From Lemma 4.1 we see that

$$V(y_n, Q_{r_1}y_n) \le V(y_n, Q_{r_1}y_n) + V(Q_{r_1}y_n, y_n) \le 2r_1\langle y_n - Q_{r_1}y_n, x^*\rangle$$

for all $x^* \in Ay_n$. From $y_n, Q_{r_1}y_n \in D(A) \subset C \subset B_r$ and $(Jx_n - Jy_n)/r_n \in Ay_n$, we see that

$$\begin{aligned} V(y_n, Q_{r_1}y_n) &\le 2r_1\left\langle y_n - Q_{r_1}y_n, \frac{Jx_n - Jy_n}{r_n}\right\rangle \\ &\le 2r_1\|y_n - Q_{r_1}y_n\|\left\|\frac{Jx_n - Jy_n}{r_n}\right\| \\ &\le 2r_1\left(\|y_n\| + \|Q_{r_1}y_n\|\right)\left\|\frac{Jx_n - Jy_n}{r_n}\right\| \\ &= 4r_1 r\left\|\frac{Jx_n - Jy_n}{r_n}\right\|. \end{aligned}$$

Since $\liminf_n r_n > 0$ and (4.4), we obtain

$$\limsup_{n\to\infty} V(y_n, Q_{r_1}y_n) \le 0.$$

This implies $\lim_n V(y_n, Q_{r_1}y_n) = 0$. From Theorem 2.5, we see that

$$\lim_{n\to\infty} \|y_n - Q_{r_1}y_n\| = 0.$$

Then, by Lemma 2.3 and (4.3), we see that $x_n \to p_0 \in \hat{F}(Q_{r_1}) = F(Q_{r_1}) = A^{-1}0$. Since $A^{-1}0 \subset C_0$, we get $p_0 = P_{C_0}u = P_{A^{-1}0}u$, which completes the proof. □

5 Applications

In this section, we give some applications of Theorems 3.1 and 4.2. We first study the convex minimization problem: Let E be a reflexive, smooth, and strictly convex Banach space with its dual E^* and let $f : E \to\ ]-\infty, \infty]$ be a proper lower semicontinuous convex function. Then the subdifferential ∂f of f is defined as follows:

$$\partial f(x) = \left\{x^* \in E^* : f(x) + \langle y - x, x^*\rangle \le f(y), \forall y \in E\right\}$$

for all $x \in E$. By Rockafellar's theorem [29, 30], the subdifferential $\partial f \subset E \times E^*$ is maximal monotone. It is easy to see that $(\partial f)^{-1}0 = \operatorname{argmin}\{f(x) : x \in E\}$. It is also known that, see, for instance, [15, 27, 28],

$$D(\partial f) \subset D(f) \subset \overline{D(\partial f)}. \tag{5.1}$$

As a direct consequence of Theorems 3.1 and 4.2, we can show the following corollaries.

Corollary 5.1 *Let E be a smooth and uniformly convex Banach space, let $f : E \to]-\infty, \infty]$ be a proper lower semicontinuous convex function with $D(f)$ being bounded, and let $r \in]0, \infty[$ such that $D(f) \subset B_r$. Let $\{\delta_n\}$ be a nonnegative real sequence and let $\delta_0 = \limsup_n \delta_n$. For a given point $u \in E$, generate a sequence $\{x_n\}$ by $x_1 = x \in \overline{D(f)}$, $C_1 = \overline{D(f)}$, and*

$$\begin{aligned}
y_n &= \operatorname*{argmin}_{y \in E} \left\{ f(y) + \frac{1}{2r_n} \|y - x_n\|^2 \right\}, \\
C_{n+1} &= \left\{ z \in \overline{D(f)} : \left\langle y_n - z, J(x_n - y_n) \right\rangle \geq 0 \right\} \cap C_n, \\
x_{n+1} &\in \left\{ z \in \overline{D(f)} : \|u - z\|^2 \leq d(u, C_{n+1})^2 + \delta_{n+1} \right\} \cap C_{n+1},
\end{aligned}$$

for all $n \in \mathbb{N}$, where $\{r_n\} \subset]0, \infty[$ such that $\liminf_n r_n > 0$. If $(\partial f)^{-1}0$ is nonempty, then

$$\limsup_{n \to \infty} \|x_n - y_n\| \leq \underline{g}_r^{-1}(\delta_0).$$

Moreover, if $\delta_0 = 0$, then $\{x_n\}$ converges strongly to $P_{(\partial f)^{-1}0} u$.

Proof Put $C = \overline{D(f)}$. Since the subdifferential $\partial f \subset E \times E^*$ is maximal monotone, we have $E = R(I + r\partial f)$ for all $r > 0$ and hence, from (5.1), we see that

$$D(\partial f) \subset \overline{D(\partial f)} = \overline{D(f)} = C \subset E = R(I + r\partial f)$$

for all $r > 0$.

Fix $r > 0$ and $z \in C$. Let P_r be the resolvent (of type (P)) of ∂f, then we also know that

$$P_r z = \operatorname*{argmin}_{y \in E} \left\{ f(y) + \frac{1}{2r} \|y - z\|^2 \right\}.$$

Therefore, we obtain the desired result by Theorem 3.1. □

Corollary 5.2 *Let E be a uniformly smooth and uniformly convex Banach space, let $f : E \to]-\infty, \infty]$ be a proper lower semicontinuous convex function with $D(f)$ being bounded and let $r \in]0, \infty[$ such that $D(f) \subset B_r$. Let $\{\delta_n\}$ be a nonnegative real sequence and let $\delta_0 = \limsup_n \delta_n$. For a given point $u \in E$, generate a sequence $\{x_n\}$ by $x_1 = x \in \overline{D(f)}$, $C_1 = \overline{D(f)}$, and*

$$\begin{aligned}
y_n &= \operatorname*{argmin}_{y \in E} \left\{ f(y) + \frac{1}{2r_n} \|y\|^2 - \frac{1}{r_n} \langle y, Jx_n \rangle \right\}, \\
C_{n+1} &= \left\{ z \in \overline{D(f)} : \langle y_n - z, Jx_n - Jy_n \rangle \geq 0 \right\} \cap C_n, \\
x_{n+1} &\in \left\{ z \in \overline{D(f)} : \|u - z\|^2 \leq d(u, C_{n+1})^2 + \delta_{n+1} \right\} \cap C_{n+1},
\end{aligned}$$

for all $n \in \mathbb{N}$, where $\{r_n\} \subset]0, \infty[$ such that $\liminf_n r_n > 0$. If $(\partial f)^{-1}0$ is nonempty, then

$$\limsup_{n \to \infty} \|x_n - y_n\| \leq \underline{g}_r^{-1}\big(\overline{g}_r\big(\underline{g}_r^{-1}(\delta_0)\big)\big).$$

Moreover, if $\delta_0 = 0$, then $\{x_n\}$ converges strongly to $P_{(\partial f)^{-1}0} u$.

Proof Fix $r > 0$ and $z \in C$. Let Q_r be the resolvent (of type (Q)) of ∂f, then we also know that

$$Q_r z = \operatorname*{argmin}_{y\in E}\left\{f(y) + \frac{1}{2r}\|y\|^2 - \frac{1}{r}\langle y, Jz\rangle\right\}.$$

In the same way as Corollary 5.1, we obtain the desired result by Theorem 4.2. □

Next, we study the approximation of fixed points for mappings of type (P) and (Q). Before show our applications, we need the following results.

Lemma 5.3 ([17]) *Let E be a reflexive, smooth, and strictly convex Banach space, let C be a nonempty subset of E, let $T : C \to E$ be a mapping, and let $A_T \subset E \times E^*$ be an operator defined by $A_T = J(T^{-1} - I)$. Then T is of mapping of type (P) if and only if A_T is monotone. In this case $T = (I + J^{-1}A_T)^{-1}$.*

Lemma 5.4 ([31]) *Let E be a reflexive, smooth, and strictly convex Banach space, let C be a nonempty subset of E and let $T : C \to E$ be a mapping, and let $A_T \subset E \times E^*$ be an operator defined by $A_T = JT^{-1} - J$. Then T is a mapping of type (Q) if and only if A_T is monotone. In this case $T = (J + A_T)^{-1}J$.*

As a direct consequence of Theorems 3.1 and 4.2, we can show the following corollaries.

Corollary 5.5 *Let E be a smooth and uniformly convex Banach space, let C be a bounded closed convex subset of E. Let $T : C \to C$ be a mapping of type (P) with $F(T)$ being nonempty and let $r \in\ ]0,\infty[$ such that $C \subset B_r$. Let $\{\delta_n\}$ be a nonnegative real sequence and let $\delta_0 = \limsup_n \delta_n$. For a given point $u \in E$, generate a sequence $\{x_n\}$ by $x_1 = x \in C$, $C_1 = C$, and*

$$C_{n+1} = \left\{z \in C : \left\langle Tx_n - z, J(x_n - Tx_n)\right\rangle \geq 0\right\} \cap C_n,$$
$$x_{n+1} \in \left\{z \in C : \|u - z\|^2 \leq d(u, C_{n+1})^2 + \delta_{n+1}\right\} \cap C_{n+1},$$

for all $n \in \mathbb{N}$, where $\{r_n\} \subset (0,\infty)$ such that $\liminf_n r_n > 0$. Then

$$\limsup_{n\to\infty} \|x_n - Tx_n\| \leq \underline{g}_r^{-1}(\delta_0).$$

Moreover, if $\delta_0 = 0$, then $\{x_n\}$ converges strongly to $P_{F(T)}u$.

Proof Put $A_T = J(T^{-1} - I)$ and $r_n = 1$ for all $n \in \mathbb{N}$. From Lemma 5.3, we see that T is the resolvent (of type (P)) of A_T for 1 and

$$D(A_T) = R(T) \subset C = D(T) = R\big(I + J^{-1}A_T\big).$$

Therefore, we obtain the desired result by Theorem 3.1. □

Corollary 5.6 *Let E be a uniformly smooth and uniformly convex Banach space, let C be a bounded closed convex subset of E. Let $T : C \to C$ be a mapping of type (Q) with $F(T)$ being nonempty and let $r \in\ ]0,\infty[$ such that $C \subset B_r$. Let $\{\delta_n\}$ be a nonnegative real sequence and* *let* $\delta_0 = \limsup_n \delta_n$. *For a given point* $u \in E$, *generate a sequence* $\{x_n\}$ *by* $x_1 = x \in C$, $C_1 = C$, *and*

$$C_{n+1} = \bigl\{z \in C : \langle Tx_n - z, Jx_n - JTx_n\rangle \geq 0\bigr\} \cap C_n,$$
$$x_{n+1} \in \bigl\{z \in C : \|u - z\|^2 \leq d(u, C_{n+1})^2 + \delta_{n+1}\bigr\} \cap C_{n+1},$$

for all $n \in \mathbb{N}$. *Then*

$$\limsup_{n\to\infty} \|x_n - Tx_n\| \leq \underline{g}_r^{-1}\bigl(\overline{g}_r\bigl(\underline{g}_r^{-1}(\delta_0)\bigr)\bigr).$$

Moreover, if $\delta_0 = 0$, *then* $\{x_n\}$ *converges strongly to* $P_{F(T)}u$.

Proof In the same way as Corollary 5.5, we obtain the desired result by Lemma 5.4 and Theorem 4.2. □

Competing interests
The author declares to have no competing interests.

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