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Wardowski type fixed point theorems in complete metric spaces

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Abstract
In this paper, we state and prove Wardowski type fixed point theorems in metric space by using a modified generalized F-contraction maps. These theorems extend other well-known fundamental metrical fixed point theorems in the literature (Dung and Hang in Vietnam J. Math. 43:743-753, 2015 and Piri and Kumam in Fixed Point Theory Appl. 2014:210, 2014, *etc.*). Examples are provided to support the usability of our results.

MSC: 74H10; 54H25

Keywords: fixed point; metric space; F-contraction

1 Introduction and preliminaries

One of the most well-known results in generalizations of the Banach contraction principle is the Wardowski fixed point theorem [3]. Before providing the Wardowski fixed point theorem, we recall that a self-map T on a metric space (X,d) is said to be an F-contraction if there exist $F \in \mathcal{F}$ and $\tau \in (0,\infty)$ such that

$$\forall x,y \in X, \quad \bigl[d(Tx,Ty) > 0 \Rightarrow \tau + F\bigl(d(Tx,Ty)\bigr) \leq F\bigl(d(x,y)\bigr)\bigr], \tag{1}$$

where $\mathcal{F}$ is the family of all functions $F:(0,\infty) \to \mathbb{R}$ such that

(F1) F is strictly increasing, *i.e.* for all $x,y \in \mathbb{R}_+$ such that $x < y$, $F(x) < F(y)$;
(F2) for each sequence $\{\alpha_n\}_{n=1}^{\infty}$ of positive numbers, $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;
(F3) there exists $k \in (0,1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

Obviously every F-contraction is necessarily continuous. The Wardowski fixed point theorem is given by the following theorem.

Theorem 1.1 [3] *Let (X,d) be a complete metric space and let $T: X \to X$ be an F-contraction. Then T has a unique fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T^n x\}_{n\in\mathbb{N}}$ converges to x^*.*

Later, Wardowski and Van Dung [4] have introduced the notion of an F-weak contraction and prove a fixed point theorem for F-weak contractions, which generalizes some results known from the literature. They introduced the concept of an F-weak contraction as follows.

Definition 1.2 Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be an F-weak contraction on (X, d) if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that, for all $x, y \in X$,

$$d(Tx, Ty) > 0 \quad \Rightarrow \quad \tau + F\big(d(Tx, Ty)\big) \leq F\big(M(x, y)\big),$$

where

$$M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right\}. \tag{2}$$

By using the notion of F-weak contraction, Wardowski and Van Dung [4] have proved a fixed point theorem which generalizes the result of Wardowski as follows.

Theorem 1.3 [4] *Let (X, d) be a complete metric space and let $T : X \to X$ be an F-weak contraction. If T or F is continuous, then T has a unique fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T^n x\}_{n \in \mathbb{N}}$ converges to x^*.*

Recently, by adding values $d(T^2x, x)$, $d(T^2x, Tx)$, $d(T^2x, y)$, $d(T^2x, Ty)$ to (2), Dung and Hang [1] introduced the notion of a modified generalized F-contraction and proved a fixed point theorem for such maps. They generalized an F-weak contraction to a generalized F-contraction as follows.

Definition 1.4 Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be a generalized F-contraction on (X, d) if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that

$$\forall x, y \in X, \quad \big[d(Tx, Ty) > 0 \Rightarrow \tau + F\big(d(Tx, Ty)\big) \leq F\big(N(x, y)\big)\big],$$

where

$$\begin{aligned} N(x, y) = \max\bigg\{& d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}, \\ & \frac{d(T^2x, x) + d(T^2x, Ty)}{2}, d\big(T^2x, Tx\big), d\big(T^2x, y\big), d\big(T^2x, Ty\big)\bigg\}. \end{aligned}$$

By using the notion of a generalized F-contraction, Dung and Hang have proved the following fixed point theorem, which generalizes the result of Wardowski and Van Dung [4].

Theorem 1.5 [1] *Let (X, d) be a complete metric space and let $T : X \to X$ be a generalized F-contraction. If T or F is continuous, then T has a unique fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T^n x\}_{n \in \mathbb{N}}$ converges to x^*.*

Very recently, Piri and Kumam [2] described a large class of functions by replacing the condition (F3) in the definition of F-contraction introduced by Wardowski with the following one:

(F3′) F is continuous on $(0, \infty)$.

They denote by $\mathfrak{F}$ the family of all functions $F : \mathbb{R}_+ \to \mathbb{R}$ which satisfy conditions (F1), (F2), and (F3′). Under this new set-up, Piri and Kumam proved some Wardowski and Suzuki type fixed point results in metric spaces as follows.

Theorem 1.6 [2] *Let T be a self-mapping of a complete metric space X into itself. Suppose there exist $F \in \mathfrak{F}$ and $\tau > 0$ such that*

$$\forall x, y \in X, \quad \bigl[d(Tx, Ty) > 0 \Rightarrow \tau + F\bigl(d(Tx, Ty)\bigr) \leq F\bigl(d(x, y)\bigr)\bigr].$$

Then T has a unique fixed point $x^* \in X$ and for every $x_0 \in X$ the sequence $\{T^n x_0\}_{n=1}^{\infty}$ converges to x^*.

Theorem 1.7 [2] *Let T be a self-mapping of a complete metric space X into itself. Suppose there exist $F \in \mathfrak{F}$ and $\tau > 0$ such that*

$$\forall x, y \in X, \quad \left[\frac{1}{2} d(x, Tx) < d(x, y) \Rightarrow \tau + F\bigl(d(Tx, Ty)\bigr) \leq F\bigl(d(x, y)\bigr)\right].$$

Then T has a unique fixed point $x^* \in X$ and for every $x_0 \in X$ the sequence $\{T^n x_0\}_{n=1}^{\infty}$ converges to x^*.

The aim of this paper is to introduce the modified generalized F-contractions, by combining the ideas of Dung and Hang [1], Piri and Kumam [2], Wardowski [3] and Wardowski and Van Dung [4] and give some fixed point result for these type mappings on complete metric space.

2 Main results

Let $\mathfrak{F}_G$ denote the family of all functions $F : \mathbb{R}_+ \to \mathbb{R}$ which satisfy conditions (F1) and (F3′) and $\mathcal{F}_G$ denote the family of all functions $F : \mathbb{R}_+ \to \mathbb{R}$ which satisfy conditions (F1) and (F3).

Definition 2.1 Let (X, d) be a metric space and $T : X \to X$ be a mapping. T is said to be modified generalized F-contraction of type (A) if there exist $F \in \mathfrak{F}_G$ and $\tau > 0$ such that

$$\forall x, y \in X, \quad \bigl[d(Tx, Ty) > 0 \Rightarrow \tau + F\bigl(d(Tx, Ty)\bigr) \leq F\bigl(M_T(x, y)\bigr)\bigr], \tag{3}$$

where

$$\begin{aligned} M_T(x, y) = \max\biggl\{ & d(x, y), \frac{d(x, Ty) + d(y, Tx)}{2}, \frac{d(T^2x, x) + d(T^2x, Ty)}{2}, d\bigl(T^2x, Tx\bigr), \\ & d\bigl(T^2x, y\bigr), d\bigl(T^2x, Ty\bigr) + d(x, Tx), d(Tx, y) + d(y, Ty)\biggr\}. \end{aligned}$$

Remark 2.2 Note that $\mathfrak{F} \subseteq \mathfrak{F}_W$. Since, for $\beta \in (0, \infty)$, the function $F(\alpha) = \frac{-1}{\alpha+\beta}$ satisfies the conditions (F1) and (F3′) but it does not satisfy (F2), we have $\mathfrak{F} \subsetneq \mathfrak{F}_W$.

Definition 2.3 Let (X, d) be a metric space and $T : X \to X$ be a mapping. T is said to be modified generalized F-contraction of type (B) if there exist $F \in \mathcal{F}_G$ and $\tau > 0$ such that

$$\forall x, y \in X, \quad \bigl[d(Tx, Ty) > 0 \Rightarrow \tau + F\bigl(d(Tx, Ty)\bigr) \leq F\bigl(M_T(x, y)\bigr)\bigr].$$

Remark 2.4 Note that $\mathcal{F} \subseteq \mathcal{F}_W$. Since, for $\beta \in (0, \infty)$, the function $F(\alpha) = \ln(\alpha + \beta)$ satisfies the conditions (F1) and (F3) but it does not satisfy (F2), we have $\mathcal{F} \subsetneq \mathcal{F}_W$.

Remark 2.5

(1) Every F-contraction is a modified generalized F-contraction.

(2) Let T be a modified generalized F-contraction. From (3) for all $x, y \in X$ with $Tx \neq Ty$, we have

$$\begin{aligned} F\big(d(Tx,Ty)\big) &< \tau + F\big(d(Tx,Ty)\big) \\ &\leq F\biggl(\max\biggl\{d(x,y), \frac{d(x,Ty)+d(y,Tx)}{2}, \frac{d(T^2x,x)+d(T^2x,Ty)}{2}, \\ &\quad d\big(T^2x,Tx\big), d\big(T^2x,y\big), d\big(T^2x,Ty\big)+d(x,Tx), \\ &\quad d(Tx,y)+d(y,Ty)\biggr\}\biggr). \end{aligned}$$

Then, by (F1), we get

$$\begin{aligned} d(Tx,Ty) < \max\biggl\{&d(x,y), \frac{d(x,Ty)+d(y,Tx)}{2}, \frac{d(T^2x,x)+d(T^2x,Ty)}{2}, d\big(T^2x,Tx\big), \\ &d\big(T^2x,y\big), d\big(T^2x,Ty\big)+d(x,Tx), d(Tx,y)+d(y,Ty)\biggr\}, \end{aligned}$$

for all $x, y \in X$, $Tx \neq Ty$.

The following examples show that the inverse implication of Remark 2.5(1) does not hold.

Example 2.6 Let $X = [0,2]$ and define a metric d on X by $d(x,y) = \mid x - y \mid$ and let $T : X \to X$ be given by

$$Tx = \begin{cases} 1, & x \in [0,2), \\ \frac{1}{2}, & x = 2. \end{cases}$$

Obviously, (X,d) is complete metric space. Since T is not continuous, T is not an F-contraction. For $x \in [0,2)$ and $y = 2$, we have

$$d(Tx,T2) = d\left(1, \frac{1}{2}\right) = \frac{1}{2} > 0$$

and

$$\begin{aligned} &\max\biggl\{d(x,y), \frac{d(x,Ty)+d(y,Tx)}{2}, \frac{d(T^2x,x)+d(T^2x,Ty)}{2}, d\big(T^2x,Tx\big), \\ &\quad d\big(T^2x,y\big), d\big(T^2x,Ty\big)+d(x,Tx), d(Tx,y)+d(y,Ty)\biggr\} \\ &\geq d(Tx,y)+d(y,Ty) \\ &= d(1,2)+d\left(2,\frac{1}{2}\right) \\ &= \frac{5}{2}. \end{aligned}$$

Therefore

$$\begin{aligned} d(Tx, T2) \leq \frac{1}{5} \max\biggl\{& d(x,y), \frac{d(x,Ty)+d(y,Tx)}{2}, \frac{d(T^2x,x)+d(T^2x,Ty)}{2}, d\bigl(T^2x,Tx\bigr), \\ & d\bigl(T^2x,y\bigr), d\bigl(T^2x,Ty\bigr)+d(x,Tx), d(Tx,y)+d(y,Ty)\biggr\}. \end{aligned}$$

So, by choosing $F(\alpha) = \ln(\alpha)$ and $\tau = \ln\frac{1}{5}$ we see that T is modified generalized F-contraction of type (A) and type (B).

Example 2.7 Let $X = \{-2,-1,0,1,2\}$ and define a metric d on X by

$$d(x,y) = \begin{cases} 0, & \text{if } x = y, \\ 2, & \text{if } (x,y) \in \{(2,-2),(-2,2)\}, \\ 1, & \text{otherwise.} \end{cases}$$

Then (X,d) is a complete metric space. Let $T : X \to X$ be defined by

$$T(-2) = T(-1) = T0 = -2, \qquad T1 = -1, \qquad T2 = 0.$$

First observe that

$$d(Tx,Ty) > 0 \quad \Leftrightarrow \quad \bigl[\bigl(x \in \{-2,-1,0\} \wedge y = 1\bigr) \vee \bigl(x \in \{-2,-1,0\} \wedge y = 2\bigr) \vee (x = 1, y = 2)\bigr].$$

Now we consider the following cases:

Case 1. Let $x \in \{-2,-1,0\} \wedge y = 1$, then

$$\begin{aligned} & d(Tx,Ty) = d(-2,-1) = 1, \qquad d(x,y) = d(x,1) = 1, \qquad d(x,Tx) = d(x,-2) = 0 \vee 1, \\ & d(y,Ty) = d(1,-1) = 1, \qquad \frac{d(x,Ty)+d(Tx,y)}{2} = \frac{d(x,-1)+d(-2,1)}{2} = \frac{1}{2} \vee 1, \\ & \frac{d(T^2x,x)+d(T^2x,Ty)}{2} = \frac{d(-2,x)+d(-2,-1)}{2} = \frac{1}{2} \vee 1, \\ & d\bigl(T^2x,Tx\bigr) = d(-2,-2) = 0, \qquad d\bigl(T^2x,y\bigr) = d(-2,1) = 1, \\ & d\bigl(T^2x,Ty\bigr) = d(-2,-1) = 1, \\ & d\bigl(T^2x,Ty\bigr) + d(x,Tx) = d(-2,-1) + d(x,-2) = 1 \vee 2, \\ & d(Tx,y) + d(y,Ty) = d(-2,1) + d(1,-1) = 2. \end{aligned}$$

Case 2. Let $x \in \{-2,-1,0\} \wedge y = 2$, then

$$\begin{aligned} & d(Tx,Ty) = d(-2,0) = 1, \qquad d(x,y) = d(x,2) = 1 \vee 2, \qquad d(x,Tx) = d(x,-2) = 0 \vee 1, \\ & d(y,Ty) = d(2,0) = 1, \qquad \frac{d(x,Ty)+d(Tx,y)}{2} = \frac{d(x,0)+d(-2,2)}{2} = 1 \vee \frac{3}{2}, \\ & \frac{d(T^2x,x)+d(T^2x,Ty)}{2} = \frac{d(-2,x)+d(-2,0)}{2} = \frac{1}{2} \vee 1, \\ & d\bigl(T^2x,Tx\bigr) = d(-2,-2) = 0, \qquad d\bigl(T^2x,y\bigr) = d(-2,2) = 2, \end{aligned}$$

$$d\big(T^2x, Ty\big) = d(-2, 0) = 1,$$
$$d\big(T^2x, Ty\big) + d(x, Tx) = d(-2, 0) + d(x, -2) = 1 \vee 2,$$
$$d(Tx, y) + d(y, Ty) = d(-2, 2) + d(2, 0) = 3.$$

Case 3. Let $x = 1 \wedge y = 2$, then

$$d(Tx, Ty) = d(-1, 0) = 1, \qquad d(x, y) = d(1, 2) = 1, \qquad d(x, Tx) = d(1, -1) = 1,$$
$$d(y, Ty) = d(2, 0) = 1, \qquad \frac{d(x, Ty) + d(Tx, y)}{2} = \frac{d(1, 0) + d(-1, 2)}{2} = 1,$$
$$\frac{d(T^2x, x) + d(T^2x, Ty)}{2} = \frac{d(-2, 1) + d(-2, 0)}{2} = 1,$$
$$d\big(T^2x, Tx\big) = d(-2, -1) = 1, \qquad d\big(T^2x, y\big) = d(-2, 2) = 2,$$
$$d\big(T^2x, Ty\big) = d(-2, 0) = 1, \qquad d\big(T^2x, Ty\big) + d(x, Tx) = d(-2, 0) + d(1, -1) = 2,$$
$$d(Tx, y) + d(y, Ty) = d(-1, 2) + d(2, 0) = 2.$$

In Case 1, we have

$$\begin{aligned} d(Tx, Ty) &= \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right\} \\ &= \max\left\{\frac{d(T^2x, x) + d(T^2x, Ty)}{2}, d\big(T^2x, Tx\big), d\big(T^2x, y\big), d\big(T^2x, Ty\big)\right\} = 1. \end{aligned}$$

This proves that for all $F \in \mathcal{F} \cup \mathfrak{F}$, T is not an F-weak contraction and generalized F-contraction. Since every F-contraction is an F-weak contraction and a generalized F-contraction, T is not an F-contraction. However, we see that

$$\begin{aligned} d(Tx, T2) \leq \frac{1}{2} \max\bigg\{ & d(x, y), \frac{d(x, Ty) + d(y, Tx)}{2}, \frac{d(T^2x, x) + d(T^2x, Ty)}{2}, d\big(T^2x, Tx\big), \\ & d\big(T^2x, y\big), d\big(T^2x, Ty\big) + d(x, Tx), d(Tx, y) + d(y, Ty)\bigg\}. \end{aligned}$$

Hence, by choosing $F(\alpha) = \ln(\alpha)$ and $\tau = \ln\frac{1}{2}$ we see that T is modified generalized F-contraction of type (A) and type (B).

Theorem 2.8 *Let* (X, d) *be a complete metric space and* $T : X \to X$ *be a modified generalized* F*-contraction of type* (A). *Then* T *has a unique fixed point* $x^* \in X$ *and for every* $x_0 \in X$ *the sequence* $\{T^n x_0\}_{n\in\mathbb{N}}$ *converges to* x^*.

Proof Let $x_0 \in X$. Put $x_{n+1} = T^n x_0$ for all $n \in \mathbb{N}$. If, there exists $n \in \mathbb{N}$ such that $x_{n+1} = x_n$, then $Tx_n = x_n$. That is, x_n is a fixed point of T. Now, we suppose that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$. Then $d(x_{n+1}, x_n) > 0$ for all $n \in \mathbb{N}$. It follows from (3) that, for all $n \in \mathbb{N}$,

$$\begin{aligned} &\tau + F\big(d(Tx_{n-1}, Tx_n)\big) \\ &\quad \leq F\bigg(\max\bigg\{d(x_{n-1}, x_n), \frac{d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})}{2}, \end{aligned}$$

$$
\begin{aligned}
&\frac{d(T^2x_{n-1},x_{n-1}) + d(T^2x_{n-1},Tx_n)}{2}, d\bigl(T^2x_{n-1},Tx_{n-1}\bigr),\\
&d\bigl(T^2x_{n-1},x_n\bigr), d\bigl(T^2x_{n-1},Tx_n\bigr) + d(x_{n-1},Tx_{n-1}), d(Tx_{n-1},x_n) + d(x_n,Tx_n)\biggr\}\biggr)\\
&= F\biggl(\max\biggl\{d(x_{n-1},x_n), \frac{d(x_{n-1},x_{n+1}) + d(x_n,x_n)}{2},\\
&\frac{d(x_{n+1},x_{n-1}) + d(x_{n+1},x_{n+1})}{2}, d(x_{n+1},x_{n+1}),\\
&d(x_{n+1},x_n), d(x_{n+1},x_{n+1}) + d(x_{n-1},x_n), d(x_n,x_n) + d(x_n,x_{n+1})\biggr\}\biggr)\\
&= F\bigl(\max\bigl\{d(x_{n-1},x_n), d(x_n,x_{n+1})\bigr\}\bigr). \qquad (4)
\end{aligned}
$$

If there exists $n \in \mathbb{N}$ such that $\max\{d(x_{n-1},x_n), d(x_n,x_{n+1})\} = d(x_n,x_{n+1})$ then (4) becomes

$$\tau + F\bigl(d(x_n,x_{n+1})\bigr) \le F\bigl(d(x_n,x_{n+1})\bigr).$$

Since $\tau > 0$, we get a contradiction. Therefore

$$\max\bigl\{d(x_{n-1},x_n), d(x_n,x_{n+1})\bigr\} = d(x_{n-1},x_n), \quad \forall n \in \mathbb{N}.$$

Thus, from (4), we have

$$
\begin{aligned}
F\bigl(d(x_n,x_{n+1})\bigr) = F\bigl(d(Tx_{n-1},Tx_n)\bigr) &\le F\bigl(d(x_{n-1},x_n)\bigr) - \tau\\
&< F\bigl(d(x_{n-1},x_n)\bigr). \qquad (5)
\end{aligned}
$$

It follows from (5) and (F1) that

$$d(x_n,x_{n+1}) < d(x_{n-1},x_n), \quad \forall n \in \mathbb{N}.$$

Therefore $\{d(x_{n+1},x_n)\}_{n\in\mathbb{N}}$ is a nonnegative decreasing sequence of real numbers, and hence

$$\lim_{n\to\infty} d(x_{n+1},x_n) = \gamma \ge 0.$$

Now, we claim that $\gamma = 0$. Arguing by contradiction, we assume that $\gamma > 0$. Since $\{d(x_{n+1},x_n)\}_{n\in\mathbb{N}}$ is a nonnegative decreasing sequence, for every $n \in \mathbb{N}$, we have

$$d(x_{n+1},x_n) \ge \gamma. \qquad (6)$$

From (6) and (F1), we get

$$
\begin{aligned}
F(\gamma) \le F\bigl(d(x_{n+1},x_n)\bigr) &\le F\bigl(d(x_{n-1},x_n)\bigr) - \tau\\
&\le F\bigl(d(x_{n-2},x_{n-1})\bigr) - 2\tau\\
&\ \ \vdots\\
&\le F\bigl(d(x_0,x_1)\bigr) - n\tau, \qquad (7)
\end{aligned}
$$

for all $n \in \mathbb{N}$. Since $F(\gamma) \in \mathbb{R}$ and $\lim_{n\to\infty}[F(d(x_0,x_1)) - n\tau] = -\infty$, there exists $n_1 \in \mathbb{N}$ such that

$$F\big(d(x_0,x_1)\big) - n\tau < F(\gamma), \quad \forall n > n_1. \tag{8}$$

It follows from (7) and (8) that

$$F(\gamma) \le F\big(d(x_0,x_1)\big) - n\tau < F(\gamma), \quad \forall n > n_1.$$

It is a contradiction. Therefore, we have

$$\lim_{n\to\infty} d(x_n, Tx_n) = \lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{9}$$

As in the proof of Theorem 2.1 in [2], we can prove that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. So by completeness of (X,d), $\{x_n\}_{n=1}^{\infty}$ converges to some point x^* in X. Therefore,

$$\lim_{n\to\infty} d\big(x_n, x^*\big) = 0. \tag{10}$$

Finally, we will show that $x^* = Tx^*$. We only have the following two cases:

(I) $\forall n \in \mathbb{N}$, $\exists i_n \in \mathbb{N}$, $i_n > i_{n-1}$, $i_0 = 1$ and $x_{i_n+1} = Tx^*$,

(II) $\exists n_3 \in \mathbb{N}$, $\forall n \ge n_3$, $d(Tx_n, Tx^*) > 0$.

In the first case, we have

$$x^* = \lim_{n\to\infty} x_{i_{n+1}} = \lim_{n\to\infty} Tx^* = Tx^*.$$

In the second case from the assumption of Theorem 2.8, for all $n \ge n_3$, we have

$$\begin{aligned}
&\tau + F\big(d\big(x_{n+1}, Tx^*\big)\big) \\
&\quad = \tau + F\big(d\big(Tx_n, Tx^*\big)\big) \\
&\quad \le F\bigg(\max\bigg\{d\big(x_n, x^*\big), \frac{d(x_n, Tx^*) + d(x^*, Tx_n)}{2}, \\
&\qquad \frac{d(T^2x_n, x_n) + d(T^2x_n, Tx^*)}{2}, d\big(T^2x_n, Tx_n\big), \\
&\qquad d\big(T^2x_n, x^*\big), d\big(T^2x_n, Tx^*\big) + d(x_n, Tx_n), \\
&\qquad d\big(Tx_n, x^*\big) + d\big(x^*, Tx^*\big)\bigg\}\bigg).
\end{aligned} \tag{11}$$

From (F3′), (10), and taking the limit as $n \to \infty$ in (11), we obtain

$$\tau + F\big(d\big(x^*, Tx^*\big)\big) \le F\big(d\big(x^*, Tx^*\big)\big).$$

This is a contradiction. Hence, $x^* = Tx^*$. Now, let us to show that T has at most one fixed point. Indeed, if $x^*, y^* \in X$ are two distinct fixed points of T, that is, $Tx^* = x^* \neq y^* = Ty^*$, then

$$d\big(Tx^*, Ty^*\big) = d\big(x^*, y^*\big) > 0.$$

It follows from (3) that

$$\begin{aligned}
F\big(d(x^*,y^*)\big) &< \tau + F\big(d(x^*,y^*)\big) \\
&= \tau + F\big(d(Tx^*,Ty^*)\big) \\
&\le F\bigg(\max\bigg\{d(x^*,y^*), \frac{d(x^*,Ty^*)+d(y^*,Tx^*)}{2}, \frac{d(T^2x^*,x^*)+d(T^2x^*,Ty^*)}{2}, \\
&\quad d\big(T^2x^*,Tx^*\big), d\big(T^2x^*,y^*\big), d\big(T^2x^*,Ty^*\big)+d\big(x^*,Tx^*\big), \\
&\quad d\big(Tx^*,y^*\big)+d\big(y^*,Ty^*\big)\bigg\}\bigg) \\
&= F\bigg(\max\bigg\{d(x^*,y^*), \frac{d(x^*,y^*)+d(y^*,x^*)}{2}, \frac{d(x^*,x^*)+d(x^*,y^*)}{2}, \\
&\quad d\big(x^*,x^*\big), d\big(x^*,y^*\big), d\big(x^*,y^*\big)+d\big(x^*,x^*\big), \\
&\quad d\big(x^*,y^*\big)+d\big(y^*,y^*\big)\bigg\}\bigg) \\
&= F\big(d(x^*,y^*)\big),
\end{aligned}$$

which is a contradiction. Therefore, the fixed point is unique. □

Theorem 2.9 *Let (X,d) be a complete metric space and $T: X \to X$ be a continuous modified generalized F-contraction of type* (B). *Then T has a unique fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T^n x\}_{n\in\mathbb{N}}$ converges to x^*.*

Proof By using a similar method to that used in the proof of Theorem 2.8, we have

$$\begin{aligned}
F\big(d(x_n,x_{n+1})\big) = F\big(d(Tx_{n-1},Tx_n)\big) &\le F\big(d(x_{n-1},x_n)\big) - \tau \\
&< F\big(d(x_{n-1},x_n)\big)
\end{aligned}$$

and

$$\lim_{n\to\infty} d(x_n,Tx_n) = \lim_{n\to\infty} d(x_n,x_{n+1}) = 0.$$

As in the proof of Theorem 2.1 in [3], we can prove that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. So, by completeness of (X,d), $\{x_n\}_{n=1}^{\infty}$ converges to some point $x^* \in X$. Since T is continuous, we have

$$d\big(x^*,Tx^*\big) = \lim_{n\to\infty} d(x_n,Tx_n) = \lim_{n\to\infty} d(x_n,x_{n+1}) = 0.$$

Again by using similar method as used in the proof of Theorem 2.8, we can prove that x^* is the unique fixed point of T. □

3 Some applications

Theorem 3.1 [2] *Let T be a self-mapping of a complete metric space X into itself. Suppose there exist $F \in \mathfrak{F}$ and $\tau > 0$ such that*

$$\forall x,y \in X, \quad \big[d(Tx,Ty) > 0 \Rightarrow \tau + F\big(d(Tx,Ty)\big) \le F\big(d(x,y)\big)\big].$$

Then T has a unique fixed point $x^* \in X$ and for every $x_0 \in X$ the sequence $\{T^n x_0\}_{n=1}^{\infty}$ converges to x^*.

Proof Since

$$\begin{aligned}
&\max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\} \\
&\quad\le \max\left\{d(x,y), \frac{d(x,Ty)+d(y,Tx)}{2}, \frac{d(T^2x,x)+d(T^2x,Ty)}{2}, d\bigl(T^2x,Tx\bigr),\right. \\
&\qquad \left. d\bigl(T^2x,y\bigr), d\bigl(T^2x,Ty\bigr)+d(x,Tx), d(Tx,y)+d(y,Ty)\right\},
\end{aligned}$$

from (F1) and Theorem 2.8 the proof is complete. □

Theorem 3.2 [3] *Let (X,d) be a complete metric space and let $T: X \to X$ be an F-contraction. Then T has a unique fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T^n x\}_{n\in\mathbb{N}}$ converges to x^*.*

Proof Since

$$\begin{aligned}
&\max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\} \\
&\quad\le \max\left\{d(x,y), \frac{d(x,Ty)+d(y,Tx)}{2}, \frac{d(T^2x,x)+d(T^2x,Ty)}{2}, d\bigl(T^2x,Tx\bigr),\right. \\
&\qquad \left. d\bigl(T^2x,y\bigr), d\bigl(T^2x,Ty\bigr)+d(x,Tx), d(Tx,y)+d(y,Ty)\right\}.
\end{aligned}$$

So from (F1) and Theorem 2.9 the proof is complete. □

Theorem 3.3 [4] *Let (X,d) be a complete metric space and let $T: X \to X$ be an F-weak contraction. If T or F is continuous, then T has a unique fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T^n x\}_{n\in\mathbb{N}}$ converges to x^*.*

Proof Since

$$\begin{aligned}
&\max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\} \\
&\quad\le \max\left\{d(x,y), \frac{d(x,Ty)+d(y,Tx)}{2}, \frac{d(T^2x,x)+d(T^2x,Ty)}{2}, d\bigl(T^2x,Tx\bigr),\right. \\
&\qquad \left. d\bigl(T^2x,y\bigr), d\bigl(T^2x,Ty\bigr)+d(x,Tx), d(Tx,y)+d(y,Ty)\right\},
\end{aligned}$$

if F is continuous, from (F1) and Theorem 2.8 the proof is complete. If T is continuous, from (F1) and Theorem 2.9 the proof is complete. □

Theorem 3.4 [1] *Let (X,d) be a complete metric space and let $T: X \to X$ be a generalized F-contraction. If T or F is continuous, then T has a unique fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T^n x\}_{n\in\mathbb{N}}$ converges to x^*.*

Proof Since

$$\begin{aligned}
&\max\biggl\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}, \\
&\quad \frac{d(T^2x,x)+d(T^2x,Ty)}{2}, d\bigl(T^2x,Tx\bigr), d\bigl(T^2x,y\bigr), d\bigl(T^2x,Ty\bigr)\biggr\} \\
&\quad \le \max\biggl\{d(x,y), \frac{d(x,Ty)+d(y,Tx)}{2}, \frac{d(T^2x,x)+d(T^2x,Ty)}{2}, d\bigl(T^2x,Tx\bigr), \\
&\quad d\bigl(T^2x,y\bigr), d\bigl(T^2x,Ty\bigr)+d(x,Tx), d(Tx,y)+d(y,Ty)\biggr\},
\end{aligned}$$

if F is continuous, from (F1) and Theorem 2.8 the proof is complete. If T is continuous, from (F1) and Theorem 2.9 the proof is complete. □

Theorem 3.5 *Let (X,d) be a complete metric space and let $T:X\to X$ be a function with the following property*:

$$d(Tx,Ty) \le \alpha d(x,y) + \beta d(x,Tx) + \gamma d(y,Ty), \tag{12}$$

where α, β, and γ are nonnegative and satisfy $\alpha+\beta+\gamma<1$. Then T has a unique fixed point.

Proof From (12), we have

$$\begin{aligned}
d(Tx,Ty) &\le (\alpha+\beta+\gamma)\max\biggl\{d(x,y), \frac{d(x,Ty)+d(y,Tx)}{2}, \frac{d(T^2x,x)+d(T^2x,Ty)}{2}, \\
&\quad d\bigl(T^2x,Tx\bigr), d\bigl(T^2x,y\bigr), d\bigl(T^2x,Ty\bigr)+d(x,Tx), d(Tx,y)+d(y,Ty)\biggr\}.
\end{aligned}$$

Then if $d(Tx,Ty)>0$, we have

$$\begin{aligned}
&\ln\frac{1}{\alpha+\beta+\gamma} + \ln\bigl(d(Tx,Ty)\bigr) \\
&\quad \le \ln\biggl(\max\biggl\{d(x,y), \frac{d(x,Ty)+d(y,Tx)}{2}, \frac{d(T^2x,x)+d(T^2x,Ty)}{2}, d\bigl(T^2x,Tx\bigr), \\
&\quad d\bigl(T^2x,y\bigr), d\bigl(T^2x,Ty\bigr)+d(x,Tx), d(Tx,y)+d(y,Ty)\biggr\}\biggr).
\end{aligned}$$

Therefore by taking $F(\alpha)=\ln(\alpha)$ and $\tau=\ln\frac{1}{\alpha+\beta+\gamma}$ in Theorem 2.8 or in Theorem 2.9 the proof is complete. □

Remark 3.6 Our theorems are extensions of the above theorems in the following aspects:

(1) Theorem 2.8 gives all consequences of Theorem 2.1 of [2] without assumption (F2) used in its proof.

(2) Theorem 2.9 gives all consequences of Theorem 2.1 of [3] without assumption (F2) used in its proof.

(3) If in Theorem 3 of [1] F is continuous, Theorem 2.8 gives all consequences of Theorem 3 of [1] without assumptions (F2) and (F3) used in its proof.

(4) If in Theorem 3 of [1] T is continuous, Theorem 2.9 gives all consequences of Theorem 3 of [1] without assumption (F2) used in its proof.

(5) Because every F-weak contraction is a generalized F-contraction, (3) and (4) are also true for Theorem 2.4 of [4].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing the article. All authors read and approved the final manuscript.

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