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Fixed point and periodic point results for α-type F-contractions in modular metric spaces

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Abstract

Motivated by Gopal *et al.* (Acta Math. Sci. 36B(3):1-14, 2016). We introduce the notion of α-type F-contraction in the setting of modular metric spaces which is independent from one given in (Hussain *et al.* in Fixed Point Theory Appl. 2015:158, 2015). Further, we establish some fixed point and periodic point results for such contraction. The obtained results encompass various generalizations of the Banach contraction principle and others.

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1 Introduction and preliminaries

The fixed point technique is one of the important tools with respect to studying the existence and uniqueness of the solution of various mathematical methods appearing in the practical problems. In particular, the Banach contraction principle provides a constructive method of finding a unique solution for models involving various types of differential and integral equations. This principle is generalized by several authors in various directions; see [3–8]. Recently, Gopal *et al.* [1] introduced the concept of α-type F-contraction in metric space by combining the ideas given in [8] and obtained some fixed point results.

On the other hand, to deal with the problems of description of superposition operators, Chistyakov [9] introduced the notion of modular metric spaces and gave some fundamental results on this topic, whereas some authors introduced the analog of the Banach contraction theorem in modular metric spaces and described the important aspects of applications of fixed point of mappings in modular metric spaces. Some recent results in this direction can be found in [2, 10–13]. In this paper, we introduce the concept of α-type F-contraction in the setting of modular metric spaces and establish fixed point and periodic point results for such a contraction. Consequently, our results generalize and improve some known results from the literature.

Consistent with Chistyakov [9], we begin with some basic definitions and results which will be used in the sequel.

Throughout this paper $\mathbb{N}$, $\mathbb{R}^+$, and $\mathbb{R}$ will denote the set of natural numbers, positive real numbers, and real numbers, respectively.

Let X be a nonempty set. Throughout this paper, for a function $w : (0,\infty) \times X \times X \to [0,\infty)$, we write

$$w_\lambda(x,y) = w(\lambda, x, y)$$

for all $\lambda > 0$ and $x, y \in X$.

Definition 1.1 [9] Let X be a nonempty set. A function $w : (0,\infty) \times X \times X \to [0,\infty]$ is said to be a metric modular on X if it satisfies, for all $x, y, z \in X$, the following conditions:

(i) $w_\lambda(x,y) = 0$ for all $\lambda > 0$ if and only if $x = y$;
(ii) $w_\lambda(x,y) = \omega_\lambda(y,x)$ for all $\lambda > 0$;
(iii) $w_{\lambda+\mu}(x,y) \le w_\lambda(x,z) + w_\mu(z,y)$ for all $\lambda, \mu > 0$.

If instead of (i) we have only the condition (i′)

$$w_\lambda(x,x) = 0 \quad \text{for all } \lambda > 0, x \in X,$$

then w is said to be a pseudomodular (metric) on X. A modular metric w on X is said to be regular if the following weaker version of (i) is satisfied:

$$x = y \quad \text{if and only if} \quad w_\lambda(x,y) = 0 \quad \text{for some } \lambda > 0.$$

Definition 1.2 [9] Let w be a pseudomodular on X. Fix $x_0 \in X$. The set

$$X_w = X_w(x_0) = \bigl\{x \in X : w_\lambda(x,x_0) \to 0 \text{ as } \lambda \to \infty\bigr\}$$

is said to be modular space (around x_0).

Definition 1.3 Let X_w be a modular metric space.

(i) The sequence $(x_n)_{n\in\mathbb{N}}$ in X_w is said to be w-convergent to $x \in X_\omega$ if and only if $w_\lambda(x_n,x) \to 0$, as $n \to \infty$ for some $\lambda > 0$.
(ii) The sequence $(x_n)_{n\in\mathbb{N}}$ in X_w is said to be w-Cauchy if $w_\lambda(x_m,x_n) \to 0$, as $m,n \to \infty$ for some $\lambda > 0$.
(iii) A subset C of X_w is said to be w-complete if any w-Cauchy sequence in C is a convergent sequence and its limit is in C.
(iv) A subset C of X_w is said to be w-bounded if for some $\lambda > 0$, we have $\delta_w(C) = \sup\{w_\lambda(x,y); x,y \in C\} < \infty$.

Next, we denote by $\mathcal{F}$ the family of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ satisfying the following conditions:

(F1) F is strictly increasing on $\mathbb{R}^+$,
(F2) for every sequence $\{s_n\}$ in $\mathbb{R}^+$, we have $\lim_{n\to\infty} s_n = 0$ if and only if $\lim_{n\to\infty} F(s_n) = -\infty$,
(F3) there exists a number $k \in (0,1)$ such that $\lim_{s\to 0^+} s^k F(s) = 0$.

Example 1.4 The following functions $F : \mathbb{R}^+ \to \mathbb{R}$ belongs to $\mathcal{F}$:

(i) $F(t) = \ln t$, with $t > 0$,
(ii) $F(t) = \ln t + t$, with $t > 0$.

Definition 1.5 [8] A mapping $T: X \to X$ is said to be α-admissible if there exists a function $\alpha: X \times X \to \mathbb{R}^+$ such that

$$x, y \in X, \quad \alpha(x,y) \geq 1 \quad \Longrightarrow \quad \alpha(Tx, Ty) \geq 1.$$

Definition 1.6 [2] Let Δ_G denote the set of all functions $G: (\mathbb{R}^+)^4 \to \mathbb{R}^+$ satisfying the condition (G) for all $t_1, t_2, t_3, t_4 \in \mathbb{R}^+$ with $t_1 t_2 t_3 t_4 = 0$, there exists $\tau > 0$ such that $G(t_1, t_2, t_3, t_4) = \tau$.

Example 1.7 The following function $G: (\mathbb{R}^+)^4 \to \mathbb{R}$ belongs to Δ_G:

(i) $G(t_1, t_2, t_3, t_4) = L \min(t_1, t_2, t_3, t_4) + \tau$,

(ii) $G(t_1, t_2, t_3, t_4) = \tau e^{L \min(t_1, t_2, t_3, t_4)}$, where $L \in \mathbb{R}^+$.

Definition 1.8 [2] Let X_ω be a modular metric space and T be a self-mapping on X_ω. Suppose that $\alpha, \eta: X_\omega \times X_\omega \to [0, \infty)$ are two functions. We say T is an α-η-GF-contraction if for $x, y \in X_\omega$ with $\eta(x, Tx) \leq \alpha(x,y)$, $\omega_{\lambda/l}(Tx, Ty) > 0$, and $\lambda, l > 0$, we have

$$G\bigl(\omega_{\lambda/l}(x, Tx), \omega\lambda/l(y, Ty), \omega\lambda/l(x, Ty), \omega\lambda/l(y, Tx)\bigr) + F\bigl(\omega\lambda/l(Tx, Ty)\bigr) \leq F\bigl(\omega\lambda/l(x,y)\bigr),$$

where $G \in \Delta_G$ and $F \in \mathcal{F}$.

2 Fixed point results for α-type F-contractions

We begin with the following definitions.

Definition 2.1 Let (X, w) be a modular metric space. Let C be a nonempty subset of X_w. A mapping $T: C \to C$ is said to be an α-type F-contraction if there exist $\tau > 0$ and two functions $F \in \mathcal{F}$, $\alpha: C \times C \to (0, \infty)$ such that, for all $x, y \in C$, satisfying $w_1(Tx, Ty) > 0$, the following inequality holds:

$$\tau + \alpha(x,y) F\bigl(w_1(Tx, Ty)\bigr) \leq F\bigl(w_1(x,y)\bigr). \tag{2.1}$$

Definition 2.2 Let (X, w) be a modular metric space. Let C be a nonempty subset of X_w. A mapping $T: C \to C$ is said to be an α-type F-weak contraction if there exist $\tau > 0$ and two functions $F \in \mathcal{F}, \alpha: C \times C \to (0, \infty)$ such that, for all $x, y \in C$, satisfying $w_1(Tx, Ty) > 0$, the following inequality holds:

$$\begin{aligned} &\tau + \alpha(x,y) F\bigl(w_1(Tx, Ty)\bigr) \\ &\quad \leq F\left(\max\left\{w_1(x,y), w_1(x, Tx), w_1(y, Ty), \frac{w_2(x, Ty) + w_2(y, Tx)}{2}\right\}\right). \end{aligned} \tag{2.2}$$

Remark 2.3 Every α-type F-contraction is an α-type F-weak contraction, but the converse is not necessarily true.

Example 2.4 Let $X_w = C = [0, \frac{9}{2}]$, $w_1 = |x - y|$, and $w_2 = |x - y|$. Define $T: C \to C$, $\alpha: C \times C \to (0, \infty)$, and $F: \mathbb{R}^+ \to \mathbb{R}$ by

$$T(x) = \begin{cases} 0, & \text{if } x \in [0, \frac{2}{9}], \\ \frac{9}{2}, & \text{otherwise.} \end{cases}$$

Then, for $x = 0$ and $y = 1$, by putting $F(t) = \ln t$ with $t > 0$, we have

$$\tau + \alpha(0,1)F\bigl(w_\lambda\bigl(T(0),T(1)\bigr)\bigr) = \tau + \alpha(0,1)\ln\left(\frac{9}{2}\right)$$

and

$$F\bigl(w_\lambda(0,1)\bigr) = \ln(1).$$

Clearly, we have

$$e^\tau\left(\frac{9}{2}\right)^{\alpha(0,1)} \nleq 1 \quad \text{for all } \tau > 0 \text{ and for all } \alpha \in (0,\infty).$$

However, since

$$\inf_{x\in[0,\frac{2}{9}],y\in(\frac{2}{9},\frac{9}{2}]}\left\{\max\left\{w_1(x,y),w_1(x,Tx),w_1(y,Ty),\frac{w_2(x,Ty)+w_2(y,Tx)}{2}\right\}\right\} = \frac{9}{4},$$

T is an α-type F-weak contraction for the choice

$$\alpha(x,y) = \begin{cases} 1, & \text{if } x,y \in [0,\frac{2}{9}] \text{ or } x,y \in (\frac{2}{9},\frac{9}{2}], \\ \frac{\log 10-\log 9}{\log 9-\log 2}, & \text{otherwise,} \end{cases}$$

and $\tau > 0$ such that $e^{-\tau} = \frac{8}{9}$.

Remark 2.5 Definition 2.1 (respectively, Definition 2.2) reduces to an F-contraction (respectively, an F-weak contraction) for $\alpha(x,y) = 1$.

The next two examples demonstrate that α-type F-contractions (defined above) and α-η-GF-contractions [2] are independent.

Example 2.6 Let $X_w = C = [0,3]$, $w_1 = |x-y|$, and $w_\lambda = \frac{1}{\lambda}|x-y|$. Define $T : C \to C$, $\alpha : C \times C \to (0,\infty)$, and $F : \mathbb{R}^+ \to \mathbb{R}$ by

$$T(x) = \begin{cases} \frac{3}{2}, & \text{if } x \in [0,3), \\ 0, & \text{if } x = 3. \end{cases}$$

So, define $F(t) = \ln t$ with $t > 0$. Then T is an α-type F-weak contraction with $\alpha(x,y) = 1$ for all $x,y \in C$ and $\tau > 0$ such that $e^{-\tau} = \frac{3}{2}$. But T is not an α-η-GF-contraction [2]. To see this, consider $\eta : C \times C \to [0,\infty)$ such that

$$\eta(x,Tx) = \begin{cases} 0, & \text{if } x = \frac{3}{2}, \\ 4, & \text{otherwise} \end{cases}$$

and

$$G(t_1,t_2,t_2,t_3,t_4) = L\min\{t_1,t_2,t_2,t_3,t_4\} + \tau.$$

Then, for $x = \frac{3}{2}$ and $y = 3$, we have

$$\eta\left(\frac{3}{2},\frac{3}{2}\right) = 0 \leq 1 = \alpha\left(\frac{3}{2},3\right) = \alpha(x,y), \qquad w_\lambda(Tx,Ty) = \frac{3}{2}\cdot\frac{1}{\lambda} > 0,$$

and

$$\begin{aligned} &G\bigl(w_\lambda(x,Tx), w_\lambda(y,Ty), w_\lambda(x,Ty), w_\lambda(y,Tx)\bigr) \\ &\quad = L\min\bigl\{w_\lambda(x,Tx), w_\lambda(y,Ty), w_\lambda(x,Ty), w_\lambda(y,Tx)\bigr\} + \tau = \tau. \end{aligned}$$

Consequently, we have

$$\begin{aligned} &G\bigl(w_\lambda(x,Tx), w_\lambda(y,Ty), w_\lambda(x,Ty), w_\lambda(y,Tx)\bigr) + F\bigl(w_\lambda(Tx,Ty)\bigr) \\ &\quad = \tau + \ln\left(\frac{3}{2}\cdot\frac{1}{\lambda}\right) \nleq \ln\left(\frac{3}{2}\cdot\frac{1}{\lambda}\right) = F\bigl(w_\lambda(x,y)\bigr), \end{aligned}$$

and thus, T is not an α-η-GF-contraction.

Example 2.7 Let $X_w = C = [0,1]$, $w_1 = |x-y|$, and $w_\lambda = \frac{1}{\lambda}|x-y|$. Define $T : C \to C$, $\alpha,\eta : C\times C \to [0,\infty)$, $G : (\mathbb{R}^+)^4 \to \mathbb{R}^+$ by

$$T(x) = \begin{cases} 0, & \text{if } x \text{ is rational,} \\ 1, & \text{if } x \text{ is irrational,} \end{cases}$$

$\alpha(x,y) = x + y$ and $\eta(x,y) = \frac{x+y}{2}$, if x and y both are rational or irrational, $\alpha(x,y) = 1$ and $\eta(x,y) = 0$ if x is irrational and y is rational (and *vice versa*), $G(t_1,t_2,t_3,t_4) = L\min\{t_1,t_2,t_3,t_4\} + \tau$ ($\tau > 0$), and $F(t) = \ln t$. Then T is an α-η-GF-contraction. But it is not an α-type F-weak contraction. To see this, consider $x = 0$ and y is any irrational.

The motivation of the following definition is in the last step of the proof of the Cauchy sequence in our theorems.

Definition 2.8 Let (X,w) be a modular metric space. Then we will say that w satisfies the Δ_M-condition if it is the case that $\lim_{m,n\to\infty} w_\lambda(x_n,x_m) = 0$, for $\lambda = m$ implies $\lim_{m,n\to\infty} w_\lambda(x_n,x_m) = 0$ ($m,n \in \mathbb{N}$, $m \geq n$), for some $\lambda > 0$.

Now, we are ready to state our first theorem which generalizes the main theorem of Gopal *et al.* [1] for modular metric spaces.

Theorem 2.9 *Let (X,w) be a modular metric space. Assume that w is regular and satisfies the Δ_M-condition. Let C be a nonempty subset of X_w. Assume that C is w-complete and w-bounded, i.e., $\delta_w(C) = \sup\{w_1(x,y) : x,y \in C\} < \infty$. Let $T : C \to C$ be an α-type F-weak contraction satisfying the following conditions*:

(i) *T is α-admissible,*
(ii) *there exists $x_0 \in C$ such that $\alpha(x_0,Tx_0) \geq 1$,*
(iii) *T is continuous.*

Then T has a fixed point $x^* \in C$ and for every $x_0 \in C$ the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ is convergent to x^*.

Proof Let $x_0 \in C$ such that $\alpha(x_0, Tx_0) \geq 1$ and define a sequence $\{x_n\}$ in C by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$.

Obviously, if there exists $n_0 \in \mathbb{N}$ such that $x_{n_0+1} = x_{n_0}$, then $Tx_{n_0} = x_{n_0}$ and the proof is finished. Hence, we suppose that $x_{n+1} \neq x_n$ for every $n \in \mathbb{N}$. Now from conditions (ii) and (i), we have

$$\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1 \quad \Longrightarrow \quad \alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1.$$

By induction we have

$$\alpha(x_n, x_{n+1}) \geq 1. \tag{2.3}$$

Since T is an α-type F-weak contraction, for every $n \in \mathbb{N}$, we have

$$\begin{aligned} F\bigl(w_1(x_{n+1}, x_n)\bigr) &= F\bigl(w_1(Tx_n, Tx_{n-1})\bigr) \\ &\leq \alpha(x_n, x_{n-1}) F\bigl(w_1(Tx_n, Tx_{n-1})\bigr), \end{aligned} \tag{2.4}$$

so

$$\begin{aligned} \tau + F\bigl(w_1(x_{n+1}, x_n)\bigr) &\leq \tau + \alpha(x_n, x_{n+1}) F\bigl(w_1(Tx_n, Tx_{n-1})\bigr) \\ &\leq F\Biggl(\max\Biggl\{w_1(x_n, x_{n-1}), w_1(x_n, Tx_n), w_1(x_{n-1}, Tx_{n-1}), \\ &\qquad \frac{w_2(x_n, Tx_{n-1}) + w_2(x_{n-1}, Tx_n)}{2}\Biggr\}\Biggr) \\ &= F\Biggl(\max\Biggl\{w_1(x_{n-1}, x_n), w_1(x_n, x_{n+1}), \frac{w_2(x_{n-1}, x_{n+1})}{2}\Biggr\}\Biggr) \\ &\leq F\Biggl(\max\Biggl\{w_1(x_{n-1}, x_n), w_1(x_n, x_{n+1}), \frac{w_1(x_{n-1}, x_n) + w_1(x_n, x_{n+1})}{2}\Biggr\}\Biggr) \\ &= F\bigl(\max\bigl\{w_1(x_{n-1}, x_n), w_1(x_n, x_{n+1})\bigr\}\bigr). \end{aligned} \tag{2.5}$$

If there exists $n \in \mathbb{N}$ such that $\max\{w_1(x_{n-1}, x_n), w_1(x_n, x_{n+1})\} = w_1(x_n, x_{n+1})$, then, from (2.5), we have

$$F\bigl(w_1(x_n, x_{n+1})\bigr) \leq F\bigl(w_1(x_n, x_{n+1})\bigr) - \tau,$$

a contradiction. Therefore $\max\{w_1(x_{n-1}, x_n), w_1(x_n, x_{n+1})\} = w_1(x_{n-1}, x_n)$, for all $n \in \mathbb{N}$. Hence, from (2.5), we have

$$F\bigl(w_1(x_n, x_{n+1})\bigr) \leq F\bigl(w_1(x_{n-1}, x_n)\bigr) - \tau,$$

This implies that

$$F\bigl(w_1(x_n, x_{n+1})\bigr) \leq F\bigl(w_1(x_0, x_1)\bigr) - n\tau \quad \text{for all } n \in \mathbb{N}. \tag{2.6}$$

Taking the limit as $n \to \infty$ in (2.6) and since C is w-bounded, we have

$$F\big(w_1(x_n, x_{n+1})\big) = -\infty;$$

from (F2), we obtain

$$\lim_{n\to\infty} \big(w_1(x_n, x_{n+1})\big) = 0. \tag{2.7}$$

From (F3), there exists $k \in (0,1)$ such that

$$\lim_{n\to\infty} \big(\big(w_1(x_{n+1}, x_n)\big)^k F\big(w_1(x_{n+1}, x_n)\big)\big) = 0. \tag{2.8}$$

From (2.8), for all $n \in \mathbb{N}$, we deduce that

$$\big(w_1(x_{n+1}, x_n)\big)^k \big(F\big(w_1(x_{n+1}, x_n)\big) - F\big(w_1(x_0, x_1)\big)\big) \leq -\big(w_1(x_{n+1}, x_n)\big)^k n\tau \leq 0. \tag{2.9}$$

By using (2.7), (2.8), and taking the limit as $n \to \infty$ in (2.9), we have

$$\lim_{n\to\infty} \big(n\big(w_1(x_{n+1}, x_n)\big)^k\big) = 0.$$

Then there exists $n_1 \in \mathbb{N}$ such that $n(w_1(x_{n+1}, x_n))^k \leq 1$ for all $n \geq n_1$, that is,

$$w_1(x_n, x_{n+1}) \leq \frac{1}{n^{1/k}} \quad \text{for all } n \geq n_1, \text{for all } \lambda > 0.$$

For all $m > n > n_1$, we have

$$\begin{aligned} w_m(x_n, x_m) &\leq w_1(x_n, x_{n+1}) + w_1(x_{n+1}, x_{n+2}) + \cdots + w_1(x_m, x_{m+1}) \\ &\leq \frac{1}{n^{1/k}} + \frac{1}{(n+1)^{1/k}} + \cdots + \frac{1}{m^{1/k}} \\ &< \sum_{i=n}^{\infty} \frac{1}{i^{1/k}}. \end{aligned}$$

Since the series $\sum_{i=n}^{\infty} \frac{1}{i^{1/k}}$ is convergent, this implies

$$\lim_{m,n\to\infty} \big(w_m(x_n, x_m)\big) = 0.$$

Since w satisfies the Δ_M-condition. Hence, we have

$$\lim_{m,n\to\infty} w_1(x_n, x_m) = 0.$$

This shows that $\{x_n\}$ is a w-Cauchy sequence. Since C is w-complete, there exists $x^* \in C$ such that $x_n \to x^*$ as $n \to \infty$. By the continuity of T and since w is regular, we have

$$w_1\big(x^*, Tx^*\big) = \lim_{n\to\infty} \big(w_1(x_n, Tx_n)\big) = \lim_{n\to\infty} \big(w_1(x_n, x_{n+1})\big) = 0.$$

Hence, x^* is a fixed point of T. □

Theorem 2.10 *Let (X, w) be a modular metric space. Assume that w is regular and satisfies the Δ_M-condition. Let C be a nonempty subset of X_w. Assume that C is w-complete modular metric space and w-bounded, i.e., $\delta_w(C) = \sup\{w_1(x,y) : x, y \in C\} < \infty$. Let $T : C \to C$ be an α-type F-weak contraction satisfying the following conditions*:

(i) *there exists $x_0 \in C$ such that $\alpha(x_0, Tx_0) \geq 1$,*
(ii) *T is α-admissible,*
(iii) *if $\{x_n\}$ is a sequence in X_w such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$,*
(iv) *F is continuous.*

Then T has a fixed point $x^* \in C$ and for every $x_0 \in C$ the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ is convergent to x^*.

Proof Let $x_0 \in C$ be such that $\alpha(x_0, Tx_0) \geq 1$ and let $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$ Following the proof of Theorem 2.9, we see that $\{x_n\}$ is a w-Cauchy sequence in the w-complete modular metric space. Then there exists $x^* \in C$ such that $x_n \to x^*$ as $n \to \infty$. From (2.3) and the hypothesis (iii), we have

$$\alpha\big(x_n, x^*\big) \geq 1 \quad \text{for all } n \in \mathbb{N}.$$

Case I: Suppose, for every $n \in \mathbb{N}$, there exists $i_n \in \mathbb{N}$ such that $x_{i_n+1} = Tx^*$ and $i_n > i_{n-1}$. Then we have

$$x^* = \lim_{n\to\infty} x_{i_n+1} = \lim_{n\to\infty} Tx^*_{i_n} = Tx^*, \tag{2.10}$$

that is, x^* is a fixed point of T.

Case II: Assume there exists $n_0 \in \mathbb{N}$ such that $x_{n+1} \neq Tx^*$ for all $n \geq n_0$, *i.e.*, $w_1(Tx_n, Tx^*) > 0$ for all $n \geq n_0$. It follows from (2.2) and (F1) that

$$\begin{aligned}
\tau + F\big(w_1\big(x_{n+1}, Tx^*\big)\big) &= \tau + F\big(w_1\big(Tx_n, Tx^*\big)\big) \leq \tau + \alpha\big(x_n, x^*\big)F\big(w_1\big(Tx_n, Tx^*\big)\big) \\
&\leq F\bigg(\max\bigg\{w_1\big(x_n, x^*\big), w_1(x_n, Tx_n), w_1\big(x^*, Tx^*\big), \\
&\qquad \frac{w_2(x_n, Tx^*) + w_2(x^*, Tx_n)}{2}\bigg\}\bigg) \\
&\leq F\bigg(\max\bigg\{w_1\big(x_n, x^*\big), w_1(x_n, x_{n+1}), w_1\big(x^*, Tx^*\big), \\
&\qquad \frac{w_1(x_n, x^*) + w_1(x^*, Tx^*) + w_1(x^*, x_n) + w_1(x_n, x_{n+1})}{2}\bigg\}\bigg).
\end{aligned} \tag{2.11}$$

If $w_1(x^*, Tx^*) > 0$ and by the fact that

$$\lim_{n\to\infty}\big(w_1\big(x_n, x^*\big)\big) = \lim_{n\to\infty}\big(w_1\big(x_{n+1}, x^*\big)\big) = 0,$$

there exists $n_1 \in \mathbb{N}$ such that, for all $n \geq n_1$, we have

$$\begin{aligned}
&\max\bigg\{w_1\big(x_n, x^*\big), w_1(x_n, x_{n+1}), w_1\big(x^*, Tx^*\big), \\
&\qquad \frac{w_1(x_n, x^*) + w_1(x^*, Tx^*) + w_1(x^*, x_n) + w_1(x_n, x_{n+1})}{2}\bigg\} = w_1\big(x^*, Tx^*\big).
\end{aligned}$$

From (2.11), we obtain

$$\tau + F\bigl(w_1\bigl(x_{n+1}, Tx^*\bigr)\bigr) \le F\bigl(w_1\bigl(x^*, Tx^*\bigr)\bigr) \tag{2.12}$$

for all $n \ge \max\{n_0, n_1\}$. Since F is continuous, taking the limit as $n \to \infty$ in (2.12), we have

$$\tau + F\bigl(w_1\bigl(x^*, Tx^*\bigr)\bigr) \le F\bigl(w_1\bigl(x^*, Tx^*\bigr)\bigr),$$

a contradiction. Thus $w_1(x^*, Tx^*) = 0$ and hence x^* is a fixed point of T. □

Indeed, uniqueness of the fixed point, we will consider the following hypothesis.
(H): for all $x, y \in \operatorname{Fix}(T)$, $\alpha(x, y) \ge 1$.

Theorem 2.11 *Adding condition* (H) *to the hypotheses of Theorem* 2.9 (*respectively, Theorem* 2.10) *the uniqueness of the fixed point is obtained.*

Proof Assume that $y^* \in C$ is an another fixed point of T, such that $w_1(x, y) < \infty$ and $w_1(Tx^*, Ty^*) = w_1(x^*, y^*) > 0$. Then we have

$$\begin{aligned}\tau + F\bigl(w_1\bigl(x^*, y^*\bigr)\bigr) &\le \tau + F\bigl(w_1\bigl(Tx^*, Ty^*\bigr)\bigr)\\ &\le \tau + \alpha\bigl(x^*, y^*\bigr)F\bigl(w_1\bigl(Tx^*, Ty^*\bigr)\bigr)\\ &\le F\bigl(w_1\bigl(x^*, y^*\bigr)\bigr),\end{aligned}$$

a contradiction. This implies that $x^* = y^*$. □

Example 2.6 satisfies all the hypotheses of Theorem 2.10, hence T has a unique fixed point $x = \frac{3}{2}$.

The following result improves the main theorem of the F-contraction for a modular metric space.

Corollary 2.12 *Let (X, w) be a modular metric space. Assume that w is regular and satisfies the Δ_M-condition. Let C be a nonempty subset of X_w. Assume that C is w-complete and w-bounded, i.e., $\delta_w(C) = \sup\{w_1(x, y) : x, y \in C\} < \infty$. Let $T : C \to C$ be an α-type F-contraction satisfying the hypotheses of Theorem* 2.11, *then T has unique fixed point.*

From Example 1.4(i) and Corollary 2.12, we obtain the following result.

Theorem 2.13 *Let (X, w) be a modular metric space. Assume that w is regular. Let C be a nonempty subset of X_w. Assume that C is w-complete and w-bounded, i.e., $\delta_w(C) = \sup\{w_1(x, y); x, y \in C\} < \infty$. Let $T : C \to C$ be a contraction. Then T has a unique fixed point x_0. Moreover, the orbit $\{T^n(x)\}$ w-converges to x_0 for $x \in C$.*

3 Periodic point results

In this section, we prove some periodic point results for self-mappings on a modular metric space. In the sequel, we need the following definition.

Definition 3.1 [14] A mapping $T : C \to C$ is said to have the *property* (P) if $\mathrm{Fix}(T^n) = \mathrm{Fix}(T)$ for every $n \in \mathbb{N}$, where $\mathrm{Fix}(T) := \{x \in X_w : Tx = x\}$.

Theorem 3.2 *Let (X, w) be a modular metric space. Assume that w is regular and satisfies the Δ_M-condition. Let C be a nonempty subset of X_w. Assume that C is w-complete and w-bounded, i.e., $\delta_w(C) = \sup\{w_1(x,y) : x,y \in C\} < \infty$. Let C be a w-complete and w-bounded subset of X. Let $T : C \to C$ be a mapping satisfying the following conditions*:

(i) *there exists $\tau > 0$ and two functions $F \in \mathcal{F}$ and $\alpha : C \times C \to (0, \infty)$ such that*

$$\tau + \alpha(x, Tx)F\bigl(w_1\bigl(Tx, T^2x\bigr)\bigr) \le F\bigl(w_1(x, Tx)\bigr)$$

holds for all $x \in C$ with $w_1(Tx, T^2x) > 0$,

(ii) *there exists $x_0 \in C$ such that $\alpha(x_0, Tx_0) \ge 1$,*

(iii) *T is α-admissible,*

(iv) *if $\{x_n\}$ is a sequence in C such that $\alpha(x_n, x_{n+1}) \le 1$ for all $n \in \mathbb{N}$ and $w_1(x_n, x) \to 0$, as $n \to \infty$, then $w_1(Tx_n, Tx) \to 0$ as $n \to \infty$,*

(v) *if $z \in \mathrm{Fix}(T^n)$ and $z \notin \mathrm{Fix}(T)$, then $\alpha(T^{n-1}z, T^n z) \ge 1$. Then T has the property (P).*

Proof Let $x_0 \in C$ be such that $\alpha(x_0, Tx_0) \ge 1$. Now, for $x_0 \in C$, we define the sequence $\{x_n\}$ by the rule $x_n = T^n x_n = Tx_{n-1}$. By (iii), we have $\alpha(x_1, x_2) = \alpha(Tx_0, Tx_1) \ge 1$ and by induction we write

$$\alpha(x_n, x_{n+1}) \ge 1 \quad \text{for all } n \in \mathbb{N}. \tag{3.1}$$

If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1} = Tx_{n_0}$, then x_{n_0} is a fixed point of T and the proof is finished. Thus, we assume $x_n \ne x_{n+1}$ or $w_1(Tx_{n-1}, T^2x_{n-1}) > 0$ for all $n \in \mathbb{N}$. From (3.1) and (i), we have

$$\begin{aligned}
\tau + F\bigl(w_1(x_n, x_{n+1})\bigr) &= \tau + F\bigl(w_1\bigl(Tx_{n-1}, T^2x_{n-1}\bigr)\bigr) \\
&\le \tau + \alpha(x_{n-1}, Tx_{n-1})F\bigl(w_1\bigl(Tx_{n-1}, T^2x_{n-1}\bigr)\bigr) \\
&\le F\bigl(w_1(x_{n-1}, Tx_{n-1})\bigr)
\end{aligned}$$

or equivalently

$$F\bigl(w_1(x_n, x_{n+1})\bigr) \le F\bigl(w_1(x_{n-1}, x_n)\bigr) - \tau.$$

By using a similar reasoning to the proof of Theorem 2.9, we see that the sequence $\{x_n\}$ is a w-Cauchy sequence and thus, by w-completeness, there exists $x^* \in X_w$ such that $x_n \to x^*$ as $n \to \infty$.

By (iv), we have $w_1(x_{n+1}, Tx^*) = w_1(Tx_n, Tx^*) \to 0$ as $n \to \infty$, that is, $x^* = Tx^*$. Hence, T has a fixed point and $\mathrm{Fix}(T^n) = \mathrm{Fix}(T)$ is true for $n = 1$. Let $n > 1$ and assume, by contradiction, that $z \in \mathrm{Fix}(T^n)$ and $z \notin \mathrm{Fix}(T)$, such that $w_1(z, Tz) > 0$. Now, applying (v) and (i), we have

$$\begin{aligned}
F\bigl(w_1(z, Tz)\bigr) &\le F\bigl(w_1\bigl(T\bigl(T^{n-1}z\bigr), T^2\bigl(T^{n-1}z\bigr)\bigr)\bigr) \\
&\le \alpha\bigl(T^{n-1}z, T^n z\bigr)F\bigl(w_1\bigl(T\bigl(T^{n-1}z\bigr), T^2\bigl(T^{n-1}z\bigr)\bigr)\bigr)
\end{aligned}$$

and

$$\begin{aligned}\tau + F\big(w_1(z, Tz)\big) &\le \tau + F\big(w_1\big(T\big(T^{n-1}z\big), T^2\big(T^{n-1}z\big)\big)\big)\\ &\le F\big(w_1\big(T^{n-1}z, T^n z\big)\big).\end{aligned}$$

Consequently, we have

$$\begin{aligned}F\big(w_1(z, Tz)\big) &\le F\big(w_1\big(T^{n-1}z, T^n z\big)\big) - \tau\\ &\le F\big(w_1\big(T^{n-2}z, T^{n-1}z\big)\big) - 2\tau\\ &\ \ \vdots\\ &\le F\big(w_1(z, Tz)\big) - n\tau.\end{aligned}$$

By taking the limit as $n \to \infty$ in the above inequality, we have $F(w_1(z, Tz)) = -\infty$, which is a contradiction until $w_1(z, Tz) = 0$ and by the regularity of w, we set $z = Tz$. Hence, $\mathrm{Fix}(T^n) = \mathrm{Fix}(T)$ for all $n \in \mathbb{N}$. □

Taking $\alpha(x, y) = 1$ for all $x, y \in C$ in Theorem 3.2, we get the following result, which is a generalization of Theorem 4 of Abbas *et al.* [14] in the setting of a modular metric.

Corollary 3.3 *Let (X, w) be a complete modular metric space. Assume that w is regular and satisfies the Δ_M-condition. Let C be a nonempty subset of X_w. Assume that C is w-complete and w-bounded, i.e., $\delta_w(C) = \sup\{w_1(x, y) : x, y \in C\} < \infty$. Let $T : C \to C$ be a continuous mapping satisfying*

$$\tau + F\big(w_1\big(Tx, T^2x\big)\big) \le F\big(w_1(x, Tx)\big)$$

for some $\tau > 0$ and for all $x \in X_w$ such that $w_1(Tx, T^2x) > 0$. Then T has property (P).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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