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Fixed point theorems of order-Lipschitz mappings in Banach algebras

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Abstract
In this paper, by introducing the concept of Picard-completeness and using the sandwich theorem in the sense of *w*-convergence, we first prove some fixed point theorems of order-Lipschitz mappings in Banach algebras with non-normal cones which improve the result of Sun's since the normality of the cone was removed. Moreover, we reconsider the case with normal cones and obtain a fixed point theorem under the assumption relating to the spectral radius, which partially improves the results of Krasnoselskii and Zabreiko's. In addition, we present some suitable examples which show the usability of our theorems.

MSC: 06A07; 47H10

Keywords: Fixed point theorem; Banach algebra; order-Lipschitz mapping; Picard-complete

1 Introduction

This work is mainly concerned with fixed point theory of order-Lipschitz mappings in Banach algebras relating to the improvements of the Banach contraction principle which states that each Banach contraction on a complete metric space has a unique fixed point. Let (X, d) be a metric space. A Banach contraction in a metric space (resp. a cone metric space) is also called a Lipschitz mapping with respect to the metric (resp. the cone metric); see [1]. Let P be a cone of a Banach algebra $(E, \|\cdot\|)$ and $\preceq$ the partial order in E introduced by P. A mapping $T : E \to E$ is called an order-Lipschitz mapping if there exist $l, k \in P$ such that the following Lipschitz condition with respect to the partial order is satisfied:

$$-l(x-y) \preceq Tx - Ty \preceq k(x-y), \quad \forall x, y \in E, \qquad y \preceq x. \tag{1}$$

In particular when k, l are nonnegative real numbers, Sun [2] obtained the following fixed point theorem of order-Lipschitz mappings in Banach spaces by using the sandwich theorem in the sense of norm-convergence.

Theorem 1 (see [2]) *Let P be a normal cone of a Banach space $(E, \|\cdot\|)$ and $u_0, v_0 \in E$ with $u_0 \preceq v_0$. Assume that $T : [u_0, v_0] \to E$ is an order-Lipschitz mapping with $l \in [0, +\infty)$ and $k \in [0, 1)$ such that*

$$u_0 \preceq Tu_0, \qquad Tv_0 \preceq v_0. \tag{2}$$

Then T has a unique fixed point $x^* \in [u_0, v_0]$. And for each $x_0 \in [u_0, v_0]$, let $\{x_n\}$ be the Picard iterative sequence (i.e., $x_n = T^n x_0$ for each n), then we must have $x_n \xrightarrow{\|\cdot\|} x^*$.

Remark 1 The normality of the cone is essential for ensuring that the sandwich theorem holds in the sense of norm-convergence which plays an important role in the proof of Theorem 1. However, if P is non-normal then the sandwich theorem does not hold in the sense of norm-convergence, and consequently, the method used in the proof of Theorem 1 may become invalid.

Krasnoselskii and Zabreiko [3] considered order-Lipschitz mappings in Banach spaces restricted with linear bounded mappings (*i.e.*, k, l are linear bounded mappings), and proved the following fixed point theorem by using the Banach contraction principle.

Theorem 2 (see [3]) *Let P be a normal solid cone of a Banach space $(E, \|\cdot\|)$ and $T : E \to E$ an order-Lipschitz mapping with $k = l$, where $k : P \to P$ is a linear bounded mapping. If $\|k\| < 1$, then T has a unique fixed point $x^* \in E$. And for each $x_0 \in E$, let $\{x_n\}$ be the Picard iterative sequence, then we must have $x_n \xrightarrow{\|\cdot\|} x^*$.*

To our knowledge, in all the works concerned with fixed points of order-Lipschitz mappings, the involving cone is necessarily assumed to be normal. In this paper, we shall remove the normality of the cone in Theorem 1 and extend Theorems 1 and 2 to Banach algebras. From Remark 1 we know that the method in [2] is not applicable for the case with non-normal cones, and so we need to find a new way to solve it. By introducing the concept of Picard-complete and using the sandwich theorem in the sense of w-convergence established in [4], we first prove some fixed point theorems of order-Lipschitz mappings in Banach algebras with non-normal cones. Motivated by [3], we reconsider the case with normal cones, and we obtain a fixed point theorem of order-Lipschitz mappings in Banach algebras under the assumption that $r(k) < 1$ by showing that there exists some n_0 such that T^{n_0} is a Banach contraction in $(E, \|\cdot\|_0)$, where $\|\cdot\|_0$ is a newly introduced norm which is equivalent to $\|\cdot\|$; see Lemma 5. In addition, some suitable examples are presented to show the usability of our theorems.

2 Preliminaries and lemmas

A Banach space $(E, \|\cdot\|)$ is called a Banach algebra [5] if there exists a multiplication in E such that, for each $x, y, z \in E$ and $a \in \mathbb{R}$, the following conditions are satisfied: (I) $(xy)z = x(yz)$; (II) $x(y + z) = xy + xz$ and $(x + y)z = xz + yz$; (III) $a(xy) = (ax)y = x(ay)$; (IV) $\|xy\| \leq \|x\|\|y\|$. If there exists some $e \in E$ such that $ex = xe = x$ for each $x \in E$ then e is called a unit (*i.e.*, a multiplicative identity) of E. A nonempty closed subset P of a Banach space $(E, \|\cdot\|)$ is a cone [5, 6] if it is such that the following conditions are satisfied: (V) $ax + by \in P$ for each $x, y \in P$ and each $a, b \geq 0$; (VI) $P \cap (-P) = \{\theta\}$, where θ is the zero element of E. A nonempty closed subset P of a Banach algebra $(E, \|\cdot\|)$ is a cone [1, 5] if it is such that (V) and (VI) are satisfied and (VII) $\{e\} \subset P$ and $P^2 = PP \subset P$.

Each cone P of a Banach space E determines a partial order $\preceq$ on E by $x \preceq y \Leftrightarrow y - x \in P$ for each $x, y \in X$. For each $u_0, v_0 \in E$ with $u_0 \preceq v_0$, we set $[u_0, v_0] = \{u \in E : u_0 \preceq u \preceq v_0\}$, $[u_0, +\infty) = \{x \in E : u_0 \preceq x\}$ and $(-\infty, v_0] = \{x \in E : x \preceq v_0\}$. A cone P is solid [5, 6] if $\operatorname{int} P \neq \emptyset$, where $\operatorname{int} P$ denotes the interior of P. For each $x, y \in E$ with $y - x \in \operatorname{int} P$, we write $x \ll y$.

Definition 1 Let P be a solid cone of a Banach space E, $\{x_n\} \subset E$ and $D \subset E$.

(i) the sequence $\{x_n\}$ is w-convergent [4, 7] if for each $\epsilon \in \operatorname{int} P$, there exist some positive integer n_0 and $x \in E$ such that $x - \epsilon \ll x_n \ll x + \epsilon$ for each $n \geq n_0$ (denote $x_n \xrightarrow{w} x$ and x is called a w-limit of $\{x_n\}$);

(ii) the sequence $\{x_n\}$ is w-Cauchy if for each $\epsilon \in \operatorname{int} P$, there exists some positive integer n_0 such that $-\epsilon \ll x_n - x_m \ll \epsilon$ for each $m, n \geq n_0$, *i.e.*, $x_n - x_m \xrightarrow{w} \theta$ $(m, n \to \infty)$;

(iii) the subset D is w-closed if for each $\{x_n\} \subset D$, $x_n \xrightarrow{w} x$ implies $x \in D$.

Lemma 1 *Let P be a solid cone of a Banach space E, $\{x_n\}$ a w-convergent sequence of E and $u_0, v_0 \in E$ with $u_0 \preceq v_0$. Then $\{x_n\}$ has a unique w-limit, and the partial order intervals $[u_0, v_0]$, $[u_0, +\infty)$ and $(-\infty, v_0]$ are w-closed.*

Proof Suppose that there exists $x, y \in E$ such that $x_n \xrightarrow{w} x$ and $x_n \xrightarrow{w} y$. From Definition 1 we find that, for each $\epsilon \in \operatorname{int} P$, there exists a positive integer n_0 such that $x - \epsilon \ll x_n \ll x + \epsilon$ and $y - \epsilon \ll x_n \ll y + \epsilon$ for each $n \geq n_0$. This forces that $x - y - 2\epsilon \ll x_n - x_n = \theta \ll x - y + 2\epsilon$ for each $n \geq n_0$. So we have $-2\epsilon \ll x - y \ll 2\epsilon$, which together with the arbitrary property of ϵ implies that $x = y$. This shows that $\{x_n\}$ has a unique w-limit.

Let $\{x_n\}$ be a sequence of $[u_0, v_0]$ such that $x_n \xrightarrow{w} x$. For each $\epsilon \in \operatorname{int} P$, there exists a positive integer n_0 such that $x - \epsilon \ll x_n \ll x + \epsilon$ for each $n \geq n_0$. Thus we get

$$\theta \preceq x_n - u_0 \ll x - u_0 + \epsilon, \qquad \theta \preceq v_0 - x_n \ll v_0 - x + \epsilon, \quad \forall n \geq n_0, \tag{3}$$

which together with the arbitrary property of ϵ implies that $u_0 \preceq x$ and $x \preceq v_0$, *i.e.*, $x \in [u_0, v_0]$. This shows that $[u_0, v_0]$ is w-closed. Similarly, we can show $[u_0, +\infty)$ and $(-\infty, v_0]$ are w-closed. The proof is complete. □

A cone P of a Banach space E is normal if there is some positive number N such that $x, y \in E$ and $\theta \preceq x \preceq y$ implies that $\|x\| \leq N\|y\|$, and the minimal N is called a normal constant of P. Note that an equivalent condition of a normal cone is that $\inf\{\|x + y\| : x, y \in P \text{ and } \|x\| = \|y\| = 1\} > 0$, then it is not hard to conclude that a cone P is non-normal if and only if there exist $\{u_n\}, \{v_n\} \subset P$ such that $u_n + v_n \xrightarrow{\|\cdot\|} \theta \nRightarrow u_n \xrightarrow{\|\cdot\|} \theta$. This implies that the sandwich theorem does not hold in the sense of norm-convergence. Recently, without using the normality of P Li and Jiang [4] proved the following sandwich theorem in the sense of w-convergence, which is very important for our further discussions.

Lemma 2 (see [4]) *Let P be a solid cone of a Banach space $(E, \|\cdot\|)$ and $\{x_n\}, \{y_n\}, \{z_n\} \subset E$ with $x_n \preceq y_n \preceq z_n$ for each n. If $x_n \xrightarrow{w} z$ and $z_n \xrightarrow{w} z$, then $y_n \xrightarrow{w} z$.*

Lemma 3 (see [7]) *Let P be a solid cone of a Banach space $(E, \|\cdot\|)$ and $x_n \subset E$. Then $x_n \xrightarrow{\|\cdot\|} x$ implies $x_n \xrightarrow{w} x$. Moreover, if P is normal then $x_n \xrightarrow{w} x \Leftrightarrow x_n \xrightarrow{\|\cdot\|} x$.*

Lemma 4 (see [3, 6]) *Let P be a solid cone of a Banach space $(E, \|\cdot\|)$. Then there is $\tau > 0$ such that, for each $x \in E$, there exist $y, z \in P$ with $\|y\| \leq \tau\|x\|$ and $\|z\| \leq \tau\|x\|$ such that $x = y - z$.*

Lemma 5 *Let P be a normal solid cone of a Banach space $(E,\|\cdot\|)$ and define $\|\cdot\|_0$ in E by*

$$\|x\|_0 = \inf_{u\in P}\bigl\{\|u\| : -u \preceq x \preceq u\bigr\}, \quad \forall x\in E. \tag{4}$$

Then $\|\cdot\|_0$ is equivalent to $\|\cdot\|$ and $(E,\|\cdot\|_0)$ is a Banach space.

Proof It follows from Lemma 4 that there is a $\tau > 0$ such that, for each $x \in E$, there exist $y,z \in P$ with $\|y\| \leq \tau\|x\|$ and $\|z\| \leq \tau\|x\|$ such that $x = y - z$, and so we have

$$-(y+z) \preceq x \preceq y+z. \tag{5}$$

Then it is clear that, for each $x \in E$, there exists $u \in P$ such that

$$-u \preceq x \preceq u, \tag{6}$$

and hence the definition of $\|\cdot\|_0$ is meaningful. It is easy to check that $\|\cdot\|_0$ is a norm in E. For each $x \in E$, by (6) and the normality of P, we get $\|x\| \leq \|x+u\| + \|u\| \leq (2N+1)\|u\|$, and hence $\|x\| \leq (2N+1)\inf_{u\in P}\|u\| = (2N+1)\|x\|_0$ by (4). On the other hand, by (5) we get $\|x\|_0 \leq \|y+z\| \leq 2\tau\|x\|$ for each $x \in E$. Thus we have $\frac{\|x\|}{2N+1} \leq \|x\|_0 \leq 2\tau\|x\|$ for each $x \in E$. This shows that $\|\cdot\|$ is equivalent to $\|\cdot\|_0$ and hence $(E,\|\cdot\|_0)$ is a Banach space. The proof is complete. □

Let P be a cone of a Banach space E and $T : E \to E$. For each $x_0 \in E$, set $O(T,x_0) = \{x_n\}$, where $\{x_n\}$ is the Picard iterative sequence (*i.e.*, $x_n = T^n x_0$ for each n).

Definition 2 Let P be a solid cone of a Banach space $(E,\|\cdot\|)$, $x_0 \in E$ and $T : E \to E$. If the Picard iterative sequence $O(T,x_0)$ is w-convergent provided that it is w-Cauchy, then T is said to be Picard-complete at x_0. Moreover, if T is Picard-complete at each $x \in E$, then T is said to be Picard-complete on E.

Remark 2
(i) If $O(T,x_0)$ is w-convergent then T is certainly Picard-complete at x_0.
(ii) If P is normal then each mapping $T : E \to E$ is Picard-complete on E by Lemma 3.

3 Fixed point theorems

We first state and prove a fixed point result of order-Lipschitz mappings in Banach algebras with non-normal cones as follows.

Theorem 3 *Let P be a solid cone of a Banach algebra $(E,\|\cdot\|)$ and $u_0, v_0 \in E$ with $u_0 \preceq v_0$. Assume that $T : [u_0, v_0] \to E$ is a nondecreasing order-Lipschitz mapping with $k,l \in P$ such that (2) is satisfied. If $r(k) < 1$ and T is Picard-complete at u_0 and v_0, then T has a unique fixed point $x^* \in [u_0, v_0]$. And for each $x_0 \in [u_0, v_0]$, we have $x_n \xrightarrow{w} x^*$, where $\{x_n\} = O(T,x_0)$.*

Proof Let $u_n = T^n u_0$ and $v_n = T^n v_0$ for each n. From (2) and the nondecreasing property of T on $[u_0, v_0]$ it follows that

$$u_0 \preceq u_1 \preceq u_2 \preceq \cdots \preceq u_n \preceq \cdots \preceq v_n \preceq \cdots \preceq v_2 \preceq v_1 \preceq v_0. \tag{7}$$

By (1) and $P^2 \subset P$, we get

$$\theta \preceq v_n - u_n \preceq k(v_{n-1} - u_{n-1}) \preceq k^2(v_{n-2} - u_{n-2}) \preceq \cdots \preceq k^n(v_0 - u_0), \quad \forall n. \tag{8}$$

Thus for each $m > n$, by (4) and (5) we have

$$\begin{aligned} &\theta \preceq u_m - u_n \preceq v_n - u_n \preceq k^n(v_0 - u_0), \\ &\theta \preceq v_n - v_m \preceq v_n - u_n \preceq k^n(v_0 - u_0). \end{aligned} \tag{9}$$

It follows from Gelfand's formula and $r(k) < 1$ that there exist n_0 and $\alpha \in [0,1)$ such that

$$\|k^n\| \le \alpha^n, \quad \forall n \ge n_0, \tag{10}$$

which implies that $k^n \xrightarrow{\|\cdot\|} \theta$. Note that $\|k^n(v_0 - u_0)\| \le \|k^n\| \|v_0 - u_0\|$ by (IV), then we obtain $k^n(v_0 - u_0) \xrightarrow{\|\cdot\|} \theta$. Moreover, by Lemma 3, we get

$$k^n(v_0 - u_0) \xrightarrow{w} \theta. \tag{11}$$

Thus it follows from (9), (11), and Lemma 2 that $\{u_n\}$ and $\{v_n\}$ are w-Cauchy sequences. Since T is Picard-complete at u_0 and v_0, there exist $u^*, v^* \in E$ such that

$$u_n \xrightarrow{w} u^*, \qquad v_n \xrightarrow{w} v^*. \tag{12}$$

Letting $n \to \infty$ in (8), by (11) and Lemma 2 we get $u^* = v^*$. Set $x^* = u^* = v^*$, then $x^* \in [u_0, v_0]$ by (7) and Lemma 1. For each $m > n$, by (7) we get

$$u_n \preceq u_m \preceq \cdots \preceq v_m \preceq v_n. \tag{13}$$

Letting $m \to \infty$ in (13), by (12) and Lemma 1 we have

$$u_n \preceq x^* \preceq v_n, \quad \forall n, \tag{14}$$

with together with the nondecreasing property of T on $[u_0, v_0]$ implies that

$$u_n = Tu_{n-1} \preceq Tx^* \preceq Tv_{n-1} \preceq v_n, \quad \forall n. \tag{15}$$

Letting $n \to \infty$ in (15), we get $x^* = Tx^*$ by (12), Lemma 1 and Lemma 2. Hence x^* is a fixed point of T.

For each $x_0 \in [u_0, v_0]$, let $x_n = T^n x_0$. It is clear that $u_1 = Tu_0 \preceq Tx_0 = x_1 \preceq Tv_0 = v_1$ since T is nondecreasing on $[u_0, v_0]$. Then by induction, we obtain

$$u_n \preceq x_n \preceq v_n, \quad \forall n,$$

which together with (12) and Lemma 2 implies that $x_n \xrightarrow{w} x^*$. Let $x \in [u_0, v_0]$ be another fixed point of T and $y_n = T^n x$. Similarly, we can show that $y_n \xrightarrow{w} x^*$. Note that $y_n = T^n x \equiv x$ implies that $y_n \xrightarrow{w} x$, then $x = x^*$ by Lemma 1. Hence T has a unique fixed point $x^* \in [u_0, v_0]$. The proof is complete. □

Example 1 Let $E = C^1_{\mathbb{R}}[0,1]$ be endowed with the norm $\|u\| = \|u\|_\infty + \|u'\|_\infty$ and $P = \{u \in E : u(t) \geq 0, \forall t \in [0,1]\}$, where $\|u\|_\infty = \max_{t\in[0,1]} u(t)$ for each $u \in C_{\mathbb{R}}[0,1]$. Define a multiplication in E by $(xy)(t) = x(t)y(t)$ for each $x, y \in E$ and $t \in [0,1]$. Clearly, $(E, \|\cdot\|)$ is a Banach algebra with a unit $e(t) \equiv 1$ and P is a non-normal solid cone.

Let $Tx = x^2$, $u_0 = \theta$ and $v_0(t) \equiv a$, where $a \in [0, \frac{1}{2})$. Clearly, $Tu_0 \preceq u_0$ and $Tv_0 \preceq v_0$. For each $x, y \in [u_0, v_0]$ with $y \preceq x$, we have $0 \leq (Tx)(t) - (Ty)(t) = x^2(t) - y^2(t) = (x(t) + y(t))(x(t) - y(t)) \leq k(x(t) - y(t))$ for each $t \in [0,1]$, where $k = 2ae$. This shows that $T : [u_0, v_0] \to E$ is a nondecreasing order-Lipschitz mapping with $r(k) = 2a < 1$. Let $\{u_n\}$ and $\{v_n\}$ be the Picard iterative sequences of u_0 and v_0, then $u_n = \theta$ and $v_n(t) \equiv a^{2^n}$ for each n, and so $\|u_n\| \equiv 0$ and $\|v_n\| = a^{2^n}$ for each n, which forces that $u_n \xrightarrow{\|\cdot\|} \theta$ and $v_n \xrightarrow{\|\cdot\|} \theta$. This together with (i) of Remark 2 and Lemma 3 implies that T is Picard-complete at u_0 and v_0. Hence by Theorem 3, T has a unique fixed point in $[u_0, v_0]$.

However, Theorems 1 and 2 are not applicable here since P is non-normal.

In analogy to the proof of Theorem 3, we can prove the following fixed point theorem of order-Lipschitz mappings in Banach space.

Theorem 4 *Let P be a solid cone of a Banach space $(E, \|\cdot\|)$ and $u_0, v_0 \in E$ with $u_0 \preceq v_0$. Assume that $T : [u_0, v_0] \to E$ is a nondecreasing order-Lipschitz mapping with $k \in [0,1)$ such that* (2) *is satisfied. If T is Picard-complete at u_0 and v_0, then T has a unique fixed point $x^* \in [u_0, v_0]$. And for each $x_0 \in [u_0, v_0]$, we have $x_n \xrightarrow{w} x^*$, where $\{x_n\} = O(T, x_0)$.*

Corollary 1 *Let P be a solid cone of a Banach space $(E, \|\cdot\|)$ and $u_0, v_0 \in E$ with $u_0 \preceq v_0$. Assume that $T : [u_0, v_0] \to E$ is an order-Lipschitz mapping with $l \in [0, +\infty)$ and $k \in [0,1)$ such that* (2) *is satisfied. If A is Picard-complete at u_0 and v_0, where $Ax = \frac{Tx+lx}{1+l}$ for each $x \in E$, then T has a unique fixed point $x^* \in [u_0, v_0]$.*

Proof Set $k_1 = \frac{l+k}{1+l}$. By (1) and (2) we get

$$u_0 \preceq Au_0, \qquad Av_0 \preceq v_0,$$

$$\theta \preceq Ax - Ay \preceq k_1(x - y), \quad \forall x, y \in [u_0, v_0], \qquad y \preceq x,$$

which indicates that $A : [u_0, v_0] \to E$ is a nondecreasing order-Lipschitz mapping. Note that $k_1 \in [0,1)$ and A is Picard-complete at u_0 and v_0, then A has a unique fixed point $x^* \in [u_0, v_0]$ by Theorem 4. Thus we have $Tx^* + lx^* = x^* + lx^*$ and so $Tx^* = x^*$. Let $x \in [u_0, v_0]$ be another fixed point of T, then $Tx = x$ and hence $Ax = x$. Moreover, by the uniqueness of fixed point of A in $[u_0, v_0]$, we get $x = x^*$. Hence x^* is the unique fixed point of T in $[u_0, v_0]$. The proof is complete. □

Remark 3 By (ii) of Remark 2, Theorem 1 immediately follows from Corollary 1, which indeed improves Theorem 1 since the normality of P has been removed.

Motivated by [3], we reconsider the case with normal cones, and we obtain the following fixed point result.

Theorem 5 *Let P be a normal solid cone of a Banach algebra $(E, \|\cdot\|)$ and $T : E \to E$ an order-Lipschitz mapping with $l = k \in P$. If $r(k) < 1$, then T has a unique fixed point $x^* \in E$. And for each $x_0 \in E$, we have $x_n \xrightarrow{\|\cdot\|} x^*$, where $\{x_n\} = O(T, x_0)$.*

Proof Since P is solid, it follows from (6) that, for each $x, y \in E$, there exists $u \in P$ such that

$$-u \preceq x - y \preceq u, \tag{16}$$

and so

$$\frac{x+y-u}{2} \preceq x, \qquad \frac{x+y-u}{2} \preceq y. \tag{17}$$

We observe that

$$-k^n u \preceq T^n x - T^n y \preceq k^n u, \quad \forall x, y \in E, \forall n, \tag{18}$$

where $u \in P$ satisfies (16). By (1) and (17), for each $x, y \in E$ we have

$$-k\left(\frac{x-y+u}{2}\right) \preceq Tx - T\left(\frac{x+y-u}{2}\right) \preceq k\left(\frac{x-y+u}{2}\right) \tag{19}$$

and

$$-k\left(\frac{y-x+u}{2}\right) \preceq Ty - T\left(\frac{x+y-u}{2}\right) \preceq k\left(\frac{y-x+u}{2}\right),$$

which can be rewritten as

$$-k\left(\frac{y-x+u}{2}\right) \preceq T\left(\frac{x+y-u}{2}\right) - Ty \preceq k\left(\frac{y-x+u}{2}\right). \tag{20}$$

Adding (19) and (20), by (II), we get

$$-ku \preceq Tx - Ty \preceq ku, \quad \forall x, y \in E,$$

which implies that (10) holds for $n = 1$. Suppose that (18) holds for n, then for each $x, y \in E$, we get

$$\frac{T^n x + T^n y - k^n u}{2} \preceq T^n x, \qquad \frac{T^n x + T^n y - k^n u}{2} \preceq T^n y.$$

Moreover, by (1), for each $x, y \in E$ we get

$$\begin{aligned} -k\left(\frac{T^n x - T^n y + k^n u}{2}\right) &\preceq T^{n+1} x - T\left(\frac{T^n x + T^n y - k^n u}{2}\right) \\ &\preceq k\left(\frac{T^n x - T^n y + k^n u}{2}\right) \end{aligned} \tag{21}$$

and

$$-k\left(\frac{T^n y - T^n x + k^n u}{2}\right) \preceq T^{n+1} y - T\left(\frac{T^n x + T^n y - k^n u}{2}\right) \preceq k\left(\frac{T^n y - T^n x + k^n u}{2}\right),$$

which can be rewritten as

$$-k\left(\frac{T^ny-T^nx+k^nu}{2}\right)\preceq T\left(\frac{T^nx+T^ny-k^nu}{2}\right)-T^{n+1}y$$
$$\preceq k\left(\frac{T^ny-T^nx+k^nu}{2}\right). \tag{22}$$

Adding (21) and (22), by (II) we get $-k^{n+1}u\preceq T^{n+1}x-T^{n+1}y\preceq k^{n+1}u$ for each $x,y\in E$, and hence (18) holds for $n+1$. Therefore (18) holds true by induction. For each $x,y\in E$ and arbitrary $u\in P$ such that (16) is satisfied, by (18) and (IV) we get $\|T^nx-T^ny\|_0\leq\|k^nu\|\leq\|k^n\|\|u\|$ for each n. Then by (4), (10), and the arbitrariness property of u, we have

$$\begin{aligned}\|T^nx-T^ny\|_0&\leq\|k^n\|\inf_{u\in P}\{\|u\|:-u\preceq x-y\preceq u\}\\&=\|k^n\|\|x-y\|_0\leq\alpha^n\|x-y\|_0,\quad\forall n\geq n_0,\end{aligned}$$

which together with $\alpha^{n_0}<1$ implies that $T^{n_0}:E\to E$ is a Banach contraction. Note that $(E,\|\cdot\|_0)$ is a Banach space by Lemma 5, then by the Banach contraction principle, T^{n_0} has a unique fixed point $x^*\in E$ (*i.e.*, $T^{n_0}x^*=x^*$). It is clear that $T^{n_0}(Tx^*)=T^{n_0+1}x^*=T(T^{n_0}x)=Tx^*$, *i.e.*, Tx^* is a fixed point of T^{n_0}, then $x^*=Tx^*$ by the uniqueness of fixed point of T^{n_0}. Let $x\in E$ be another fixed point of T. Then x is also a fixed point of T^{n_0}, and so $x=x^*$ by the unique existence of fixed point of T^{n_0}. Hence x^* is the unique fixed point of T.

For each $x_0\in E$, let $x_n=T^nx_0$ and $y_n^i=T^{nn_0}x_i$ for each n and each $0\leq i\leq n_0-1$. It is clear that $y_n^i\overset{\|\cdot\|_0}{\to}x^*$ for each i, and so for each $\varepsilon>0$, there exists a positive integer m_0^i such that

$$\|y_n^i-x^*\|_0<\varepsilon,\quad\forall n\geq m_0^i.$$

Set $m_0=\max_{1\leq i\leq n_0-1}m_0^i$, then for each i, we get

$$\|y_n^i-x^*\|_0<\varepsilon,\quad\forall n\geq m_0,$$

which together with $\{x_n\}=\bigcup_{i=0}^{n_0-1}\{y_n^i\}$ implies that

$$\|x_n-x^*\|_0<\varepsilon,\quad\forall n\geq m_0n_0.$$

This shows that $x_n\overset{\|\cdot\|_0}{\to}x^*$ and hence $x_n\overset{\|\cdot\|}{\to}x^*$ since $\|\cdot\|$ and $\|\cdot\|_0$ are equivalent by Lemma 5. The proof is complete. □

Remark 4 Theorem 5 partially improves Theorem 2 since the norm condition $\|k\|<1$ is replaced by the spectral radius condition $r(k)<1$.

The following example will show Theorem 5 is more applicable than many other fixed point results.

Example 2 Let $E=P=\mathbb{R}_+^2=\{x=(x_1,x_2):x_1,x_2\geq 0\}$ with the norm $\|x\|=|x_1|+|x_2|$. Clearly, P is a normal solid cone. Define a multiplication in E by

$$xy=(x_1y_1,x_1y_2+x_2y_1),\quad\forall x=(x_1,x_2),y=(y_1,y_2)\in E,$$

then E is a Banach algebra with $e = (1,0)$. Define a mapping $T : E \to E$ by

$$T(x_1,x_2) = \bigl(\ln(a+x_1), \arctan(b+x_2) + cx_1\bigr), \quad \forall x \in E,$$

where $a > 1$, $b \geq \sqrt{a-1}$ and $c > 0$, then for each $x, y \in E$ with $y \preceq x$, by the Lagrange mean value theorem, we have

$$\begin{aligned}
-k(x-y) \preceq \theta \preceq Tx - Ty \\
&= \bigl(\ln(a+x_1) - \ln(a+y_1), \arctan(b+x_2) - \arctan(b+y_2) + c(x_1-y_1)\bigr) \\
&\preceq \left(\frac{x_1-y_1}{a}, \frac{x_2-y_2}{1+b^2} + c(x_1-y_1)\right) \\
&\preceq \left(\frac{x_1-y_1}{a}, \frac{x_2-y_2}{a} + c(x_1-y_1)\right) \\
&= \left(\frac{1}{a}, c\right)(x_1-y_1, x_2-y_2) = k(x-y),
\end{aligned}$$

where $k = (\frac{1}{a}, c) \in P$. This implies that T is an order-Lipschitz mapping with $k = l = (\frac{1}{a}, c)$. Note that $k^2 = (\frac{1}{a^2}, \frac{2c}{a})$ and $k^3 = (\frac{1}{a^3}, \frac{3c}{a^2})$, then by induction we obtain $k^n = (\frac{1}{a^n}, \frac{nc}{a^{n-1}})$ for each n. Moreover, by Gelfand's formula, we get

$$r(k) = \lim_{n\to\infty} \bigl\|k^n\bigr\|^{\frac{1}{n}} = \lim_{n\to\infty}\left(\frac{1}{a^n} + \frac{nc}{a^{n-1}}\right)^{\frac{1}{n}} = \frac{1}{a}\lim_{n\to\infty}\left(1 + \frac{nc}{a}\right)^{\frac{1}{n}} = \frac{1}{a} < 1.$$

Hence by Theorem 5, T has a unique fixed point.

In the case that $c > 1 - \frac{1}{a}$, we get $\|k\| = \frac{1}{a} + c > 1$, and hence Theorem 2 is not applicable even taking k as a linear bounded mapping.

In the case that $c > 2$, we get $\|Tx - Ty\| \geq c|x_1 - y_1| > 2|x_1 - y_1| \geq |x_1 - y_1| + |x_2 - y_2| = \|x - y\|$ for each $x, y \in E$ with $|x_1 - y_1| \geq |x_2 - y_2|$, and hence the Banach contraction principle is not applicable.

In the case that $c > 1$, we get $\arctan(b+x_2) - \arctan(b+y_2) + c(x_1 - y_1) \geq c(x_2 - y_2) > x_2 - y_2$ for each $x, y \in E$ with $y \preceq x$ and $x_1 - y_1 \geq x_2 - y_2$. This implies that there does not exist $l \in [0,1)$ such that $Tx - Ty \preceq l(x-y)$. Consequently, Theorem 1 is not applicable.

Remark 5 The normality of P is essential for the completeness of $(E, \|\cdot\|_0)$ (see Lemma 5), which leads to that the Banach contraction principle is applicable in Theorem 5. Naturally, one may wonder whether the normality of P in Theorem 5 could be removed by the method used in Theorem 1 or other methods.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors have contributed in this work on an equal basis. All authors read and approved the final manuscript.

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References

1. Xu, S, Radenović, S: Fixed point theorems of generalized Lipschitz mappings on cone metric spaces over Banach algebras without assumption of normality. Fixed Point Theory Appl. **2014**, Article ID 102 (2014)
2. Sun, J: Iterative solutions for a class of nonlinear operator equations. Chinese J. Engrg. Math. **6**, 12-17 (1989)
3. Krasnoselskii, MA, Zabreiko, PP: Geometrical Methods of Nonlinear Analysis. Springer, Berlin (1984)
4. Li, Z, Jiang, S: Common fixed point theorems of contractions in partial cone metric spaces over nonnormal cones. Abstr. Appl. Anal. **2014**, Article ID 653841 (2014)
5. Rudin, W: Functional Analysis, 2nd edn. McGraw-Hill, New York (1991)
6. Deimling, K: Nonlinear Functional Analysis. Springer, Berlin (1985)
7. Jiang, S, Li, Z: Extensions of Banach contraction principle to partial cone metric spaces over a non-normal solid cone. Fixed Point Theory Appl. **2013**, Article ID 250 (2013)