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Convergence theorems for finite families of total asymptotically nonexpansive mappings in hyperbolic spaces

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Abstract

In this paper, using a multistep iterative scheme, we establish strong and Δ-convergence theorems for finite families of total asymptotically quasi-nonexpansive mappings in uniformly convex hyperbolic spaces. We then establish Δ- and polar convergence theorems for finite families of total asymptotically nonexpansive mappings in CAT(0) spaces. These new theorems are extensions, improvements, and generalizations of some recently announced results by many authors.

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Keywords: total asymptotically quasi-nonexpansive mappings; common fixed point; uniformly convex hyperbolic space; Δ-convergence

1 Introduction

Let (X,d) be a metric space, $x,y \in X$, and $d(x,y) = l$. A geodesic path from x to y is an isometry $c : [0,l] \to c([0,l]) \subset X$ such that $c(0) = x$ and $c(l) = y$. The image of a geodesic path between two points is called a geodesic segment. A metric space (X,d) is called a geodesic space if every two points of X are joined by a geodesic segment.

A geodesic triangle represented by $\Delta(x,y,z)$ in a geodesic space consists of three points x, y, z and the three segments joining each pair of the points. A comparison triangle of a geodesic triangle $\Delta(x,y,z)$, denoted by $\overline{\Delta}(x,y,z)$ or $\Delta(\overline{x},\overline{y},\overline{z})$, is a triangle in the Euclidean space $\mathbb{R}^2$ such that $d(x,y) = d_{\mathbb{R}^2}(\overline{x},\overline{y})$, $d(x,z) = d_{\mathbb{R}^2}(\overline{x},\overline{z})$, and $d(y,z) = d_{\mathbb{R}^2}(\overline{y},\overline{z})$. This is obtainable by using the triangle inequality, and it is unique up to isometry on $\mathbb{R}^2$. A geodesic segment joining two points x, y in a geodesic space X is represented by $[x,y]$. Every point z in the segment is represented by $\alpha x \oplus (1-\alpha)y$ where $\alpha \in [0,1]$, that is, $[x,y] := \{\alpha x \oplus (1-\alpha)y : \alpha \in [0,1]\}$. A subset K of a metric space X is called convex if for all $x,y \in K$, $[x,y] \subset K$. A geodesic space is called a CAT(0) space if for every geodesic triangle Δ and its comparison $\overline{\Delta}$, the following inequality is satisfied: $d(x,y) \leq d_{\mathbb{R}^2}(\overline{x},\overline{y})$ for all $x,y \in \Delta$ and $\overline{x},\overline{y} \in \overline{\Delta}$. Examples of CAT(0) spaces include the $\mathbb{R}$-tree, Hadamard manifold, and Hilbert ball equipped with hyperbolic metric. For more details on these spaces, see, for example, [1–5].

A geodesic space (X,d) is called hyperbolic (see [6, 7]) if, for any $x,y,z \in X$,

$$d\left(\frac{1}{2}z \oplus \frac{1}{2}x, \frac{1}{2}z \oplus \frac{1}{2}y\right) \leq \frac{1}{2}d(x,y).$$

The class of hyperbolic spaces include the normed spaces, $CAT(0)$ spaces, and some others. The following is an example of a hyperbolic space that is not a normed space.

Example 1.1 Let $\mathbb{D}$ be a unit disc in a complex plane $\mathbb{C}$. Define $d : \mathbb{D} \times \mathbb{D} \to \mathbb{R}$ by

$$d(z,w) = \log\left(\frac{1 + |\frac{z-w}{1-z\overline{w}}|}{1 - |\frac{z-w}{1-z\overline{w}}|}\right).$$

Then $(\mathbb{D},d)$ is a complete hyperbolic metric space.

It is then clear that the class of hyperbolic spaces is more general than the class of normed spaces.

Definition 1.2 Let (X,d) be a hyperbolic metric space. Then X is called uniformly convex if for all $a \in X$, $r > 0$, and $\epsilon > 0$,

$$\delta_a(r,\epsilon) = \inf\left\{1 - \frac{1}{r}d\left(\frac{1}{2}x \oplus \frac{1}{2}y, a\right); d(x,a) \leq r, d(y,a) \leq r, d(x,y) \geq r\epsilon\right\} > 0.$$

Let (X,d) be a metric space. A self-mapping $T : X \to X$ is called *nonexpansive* if $d(Tx,Ty) \leq d(x,y)$ for all $x,y \in X$ and *quasi-nonexpansive* if $F(T) := \{x \in X : Tx = x\} \neq \emptyset$ and $d(Tx,p) \leq d(x,p)$ for all $x \in X$ and $p \in F(T)$. The class of quasi-nonexpansive mappings properly contains the class of nonexpansive mappings with fixed points; see, for example, [8].

A mapping T is called *asymptotically nonexpansive* [9] if there exists a sequence $\{k_n\} \subset [1,\infty)$ such that $k_n \to 1$ as $n \to \infty$ and, for every $n \in \mathbb{N}$,

$$d\bigl(T^n x, T^n y\bigr) \leq k_n d(x,y) \quad \text{for all } x,y \in X.$$

If $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1,\infty)$ such that $k_n \to 1$ as $n \to \infty$ and, for $n \in \mathbb{N}$,

$$d\bigl(T^n x, p\bigr) \leq k_n d(x,p) \quad \text{for all } x \in X \text{ and } p \in F(T),$$

then T is called an *asymptotically quasi-nonexpansive mapping*. A mapping T is called *total asymptotically nonexpansive* if there exist infinitesimal real sequences $\{u_n\}$ and $\{v_n\}$ of nonnegative numbers (*i.e.*, $u_n, v_n \to 0$ as $n \to \infty$) and a strictly increasing function $\psi : [0,\infty) \to [0,\infty)$ with $\psi(0) = 0$ such that

$$d\bigl(T^n x, T^n y\bigr) \leq d(x,y) + u_n \psi\bigl(d(x,y)\bigr) + v_n \quad \text{for all } x,y \in X.$$

A mapping T is *total asymptotically quasi-nonexpansive* if $F(T) \neq \emptyset$ and there exist infinitesimal real sequences $\{u_n\}$ and $\{v_n\}$ and a strictly increasing function $\psi : [0,\infty) \to$

$[0,\infty)$ with $\psi(0) = 0$ such that

$$d(T^n x, p) \leq d(x,p) + u_n \psi\big(d(x,p)\big) + v_n \quad \text{for all } x \in X, p \in F(T).$$

The concept of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [9] as an important generalization of nonexpansive mappings. Alber *et al.* [10] introduced the class of total asymptotically nonexpansive mappings that generalizes several classes of maps that are extensions of asymptotically nonexpansive mappings. These classes of maps were extensively studied by many authors (see, *e.g.*, [9, 11–16], to list a few) by virtue of important generalizations of nonexpansive mappings. Example 1 of [17] shows that the class of total asymptotically nonexpansive mappings properly contains the class of asymptotically nonexpansive mappings.

Remark 1.3 In what follows, for a closed convex and nonempty subset K of a uniformly convex metric space X and a bounded sequence $\{x_n\}$, we shall write $x_n \rightharpoonup x$ if and only if $\phi(x) = \inf_{y\in K} \phi(y)$ where $\phi(y) = \limsup_{n\to\infty} d(x_n, y)$; see, for example, [11].

A mapping T is said to be demiclosed at zero if for any sequence $\{x_n\}$ in X such that $x_n \rightharpoonup x$ and $d(x_n, Tx_n) \to 0$ as $n \to \infty$, we have $Tx = x$.

Let K be a nonempty subset of a metric space X, and let $\{x_n\}$ be any bounded sequence in K. For $x \in X$, define $r(x, \{x_n\}) := \limsup_{n\to\infty} d(x_n, x)$. The asymptotic radius of the sequence $\{x_n\}$ in K denoted by $r(K, \{x_n\})$ is defined by $r(K, \{x_n\}) := \inf\{r(x, \{x_n\}) : x \in K\}$. A point z is called an asymptotic center of a sequence $\{x_n\}$ in K if $r(z, \{x_n\}) = r(K, \{x_n\})$. The set of all asymptotic centers of the sequence $\{x_n\}$ in K is denoted by $A(K, \{x_n\})$. The asymptotic radius and asymptotic center of the sequence $\{x_n\}$ with respect to the whole space are denoted by $r(\{x_n\})$ and $A(\{x_n\})$, respectively. It is known that $r(\{x_n\}) = 0$ if and only if $\lim_{n\to\infty} x_n = x$.

A sequence $\{x_n\}$ in X is said to be Δ-convergent to a point x if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. This is written as $\Delta\text{-}\lim_{n\to\infty} x_n = x$. A sequence $\{x_n\}$ is said to polar converge to a point $x \in X$ (see [18]) if for every $y \in X$ such that $y \neq x$, there exists $N_y \in \mathbb{N}$ such that $d(x_n, x) < d(x_n, y)$ for all $n \geq N_y$. A sequence $\{x_n\}$ is said to converge Δ-strongly to a point x if the limit $\lim_{n\to\infty} d(x_n, x)$ exists and for any $y \neq x$, $\lim_{n\to\infty} d(x_n x) \leq \liminf_{n\to\infty} d(x_n, y)$.

The notion of polar convergence was introduced by Devillanova *et al.* [18]. They discussed various relations between polar convergence and Δ-convergence in metric spaces. By definition, if $\{x_n\}$ Δ-converges strongly to x, then the limit $\lim_{n\to\infty} d(x_n, x)$ exists. Thus, for any subsequence $\{x_{n_k}\}$ of $\{x_n\}$, $\lim_{n\to\infty} d(x_n, x) = \lim_{k\to\infty} d(x_{n_k}, x)$. This implies that x is an asymptotic center of $\{x_{n_k}\}$, and hence $\{x_n\}$ Δ-converges to x.

Chang *et al.* [11] established relations between the weak convergence and Δ-convergence in their attempt to establish the demiclosedness principle for total asymptotically nonexpansive mappings.

Recently, new fixed point results were studied by many authors in the setting of hyperbolic and CAT(0) metric spaces; see, for example, [11, 12, 16, 19–24], and the references therein.

In 1976, Lim [25] introduced the concept of Δ-convergence in general metric spaces. In 2008, Kirk and Panyanak [24] studied Δ-convergence in the setting of hyperbolic and

CAT(0) spaces. Basarir and Sahin [19] studied a multistep iterative process for fixed points of generalized nonexpansive mappings in a CAT(0) space. They established the demicloseness principle for this class of maps in a CAT(0) space. Kim *et al.* [22] proved strong and Δ-convergence theorems for generalized nonexpansive mappings in hyperbolic spaces. Chang *et al.* [11] proved strong and Δ-convergence theorems for total asymptotically nonexpansive mappings in CAT(0) spaces. They also established the demiclosedness principle for this class of maps in a CAT(0) space.

In 1936, Markov [26] (see also Kakutani [27]) showed that if a commuting family of bounded *linear* transformations T_α, $\alpha \in \Delta$ (Δ an arbitrary index set), of a locally convex Hausdorff space E into itself leaves some nonempty *compact convex* subset K of E invariant, then the family has at least one common fixed point in the set K.

Chidume and the author [13] introduced the scheme

$$\begin{cases} x_1 \in K, \\ x_{n+1} = P[(1-\alpha_{1n})x_n + \alpha_{1n}T_1(PT_1)^{n-1}y_{n+m-2}], \\ y_{n+m-2} = P[(1-\alpha_{2n})x_n + \alpha_{2n}T_2(PT_2)^{n-1}y_{n+m-3}], \\ \vdots \\ y_n = P[(1-\alpha_{mn})x_n + \alpha_{mn}T_m(PT_m)^{n-1}x_n], \qquad n \geq 1, m \geq 2, \end{cases}$$

and studied the convergence of this scheme to a common fixed point of finite family of nonself-asymptotically nonexpansive mappings in a uniformly convex Banach space.

Let $\{\alpha_n\}$ be a real sequence in $[\epsilon, 1-\epsilon]$, $\epsilon \in (0,1)$. Let $T_1, T_2, \ldots, T_m : K \to K$ be a family of mappings. Define the sequence $\{x_n\}$ by

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)x_n \oplus \alpha_n T_1^n x_n, \qquad n \geq 1, m = 1, \\ x_{n+1} = (1-\alpha_n)x_n \oplus \alpha_n T_1^n y_{n+m-2}, \\ y_{n+m-2} = (1-\alpha_n)x_n \oplus \alpha_n T_2^n y_{n+m-3}, \\ \vdots \\ y_n = (1-\alpha_n)x_n \oplus \alpha_n T_m^n x_n, \qquad n \geq 1, m \geq 2. \end{cases} \tag{1.1}$$

Our purpose in this paper to prove necessary and sufficient conditions for the strong convergence of the scheme defined by (1.1) to a common fixed point of finite family $T_1, T_2, \ldots, T_m$ of *total asymptotically quasi-nonexpansive mappings* in a complete hyperbolic space. We also prove Δ-convergence and polar convergence theorems for finite family of uniformly L-Lipschitzian *total asymptotically nonexpansive mappings* in a CAT(0) space. Our results generalized and improved some recent important results announced.

2 Preliminaries

In what follows, we shall use the following results.

Theorem 2.1 ([11]) *Let K be a closed and convex subset of a complete CAT(0) space X, and $T : K \to X$ be a uniformly L-Lipschitzian and total asymptotically nonexpansive mapping. Let $\{x_n\}$ be a bounded sequence in K such that $x_n \rightharpoonup x$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Then $x = Tx$.*

Lemma 2.2 ([5]) *Let E be a complete $CAT(0)$ space, $\{x_n\}$ be a bounded sequence in E with $A(\{x_n\}) = \{p\}$, and $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$. If the sequence $\{d(x_n, u)\}$ converges, then $p = u$.*

Lemma 2.3 ([5]) *Let E be a $CAT(0)$ space. Then*

$$d^2\big((1-\alpha)x \oplus \alpha y, a\big) \leq (1-\alpha)d^2(x,a) + \alpha d^2(y,a) - \alpha(1-\alpha)d^2(x,y)$$

for all $\alpha \in [0,1]$ and $x, y, a \in E$.

Lemma 2.4 ([28]) *Let E be a complete $CAT(0)$ space. Let K be a closed convex subset of E. If $\{x_n\}$ is a bounded sequences in K, then the asymptotic center of $\{x_n\}$ is in K.*

Lemma 2.5 ([29]) *Let (E, d) be a uniformly convex hyperbolic space. Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in E. For any $\lambda \in (0,1)$, if there exists $r \in [0,\infty)$ such that*

$$\limsup_{n\to\infty} d(x_n, a) \leq r, \qquad \limsup_{n\to\infty} d(y_n, a) \leq r, \quad \text{and} \quad \limsup_{n\to\infty} d\big((1-\lambda)x_n \oplus \lambda y_n, a\big) = r,$$

then $\lim_{n\to\infty} d(x_n, y_n) = 0$.

Lemma 2.6 ([24]) *Every bounded sequence in a complete $CAT(0)$ space has a Δ-convergent subsequence.*

Lemma 2.7 ([30]) *Let $\{\lambda_n\}$ and $\{\sigma_n\}$ be sequences of nonnegative real numbers such that $\lambda_{n+1} \leq \lambda_n + \sigma_n$ for all $n \geq 1$ and $\sum_{n=1}^{\infty} \sigma_n < \infty$. Then $\lim_{n\to\infty} \lambda_n$ exists. Moreover, if there exists a subsequence $\{\lambda_{n_j}\}$ of $\{\lambda_n\}$ such that $\lambda_{n_j} \to 0$ as $j \to \infty$, then $\lambda_n \to 0$ as $n \to \infty$.*

3 Main results

In this section, we state and prove the main results of this paper. In the sequel, we denote the set $\{1, 2, \ldots, m\}$ by I, and we always assume that $F := \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$.

Lemma 3.1 *Let (X, d) be a hyperbolic space, and K be a nonempty closed convex subset of X. Let $T_1, T_2, \ldots, T_m : K \to K$ be total asymptotically quasi-nonexpansive mappings with sequences $\{u_{in}\}_{n=1}^{\infty}$, $\{v_{in}\}_{n=1}^{\infty}$ and mappings $\psi_i : [0,\infty) \to [0,\infty)$ satisfying $\sum_{n=1}^{\infty} u_{in} < \infty$ and $\sum_{n=1}^{\infty} v_{in} < \infty$, $i \in I$. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequences in $[\epsilon, 1-\epsilon]$, $\epsilon \in (0,1)$. Assume there exist constants M_i, $\overline{M}_i$ such that $\psi_i(r_i) \leq M_i r_i$ for all $r_i \geq \overline{M}_i$, $i \in I$. Let $\{x_n\}$ be the sequence defined iteratively by*

$$\begin{cases} x_1 \in K, & \\ x_{n+1} = (1-\alpha_n)x_n \oplus \alpha_n T_1^n x_n, & n \geq 1, m = 1, \\ x_{n+1} = (1-\alpha_n)x_n \oplus \alpha_n T_1^n y_{n+m-2}, & \\ y_{n+m-2} = (1-\alpha_n)x_n \oplus \alpha_n T_2^n y_{n+m-3}, & \\ \vdots & \\ y_n = (1-\alpha_n)x_n \oplus \alpha_n T_m^n x_n, & n \geq 1, m \geq 2. \end{cases} \tag{3.1}$$

Then, $\{x_n\}$ is bounded, and the limits $\lim_{n\to\infty} d(x_n, x^*)$ and $\lim_{n\to\infty} d(x_n, F)$ exist.

Proof We start the proof by considering the case $m \geq 2$. Since ψ_i is increasing for each $i \in I$, $\psi_i(r_i) \leq \psi(\overline{M}_i)$ whenever $r_i \leq \overline{M}_i$, and, by hypothesis, $\psi_i(r_i) \leq M_i r_i$ when $r_i \geq \overline{M}_i$. In any case, $\psi_i(r_i) \leq \psi_i(\overline{M}_i) + M_i r_i$, $i \in I$. Now set $w_n := \sum_{i=1}^m u_{in} M_i$ and let $x^* \in F$. Then we have

$$\begin{aligned}
d\bigl(x_{n+1}, x^*\bigr) &= d\bigl((1-\alpha_n)x_n \oplus \alpha_n T_1^n y_{n+m-2}, x^*\bigr) \\
&\leq (1-\alpha_n) d\bigl(x_n, x^*\bigr) + \alpha_n d\bigl(T_1^n y_{n+m-2}, x^*\bigr) \\
&\leq (1-\alpha_n) d\bigl(x_n, x^*\bigr) + \alpha_n \bigl[d\bigl(y_{n+m-2}, x^*\bigr) \\
&\quad + u_{1n}\psi_1\bigl(d\bigl(y_{n+m-2}, x^*\bigr)\bigr) + v_{1n}\bigr] \\
&\leq (1-\alpha_n) d\bigl(x_n, x^*\bigr) + \alpha_n(1 + u_{1n}M_1) d\bigl(y_{n+m-2}, x^*\bigr) + \alpha_n u_{1n}\psi_1(\overline{M}_1) \qquad (3.2) \\
&\quad + \alpha_n v_{1n} \\
&\leq (1-\alpha_n) d\bigl(x_n, x^*\bigr) + \alpha_n(1 + u_{1n}M_1)\bigl[(1-\alpha_n) d\bigl(x_n, x^*\bigr) \\
&\quad + \alpha_n\bigl[d\bigl(y_{n+m-3}, x^*\bigr) + u_{2n}\psi_2\bigl(d\bigl(y_{n+m-3}, x^*\bigr)\bigr) + v_{2n}\bigr]\bigr] \\
&\quad + \alpha_n u_{1n}\psi_1(\overline{M}_1) + \alpha_n v_{1n} \\
&\leq (1-\alpha_n) d\bigl(x_n, x^*\bigr) + \alpha_n(1 + u_{1n}M_1)\bigl[(1-\alpha_n) d\bigl(x_n, x^*\bigr) \qquad (3.3) \\
&\quad + \alpha_n\bigl[(1 + u_{2n}M_2) d\bigl(y_{n+m-3}, x^*\bigr) + \alpha_n u_{2n}\psi_2(\overline{M}_2) + v_{2n}\bigr]\bigr] \\
&\quad + \alpha_n u_{1n}\psi_1(\overline{M}_1) + \alpha_n v_{1n} \\
&\leq (1-\alpha_n) d\bigl(x_n, x^*\bigr) + (1-\alpha_n)\alpha_n(1 + u_{1n}M_1) d\bigl(x_n, x^*\bigr) \\
&\quad + \alpha_n^2(1 + u_{1n}M_1)(1 + u_{2n}M_2) d\bigl(y_{n+m-3}, x^*\bigr) + \alpha_n^2(1 + u_{1n}M_1)v_{2n} \\
&\quad + \alpha_n^2(1 + u_{1n}M_1)u_{2n}\psi_2(\overline{M}_2) + \alpha_n\bigl(v_{1n} + u_{1n}\psi_1(\overline{M}_1)\bigr) \\
&\leq (1-\alpha_n) d\bigl(x_n, x^*\bigr) + \alpha_n(1-\alpha_n)(1 + u_{1n}M_1) d\bigl(x_n, x^*\bigr) + \cdots \\
&\quad + (\alpha_n)^{h-1}(1-\alpha_n)(1 + u_{1n}M_1)(1 + u_{2n}M_2)\cdots(1 + u_{h-1n}M_{h-1}) d\bigl(x_n, x^*\bigr) \\
&\quad + \cdots + (\alpha_n)^m(1-\alpha_n)(1 + u_{1n}M_1)(1 + u_{2n}M_2)\cdots(1 + u_{mn}M_m) d\bigl(x_n, x^*\bigr) \\
&\quad + \alpha_n v_{1n} + \alpha_n^2(1 + u_{1n}M_1)v_{2n} + \cdots + \alpha_n^m(1 + u_{1n}M_1)(1 + u_{2n}M_2) \\
&\quad \cdots(1 + u_{m-1n}M_m)v_{mn} \\
&\quad + \alpha_n v_{1n} + \alpha_n u_{1n}\psi_1(\overline{M}_1) + \alpha_n^2(1 + u_{1n}M_1)v_{2n} + \alpha_n^2(1 + u_{1n})u_{2n}\psi_2(\overline{M}_2) \\
&\quad + \cdots + \alpha_n^m(1 + u_{1n}M_1)(1 + u_{2n}M_2) + \cdots + (1 + u_{m-1n}M_m)v_{mn} \\
&\quad + \cdots + \alpha_n^m(1 + u_{1n}M_1)(1 + u_{2n}M_2) + \cdots + (1 + u_{m-1n}M_m)u_{mn}\psi_m(\overline{M}_m) \\
&\leq d\bigl(x_n, x^*\bigr)\bigl[1 + u_{1n}M_1 + u_{2n}M_2(1 + u_{1n}M_1) \\
&\quad + u_{3n}M_3(1 + u_{1n}M_1)(1 + u_{2n}M_2) + \cdots \\
&\quad + u_{mn}M_m(1 + u_{1n}M_1)(1 + u_{2n}M_2)\ldots(1 + u_{m-1n}M_{m-1})\bigr] \\
&\quad + \alpha_n v_{1n} + \sum_{j=1}^m \alpha_n^j\bigl[v_{jn} + u_{jn}\psi_j(\overline{M}_j)\bigr]\prod_{k=1}^j(1 + u_{kn}M_k) \\
&\leq d\bigl(x_n, x^*\bigr)\left[1 + \binom{m}{1} w_n + \binom{m}{2} w_n^2 + \cdots + \binom{m}{m} w_n^m\right] \qquad (3.4)
\end{aligned}$$

$$+ \alpha_n v_{1n} + \sum_{j=1}^{m} \alpha_n^j \big[v_{jn} + u_{jn} \psi_j(\overline{M}_j) \big] \prod_{k=1}^{j} (1 + u_{kn} M_k)$$

$$\le d\big(x_n, x^*\big)(1 + \delta_m w_n) + \alpha_n v_{1n} + \sum_{j=1}^{m} \alpha_n^j \big[v_{jn} + u_{jn} \psi_j(\overline{M}_j) \big] \prod_{k=1}^{j} (1 + u_{kn} M_k)$$

$$\le d\big(x_n, x^*\big) e^{\delta_m w_n} + \alpha_n v_{1n} + \sum_{j=1}^{m} \alpha_n^j \big[v_{jn} + u_{jn} \psi_j(\overline{M}_j) \big] \prod_{k=1}^{j} (1 + u_{kn} M_k)$$

$$\le d\big(x_1, x^*\big) e^{\delta_m \sum_{n=1}^{\infty} w_n} + \alpha_n v_{1n} + \sum_{j=1}^{m} \alpha_n^j \big[v_{jn} + u_{jn} \psi_j(\overline{M}_j) \big] \prod_{k=1}^{j} (1 + u_{kn} M_k)$$

$$\le d\big(x_1, x^*\big) e^{\delta_m \sum_{n=1}^{\infty} w_n} + \alpha_n v_{1n} + \sum_{j=1}^{m} \alpha_n^j \big[v_{jn} + u_{jn} \psi_j(\overline{M}_j) \big] e^{\sum_{k=1}^{j} u_{kn} M_k} < \infty, \tag{3.5}$$

where δ_m is a positive real number defined by $\delta_m := \big[\binom{m}{1} + \binom{m}{2} + \cdots + \binom{m}{m} \big]$.

This implies that $\{x_n\}$ is bounded, and so setting $v_n := \max_{1 \le j \le m} \{ v_{in} + u_{jn} \psi_i(\overline{M}_i) \}$, we have that there exists a positive integer M such that

$$d\big(x_{n+1}, x^*\big) \le d\big(x_n, x^*\big) + (\delta_m w_n + v_n) M. \tag{3.6}$$

Since (3.6) is true for each x^* in F, we have

$$d(x_{n+1}, F) \le d(x_n, F) + (\delta_m w_n + v_n) M. \tag{3.7}$$

By Lemma 2.7, $\lim_{n\to\infty} d(x_n, x^*)$ and $\lim_{n\to\infty} d(x_n, F)$ exist.

For $m = 1$, we have

$$\begin{aligned} d\big(x_{n+1}, x^*\big) &= d\big((1-\alpha_n)x_n \oplus \alpha_n T_1^n x_n, x^*\big) \\ &\le (1-\alpha_n) d\big(x_n, x^*\big) + \alpha_n d\big(T_1^n x_n, x^*\big) \\ &\le (1-\alpha_n) d\big(x_n, x^*\big) + \alpha_n \big[d\big(x_n, x^*\big) + u_{1n} \psi_1\big(d\big(x_n, x^*\big)\big) + v_{1n} \big] \\ &\le (1 + \alpha_n u_{1n} M_1) d\big(x_n, x^*\big) + \alpha_n \big[v_{1n} + u_{1n} \psi_1(\overline{M}_1) \big] \\ &\le d\big(x_n, x^*\big)(1 + w_n) + \alpha_n \big[v_{1n} + u_{1n} \psi_1(\overline{M}_1) \big] \\ &\le d\big(x_n, x^*\big) e^{w_n} + \alpha_n \big[v_{1n} + u_{1n} \psi_1(\overline{M}_1) \big] \\ &\le d\big(x_1, x^*\big) e^{\sum_{n=1}^{\infty} w_n} + \alpha_n \big[v_{1n} + u_{1n} \psi_1(\overline{M}_1) \big] \\ &\le d\big(x_1, x^*\big) e^{\sum_{n=1}^{\infty} w_n} + \alpha_n \big[v_{1n} + u_{1n} \psi_1(\overline{M}_1) \big] < \infty. \end{aligned} \tag{3.8}$$

Hence, $\{x_n\}$ is bounded, and using (3.8), Lemma 2.7, and similar arguments as before, we get that the limits $\lim_{n\to\infty} d(x_n, x^*)$ and $\lim_{n\to\infty} d(x_n, F)$ exist. This completes the proof. □

Theorem 3.2 *Let K be a nonempty closed convex subset of a hyperbolic space E. Let $T_1, T_2, \dots, T_m : K \to K$ be total asymptotically quasi-nonexpansive mappings with sequences and functions satisfying the conditions of Lemma* 3.1. *Let $\{x_n\}$ be defined by* (3.1).

Then, $\{x_n\}$ *converges to a common fixed point of the family* $T_1, T_2, \dots, T_m$ *if and only if* $\liminf_{n\to\infty} d(x_n, F) = 0$.

Proof The necessity is trivial. We prove the sufficiency. Let $\liminf_{n\to\infty} d(x_n, F) = 0$. Since $\lim_{n\to\infty} d(x_n, F)$ exists by Lemma 2.7, we have that $\lim_{n\to\infty} d(x_n, F) = 0$. Thus, given $\epsilon > 0$, there exist a positive integer N_0 and $b^* \in F$ such that for all $n \geq N_0$, $d(x_n, b^*) < \frac{\epsilon}{2}$. Then, for any $k \in \mathbb{N}$ and $n \geq N_0$, we have

$$d(x_{n+k}, x_n) \leq d\bigl(x_{n+k}, b^*\bigr) + d\bigl(b^*, x_n\bigr) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

and so $\{x_n\}$ is Cauchy. Let $\lim_{n\to\infty} x_n = b$. We need to show that $b \in F$. Let $T_i \in \{T_1, T_2, \dots, T_m\}$. Since $\lim_{n\to\infty} d(x_n, F) = 0$, there exists $N \in \mathbb{N}$ sufficiently large and $b^* \in F$ such that $n \geq N$ implies $d(b, x_n) < \frac{\epsilon}{8(1+w_1)}$, $d(b^*, x_n) < \frac{\epsilon}{8(1+w_1)}$ and $v_{in} + u_{in}\psi_i(\overline{M}_i) < \frac{\epsilon}{4}$. Then, $d(b^*, b) < \frac{\epsilon}{4(1+w_1)}$. Thus, we have the following estimates for $n \geq N$ and arbitrary T_i, $i = 1, 2, \dots, m$:

$$\begin{aligned} d(b, T_i b) &\leq d(b, x_n) + d\bigl(x_n, b^*\bigr) + d\bigl(b^*, T_i b\bigr) \\ &\leq d(b, x_n) + d\bigl(x_n, b^*\bigr) + (1 + w_1) d\bigl(b^*, b\bigr) + v_{in} + u_{in}\psi_i(\overline{M}_i) \\ &< \frac{\epsilon}{4(1+w_1)} + \frac{\epsilon}{4(1+w_1)} + \frac{\epsilon}{4} + \frac{\epsilon}{4} \leq \epsilon. \end{aligned}$$

This implies that $b \in \mathrm{Fix}(T_i)$ for all $i = 1, 2, \dots, m$, and thus $b \in F$. This completes the proof. □

Corollary 3.3 *Let* K *be a nonempty closed convex subset of a complete hyperbolic space* X. *Let* $T_1, T_2, \dots, T_m : K \to K$ *be total asymptotically nonexpansive mappings with* $F \neq \emptyset$. *Let the sequence* $\{\alpha_n\}_{n=1}^{\infty}$ *be as in Lemma* 3.1. *Let* $\{x_n\}$ *be defined by* (3.1). *Then,* $\{x_n\}$ *converges to a common fixed point of the family* $T_1, T_2, \dots, T_m$ *if and only if* $\liminf_{n\to\infty} d(x_n, F) = 0$.

For our next theorems, we start by proving the following auxiliary lemma.

Lemma 3.4 *Let* X *be a uniformly convex hyperbolic space, and* K *be a closed, convex, and nonempty subset of* X. *Let* $T_1, T_2, \dots, T_m : K \to K$ *be uniformly continuous total asymptotically quasi-nonexpansive mappings with sequences and functions satisfying the conditions of Lemma* 3.1. *Let* $\{\alpha_n\}_{n=1}^{\infty}$ *be a sequence in* $[\epsilon, 1-\epsilon]$, $\epsilon \in (0,1)$. *Let* $\{x_n\}$ *be a sequence defined iteratively by* (3.1). *Then,*

$$\lim_{n\to\infty} d(x_n, T_1 x_n) = \lim_{n\to\infty} d(x_n, T_2 x_n) = \cdots = \lim_{n\to\infty} d(x_n, T_m x_n) = 0.$$

Proof Since for some $x^* \in F$, the limit $\lim_{n\to\infty} d(x_n, x^*)$ exists by Lemma 3.1, let $\lim_{n\to\infty} d(x_n, x^*) = l$. From (3.2), (3.3), and (3.4) we obtain the following relation by taking the limit superior through the inequalities:

$$\begin{aligned} l &= \limsup_{n\to\infty} d\bigl(x_{n+1}, x^*\bigr) \leq \limsup_{n\to\infty} d\bigl(y_{n+m-2}, x^*\bigr) \\ &\leq \limsup_{n\to\infty} d\bigl(y_{n+m-3}, x^*\bigr) \end{aligned}$$

$$\begin{aligned}
&\vdots \\
&\le \limsup_{n\to\infty} d\bigl(y_{n+1}, x^*\bigr) \\
&\le \limsup_{n\to\infty} d\bigl(x_n, x^*\bigr) = l.
\end{aligned}$$

This implies that for $2 \le h \le m$, we have $\limsup_{n\to\infty} d((1-\alpha_n)x_n \oplus \alpha_n T_h^n y_{n+m-h-1}, x^*) \le l$. From this and from $\lim_{n\to\infty} d(x_n, x^*) = l$, using Lemma 2.5, we have $\lim_{n\to\infty} d(x_n, T_h^n y_{n+m-h-1}) = 0$, $2 \le h \le m$. Observe that

$$\begin{aligned}
d(x_n, y_{n+m-h-1}) &= d\bigl((1-\alpha_n)x_n \oplus \alpha_n T_h^n y_{n+m-h-2}, x_n\bigr) \\
&\le \alpha_n d\bigl(T_h^n y_{n+m-h-2}, x_n\bigr) \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Thus,

$$\begin{aligned}
d\bigl(x_n, T_h^n x_n\bigr) &\le d\bigl(x_n, T_h^n y_{n+m-h-1}\bigr) + d\bigl(T_h^n y_{n+m-h-1}, T_h^n x_n\bigr) \\
&\le d\bigl(x_n, T_h^n y_{n+m-h-1}\bigr) + (1 + u_{hn}M)d(y_{n+m-h-1}, x_n) \\
&\quad + v_{hn} + u_{hn}\psi_h(\overline{M}_h) \to 0 \quad \text{as } n \to \infty,
\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}
d(x_{n+1}, x_n) &= d\bigl((1-\alpha_n)x_n \oplus \alpha_n T_1^n y_{n+m-2}, x_n\bigr) \\
&\le \alpha_n d\bigl(x_n, T_h^n y_{n+m-2}\bigr) \to 0 \quad \text{as } n \to \infty.
\end{aligned} \tag{3.10}$$

Now

$$\begin{aligned}
d(x_n, T_h x_n) &\le d\bigl(x_n, T_h^n x_n\bigr) + d\bigl(T_h^n x_n, T_h^n y_{n+m-h-1}\bigr) \\
&\quad + d\bigl(T_h^n y_{n+m-h-1}, T_h x_n\bigr) \\
&\le d\bigl(x_n, T_h^n x_n\bigr) + (1 + u_{hn}M)d(y_{n+m-h-1}, x_n) \\
&\quad + v_{hn} + u_{hn}\psi_h(\overline{M}_h) + d\bigl(T_h^n y_{n+m-h-1}, T_h x_n\bigr).
\end{aligned} \tag{3.11}$$

Consider the following:

$$\begin{aligned}
d\bigl(T_h^{n-1} y_{n+m-h-1}, x_n\bigr) &\le d\bigl(T_h^{n-1} y_{n+m-h-1}, x_{n-1}\bigr) \\
&\quad + d(x_{n-1}, x_n) \to 0 \quad \text{as } n \to \infty.
\end{aligned} \tag{3.12}$$

Since T_h is uniformly continuous and $d(T_h^{n-1} y_{n+m-h-1}, x_n) \to 0$ as $n \to \infty$, we get $d(T_h^n y_{n+m-h-1}, T_h x_n) \to 0$ as $n \to \infty$. So from (3.11) we get

$$\lim_{n\to\infty} d(x_n, T_h x_n) = 0. \tag{3.13}$$

□

Theorem 3.5 *Let X be a uniformly convex hyperbolic space, and K be a closed convex nonempty subset of X. Let $T_1, T_2, \ldots, T_m : K \to K$ be uniformly L- Lipschitzian total asymptotically quasi-nonexpansive mappings with sequences and functions satisfying the conditions of Lemma* 3.1. *Let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence in $[\epsilon, 1-\epsilon]$, $\epsilon \in (0,1)$ and assume that each*

T_i is demiclosed at 0 *for each $i \in I$. Let $\{x_n\}$ be a sequence defined iteratively by* (3.1). *Then, $\{x_n\}$ Δ-converges to an element of F.*

Proof Let $W_\Delta(\{x_n\}) := \bigcup_{\{u_n\}\subset\{x_n\}} A(\{u_n\})$. We now show that $W_\Delta(\{x_n\}) \subset F$ and also that $W_\Delta(\{x_n\})$ consists only of a single point. Now let $u \in W_\Delta(\{x_n\})$. Then there exists a subsequence say $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemma 2.6 there exists a convergence subsequence $\{v_n\}$ of $\{u_n\}$ such that Δ-$\lim v_n = v$ for some $v \in K$. But $\lim_{n\to\infty} d(v_n, T_i v_n) = 0$ for each $i \in \{1, 2, 3, \dots, m\}$. By the demiclosedness property of each T_i we have $v \in F$. Since the limit $\lim_{n\to\infty} d(v_n, v)$ exists, $u = v \in F$, and this implies $W_\Delta(\{x_n\}) \subset F$. Next, we show that $W_\Delta(\{x_n\})$ is singletone. Let $\{u_n\}$ be a subsequence of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$, and let $A(\{x_n\}) = \{x\}$. Since $u \in W_\Delta(\{x_n\}) \subset F$, the limit $\lim_{n\to\infty} d(x_n, u)$ exists, by Lemma 2.2, $x = u$, and so $W_\Delta(\{x_n\})$ is singletone, which implies that $\{x_n\}$ Δ-converges to an element of F. □

Next, we present Δ- and polar convergence theorems for finite families of total asymptotically nonexpansive mappings in the framework of a complete CAT(0) space. This next result is a corollary of the previous Lemma 3.4, but we shall present them using a different method of proof.

Corollary 3.6 *Let X be a complete CAT*(0) *space, and K be a closed, convex, and nonempty subset of X. Let $T_1, T_2, \dots, T_m : K \to K$ be uniformly continuous total asymptotically nonexpansive mappings with sequences and functions satisfying the conditions of Lemma* 3.1. *Let $\{\alpha_n\}_{n=1}^\infty$ be a sequence in $[\epsilon, 1-\epsilon]$, $\epsilon \in (0,1)$. Let $\{x_n\}$ be a sequence defined iteratively by* (3.1). *Then,*

$$\lim_{n\to\infty} d(x_n, T_1 x_n) = \lim_{n\to\infty} d(x_n, T_2 x_n) = \cdots = \lim_{n\to\infty} d(x_n, T_m x_n) = 0.$$

Proof Since $\{x_n\}$ is bounded, for some $x^* \in F$, there exist positive real numbers γ and M with $d^2(x_n, x^*) \le \gamma$ for all $n \ge 1$, and by using Lemma 2.3, the recursion formula (3.1), we have

$$\begin{aligned}
d^2(y_n, x^*) &= d^2\big((1-\alpha_n)x_n \oplus \alpha_n T_m^n x_n, x^*\big) \\
&\le (1-\alpha_n)d^2(x_n, x^*) + \alpha_n d^2(T_m^n x_n, x^*) \\
&\quad - \alpha_n(1-\alpha_n)d^2(x_n, T_m^n x_n) \\
&\le (1-\alpha_n)d^2(x_n, x^*) + \alpha_n\big[(1 + u_{mn}M_m)d(x_n, x^*) \\
&\quad + \big[v_{mn} + u_{mn}\psi_m(\overline{M}_m)\big]\big]^2 \\
&\quad - \alpha_n(1-\alpha_n)d^2(x_n, T_m^n x_n) \\
&\le (1-\alpha_n)d^2(x_n, x^*) + \alpha_n\big[(1 + u_{mn}M_m)^2 d^2(x_n, x^*) \\
&\quad + 2\big(v_{mn} + u_{mn}\psi_m(\overline{M}_m)\big)(1 + u_{mn}M_m)d(x_n, x^*) + \big[v_{mn} + u_{mn}\psi_m(\overline{M}_m)\big]^2\big] \\
&\quad - \alpha_n(1-\alpha_n)d^2(x_n, T_m^n x_n) \\
&= (1-\alpha_n)d^2(x_n, x^*) + \alpha_n\big[d^2(x_n, x^*) + (2u_{mn}M_m + u_{mn}^2 M_m^2)d^2(x_n, x^*) \\
&\quad + 2\big(v_{mn} + u_{mn}\psi_m(\overline{M}_m)\big)(1 + u_{mn}M_m)d(x_n, x^*) + \big[v_{mn} + u_{mn}\psi_m(\overline{M}_m)\big]^2\big]
\end{aligned}$$

$$
\begin{aligned}
&- \alpha_n(1-\alpha_n)d^2\bigl(x_n, T_m^n x_n\bigr) \\
&\le d^2\bigl(x_n, x^*\bigr) + 7\alpha_n\omega_n M^2\gamma + \alpha_n\omega_n^2 - \alpha_n(1-\alpha_n)d^2\bigl(x_n, T_m^n x_n\bigr).
\end{aligned}
$$

Also,

$$
\begin{aligned}
d^2\bigl(y_{n+1}, x^*\bigr) &= d^2\bigl((1-\alpha_n)x_n \oplus \alpha_n T_{m-1}^n y_n, x^*\bigr) \\
&\le (1-\alpha_n)d^2\bigl(x_n, x^*\bigr) + \alpha_n d^2\bigl(T_{m-1}^n y_n, x^*\bigr) - \alpha_n(1-\alpha_n)d^2\bigl(x_n, T_{m-1}^n y_n\bigr) \\
&\le (1-\alpha_n)d^2\bigl(x_n, x^*\bigr) + \alpha_n\bigl[(1+u_{m-1n}M_{m-1})d\bigl(y_n, x^*\bigr) \\
&\quad + \bigl[v_{m-1n} + u_{m-1n}\psi_{m-1}(\overline{M}_{m-1})\bigr]\bigr]^2 - \alpha_n(1-\alpha_n)d^2\bigl(x_n, T_{m-1}^n y_n\bigr) \\
&\le (1-\alpha_n)d^2\bigl(x_n, x^*\bigr) + \alpha_n\bigl[(1+u_{m-1n}M_{m-1})^2 d^2\bigl(y_n, x^*\bigr) \\
&\quad + 2\bigl[v_{m-1n} + u_{m-1n}\psi_{m-1}(\overline{M}_{m-1})\bigr](1+u_{m-1n}M_{m-1})d\bigl(y_n, x^*\bigr) \\
&\quad + \bigl[v_{m-1n} + u_{m-1n}\psi_{m-1}(\overline{M}_{m-1})\bigr]^2\bigr] - \alpha_n(1-\alpha_n)d^2\bigl(x_n, T_{m-1}^n y_n\bigr) \\
&= (1-\alpha_n)d^2\bigl(x_n, x^*\bigr) + \alpha_n\bigl[d^2\bigl(y_n, x^*\bigr) \\
&\quad + \bigl(2u_{m-1n}M_{m-1} + u_{m-1n}^2 M_{m-1}^2\bigr)d^2\bigl(y_n, x^*\bigr) \\
&\quad + 2\bigl[v_{m-1n} + u_{m-1n}\psi_{m-1}(\overline{M}_{m-1})\bigr](1+u_{m-1n}M_{m-1})d\bigl(y_n, x^*\bigr) \\
&\quad + \bigl[v_{m-1n} + u_{m-1n}\psi_{m-1}(\overline{M}_{m-1})\bigr]^2\bigr] - \alpha_n(1-\alpha_n)d^2\bigl(x_n, T_{m-1}^n y_n\bigr) \\
&\le (1-\alpha_n)d^2\bigl(x_n, x^*\bigr) + \alpha_n d^2\bigl(y_n, x^*\bigr) + 7\alpha_n\omega_n M^2\gamma + \alpha_n\omega_n^2 \\
&\quad - \alpha_n(1-\alpha_n)d^2\bigl(x_n, T_{m-1}^n y_n\bigr) \\
&\le (1-\alpha_n)d^2\bigl(x_n, x^*\bigr) + \alpha_n\bigl[d^2\bigl(x_n, x^*\bigr) + 7\alpha_n\omega_n M^2\gamma \\
&\quad + \alpha_n\omega_n^2 - \alpha_n(1-\alpha_n)d^2\bigl(x_n, T_m^n x_n\bigr)\bigr] \\
&\quad + 7\alpha_n\omega_n M^2\gamma + \alpha_n\omega_n^2 - \alpha_n(1-\alpha_n)d^2\bigl(x_n, T_{m-1}^n y_n\bigr) \\
&\le d^2\bigl(x_n, x^*\bigr) + \alpha_n\bigl(7\omega_n M^2\gamma + \omega_n^2\bigr)(1+\alpha_n) \\
&\quad - \alpha_n^m(1-\alpha_n)\bigl[d^2\bigl(x_n, T_m^n x_n\bigr) + d^2\bigl(x_n, T_{m-1}^n y_n\bigr)\bigr].
\end{aligned}
$$

Continuing in this fashion, we get, using $x_{n+1} = (1-\alpha_n)x_n \oplus \alpha_n T_1 y_{n+m-2}$, that

$$
\begin{aligned}
d^2\bigl(x_{n+1}, x^*\bigr) &\le d^2\bigl(x_n, x^*\bigr) + \alpha_n\bigl(\omega_n^2 + 7\omega_n^2 M^2\gamma\bigr)\sum_{j=0}^{m-1}\alpha_n^j \\
&\quad - \alpha_n^m(1-\alpha_n)\left[d^2\bigl(x_n, T_m^n x_n\bigr) + \sum_{j=1}^{m-1} d^2\bigl(x_n, T_{m-j}^n y_{n+j-1}\bigr)\right],
\end{aligned}
$$

so that

$$
\begin{aligned}
&\alpha_n^m(1-\alpha_n)\left[d^2\bigl(x_n, T_m^n x_n\bigr) + \sum_{j=1}^{m-1} d^2\bigl(x_n, T_{m-j}^n y_{n+j-1}\bigr)\right] \\
&\quad \le d^2\bigl(x_n, x^*\bigr) - d^2\bigl(x_{n+1}, x^*\bigr) + \alpha_n\bigl(\omega_n^2 + 7\omega_n^2 M^2\gamma\bigr)\sum_{j=0}^{m-1}\alpha_n^j.
\end{aligned}
$$

This implies that

$$\sum_{n=1}^{\infty}\left(\alpha_n^m(1-\alpha_n)\left[d^2\big(x_n,T_m^nx_n\big)+\sum_{j=1}^{m-1}d^2\big(x_n,T_{m-j}^ny_{n+j-1}\big)\right]\right)<\infty,$$

and by the choice of the sequence $\{\alpha_n\}$ we have

$$\begin{aligned}\lim_{n\to\infty}d\big(x_n,T_m^nx_n\big)&=\lim_{n\to\infty}d\big(x_n,T_{m-1}^ny_n\big)\\&\vdots\\&=\lim_{n\to\infty}d\big(x_n,T_h^ny_{n+m-h-1}\big)\\&\vdots\\&=\lim_{n\to\infty}d\big(x_n,T_1^ny_{n+m-2}\big)=0\end{aligned}\tag{3.14}$$

for $2\le h<m$.

The remaining part of the proof follows as in Lemma 3.4. □

Theorem 3.7 *Let E be a complete CAT(0) space, and K be a closed convex nonempty subset of E. Let $T_1,T_2,\dots,T_m:K\to K$ be uniformly L-Lipschitzian total asymptotically quasi-nonexpansive mappings with sequences and functions satisfying the conditions of Lemma* 3.1. *Let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence in $[\epsilon,1-\epsilon]$, $\epsilon\in(0,1)$. Let $\{x_n\}$ be the sequence defined iteratively by* (3.1). *Then, $\{x_n\}$ Δ-converges to an element of $\mathcal{F}$.*

Proof Let $W_\Delta(\{x_n\}):=\bigcup_{\{u_n\}\subset\{x_n\}}A(\{u_n\})$. We now show that $W_\Delta(\{x_n\})\subset\mathcal{F}$ and also that $W_\Delta(\{x_n\})$ consists only of a single point. Now let $u\in W_\Delta(\{x_n\})$. Then there exists a subsequence say $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\})=\{u\}$. By Lemma 2.6 there exists a convergent subsequence $\{v_n\}$ of $\{u_n\}$ such that Δ-$\lim v_n=v$ for some $v\in K$. But $\lim_{n\to\infty}d(v_n,T_iv_n)=0$ for each $i\in\{1,2,3,\dots,m\}$. By the demiclosedness property of each T_i we have $v\in\mathcal{F}$. Since the limit $\lim_{n\to\infty}d(v_n,v)$ exists, $u=v\in\mathcal{F}$, and this implies $W_\Delta(\{x_n\})\subset\mathcal{F}$. Next, we show that $W_\Delta(\{x_n\})$ is singletone. Let $\{u_n\}$ be a subsequence of $\{x_n\}$ such that $A(\{u_n\})=\{u\}$, and let $A(\{x_n\})=\{x\}$. Since $u\in W_\Delta(\{x_n\})\subset\mathcal{F}$, the limit $\lim_{n\to\infty}d(x_n,u)$ exists, by Lemma 2.2, $x=u$, and so $W_\Delta(\{x_n\})$ is singletone, which implies that $\{x_n\}$ Δ-converges to an element of $\mathcal{F}$. □

Remark 3.8 The CAT(0) spaces are rotund metric ('staple rotund,' see [31]) spaces. The polar and Δ-convergence coincide in a complete rotund metric space; see Lemma 3.6 of [18].

As a consequence of Remark 3.8 and Theorem 3.7, we have the following theorem.

Theorem 3.9 *Let E be a complete CAT(0) space, and K be a closed convex nonempty subset of E. Let $T_1,T_2,\dots,T_m:K\to K$ be uniformly L-Lipschitzian total asymptotically quasi-nonexpansive mappings with sequences and functions satisfying the conditions of Lemma* 3.1. *Let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence in $[\epsilon,1-\epsilon]$, $\epsilon\in(0,1)$. Let $\{x_n\}$ be the sequence defined iteratively by* (3.1). *Then, $\{x_n\}$ polar converges to an element of $\mathcal{F}$.*

Competing interests
The author declares that he has no competing interests.

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