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Best proximity point theorems via fixed point theorems for multivalued mappings

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Abstract

It is well known that the concept of a best proximity point includes that of a fixed point as a special case. In this paper, we show that the best proximity point theorems of Basha and Shahzad (Fixed Point Theory Appl. 2012:42, 2012) and of Fernández-León (J. Nonlinear Convex Anal. 15(2):313-324, 2014) can be regarded as a fixed point theorem for multivalued mappings which is modified as regards the results of Mizoguchi and Takahashi (J. Math. Anal. Appl. 141(1):177-188, 1989) and of Kada *et al.* (Math. Jpn. 44(2):381-391, 1996).

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1 Introduction

Let X be any nonempty set and $T : X \to X$ be a given mapping. A point $x \in X$ such that $x = Tx$ is called a *fixed point* of T. Many problems can be reformulated to the problem of finding a fixed point of a certain mapping. If T is not a self-mapping, it is plausible that the equation $x = Tx$ has no solution. In this situation, we may find an element $x \in X$ which is close to Tx in some sense.

Now, we suppose that X is equipped with a metric d, that is, (X, d) is a metric space. For two subsets A and B of X and $T : A \to B$, we are interested in finding an element $x \in A$ such that

$$d(x, Tx) = \inf\bigl\{d(a, b) : a \in A, b \in B\bigr\} =: d(A, B).$$

Such an element x is called a *best proximity point* of T. It follows immediately that the problem of finding a best proximity point is more general than that of finding a fixed point. In fact, if $A = B$, then $d(A, B) = 0$ and hence a best proximity point of T becomes a fixed point of T. In this setting, we recall the following notions:

$$A_0 := \bigl\{a \in A : d(a, b) = d(A, B) \text{ for some } b \in B\bigr\}$$
$$B_0 := \bigl\{b \in B : d(a, b) = d(A, B) \text{ for some } a \in A\bigr\}.$$

Basha [5] proposed the following result for the existence of a best proximity point of a non-self-mapping.

Theorem 1 ([5], Theorem 3.1) *Let (X,d) be a complete metric space and A, B be two subsets of X such that $A_0 \neq \varnothing$ (and hence $B_0 \neq \varnothing$). Suppose that $T : A \to B$ is a mapping such that $T(A_0) \subset B_0$. We make the following assumptions*:

- *A and B are closed*;
- *B is approximatively compact with respect to A*;
- *T is a proximal contraction, that is, there exists $\alpha \in [0,1)$ such that, for all $u, v, x, y \in A$,*

$$d(u,Tx) = d(A,B) = d(v,Ty)$$

implies

$$d(u,Tx) + d(Tx,Ty) + d(Ty,v) \leq \alpha d(x,y).$$

Then the following hold:

(a) *there exists a unique element $x \in A$ such that $d(x,Tx) = d(A,B)$*;

(b) *if $\{x_n\}$ is a sequence in A_0 satisfying $d(x_{n+1},Tx_n) = d(A,B)$ for all $n \geq 0$, then $\lim_{n\to\infty} x_n = x$.*

It is clear that Theorem 1 extends Banach's contraction principle in the setting that $A = B = X$. By the way, there are plenty of papers which had generalized this result (for example, see [1, 2, 6]).

Basha and Shahzad [1] introduced the following two concepts of contractiveness for non-self-mappings.

Definition 2 ([1]) Let (X,d) be a metric space. Let A and B be nonempty subsets of X. We say that $T : A \to B$ is

(a) a *generalized proximal contraction of the first kind* if there exist non-negative numbers α, β, γ with $\alpha + 2\beta + 2\gamma < 1$ such that the condition

$$d(u,Tx) = d(A,B) = d(v,Ty)$$

implies

$$d(u,v) \leq \alpha d(x,y) + \beta d(x,u) + \beta d(y,v) + \gamma d(x,v) + \gamma d(y,u);$$

(b) a *generalized proximal contraction of the second kind* if there exist non-negative numbers α, β, γ with $\alpha + 2\beta + 2\gamma < 1$ such that the condition

$$d(u,Tx) = d(A,B) = d(v,Ty)$$

implies

$$\begin{aligned} d(Tu,Tv) \leq{}& \alpha d(Tx,Ty) + \beta d(Tx,Tu) + \beta d(Ty,Tv) \\ &+ \gamma d(Tx,Tv) + \gamma d(Ty,Tu). \end{aligned}$$

Remark 3 Every proximal contraction is a generalized proximal contraction of the first kind.

In this paper, we show that the problem of finding a best proximity point recently established by Fernández-León [2] and Basha and Shahzad [1] reduces to a problem of finding a fixed point of a multivalued mapping. Recall that $x \in X$ is a *fixed point* of a multivalued mapping $T : X \to 2^X \setminus \{\varnothing\}$ if $x \in Tx$. There are many conditions guaranteeing the existence of a fixed point of a multivalued mapping. Two of the classical works in this research are due to Nadler [7] and Caristi [8]. The interested reader is referred to [9], Chapter 5, for more discussion.

2 Main results

By studying the works of [4] and [3], we obtain the following fixed point theorem for a multivalued mapping.

Theorem 4 *Let (X, d) be a complete metric space. Let Y be a nonempty subset of X and let $F : Y \to (-\infty, \infty]$ be a proper function which is bounded below. Let $S : Y \to 2^Y \setminus \{\varnothing\}$ be a multivalued mapping such that for each $x \in Y$ there exists $y \in Sx$ satisfying*

$$F(y) + d(x,y) \le F(x). \tag{2.1}$$

Assume that for $z \in X$

$$\inf\bigl\{d(x,z) + d(x,Sx) : x \in Y\bigr\} = 0 \quad \Longrightarrow \quad z \in Sz \cap Y. \tag{2.2}$$

Then there exists $w \in Y$ such that $w \in Sw$.

Proof Let x_0 be an element in Y such that $F(x_0) < \infty$. By the condition (2.1), there is an $x_1 \in Sx_0$ such that $F(x_1) + d(x_0, x_1) \le F(x_0)$. By induction, we have a sequence $\{x_n\}$ in Y such that

$$x_{n+1} \in Sx_n \quad \text{and} \quad F(x_{n+1}) + d(x_n, x_{n+1}) \le F(x_n) \quad \text{for all } n \ge 0.$$

So $\{F(x_n)\}$ is a decreasing sequence. Since F is bounded below, $\lim_{n\to\infty} F(x_n) = \alpha$ for some $\alpha \in \mathbb{R}$. Let $m \ge 0$. We have

$$\begin{aligned}\sum_{n=0}^{n=m} d(x_n, x_{n+1}) &\le \sum_{n=0}^{n=m} \bigl(F(x_n) - F(x_{n+1})\bigr) \\ &= F(x_0) - F(x_{m+1}) \\ &\le F(x_0) - \alpha.\end{aligned}$$

Then $\sum_{n=0}^{\infty} d(x_n, x_{n+1}) = \lim_{m\to\infty} \sum_{n=0}^{n=m} d(x_n, x_{n+1}) < \infty$ and hence $\{x_n\}$ is a Cauchy sequence. So $\lim_{n\to\infty} x_n = w$ for some $w \in X$. Note that

$$\lim_{n\to\infty} d(x_n, w) = 0 \quad \text{and} \quad \lim_{n\to\infty} d(x_n, Sx_n) \le \lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

By the condition (2.2), we have $w \in Sw \cap Y$. $\square$

2.1 Results for a generalized proximal contraction of the first kind

We show that the following result of Fernández-León [2] is a consequence of our Theorem 4.

Theorem 5 ([2], Proposition 3.5) *Let (X,d) be a complete metric space. Let A and B be nonempty subsets of X such that A_0 is nonempty. Let $T : A \to B$ be a mapping such that $T(A_0) \subset B_0$. Let us assume the following conditions*:

- *A_0 is closed*;
- *T is a generalized proximal contraction of the first kind.*

Then the following hold:

(a) *there exists a unique element x in A such that $d(x, Tx) = d(A,B)$*;

(b) *if $\{x_n\}$ is a sequence in A_0 satisfying $d(x_{n+1}, Tx_n) = d(A,B)$ for each $n \geq 0$, then* $\lim_{n\to\infty} x_n = x$.

Proof For each $x \in A_0$, we let

$$Sx = \{y : y \in A_0 \text{ and } d(y, Tx) = d(A,B)\}.$$

It follows that $S : A_0 \to 2^{A_0} \setminus \{\varnothing\}$.

Since T is a generalized proximal contraction of the first kind, there are $\alpha, \beta, \gamma \geq 0$ with $\alpha + 2\beta + 2\gamma < 1$ such that $d(u, Tx) = d(A,B) = d(v, Ty)$ implies

$$d(u,v) \leq \alpha d(x,y) + \beta d(x,u) + \beta d(y,v) + \gamma d(x,v) + \gamma d(y,u)$$

for all $u, v, x, y \in A$. Put $c = \frac{\alpha+\beta+\gamma}{1-\beta-\gamma}$ and $b = \frac{c+1}{2}$. Then $0 \leq c < b < 1$.

Claim that, for all $x, y, z \in A_0$, if $y \in Sx$ and $z \in Sy$, then $d(z,y) \leq cd(y,x)$. To see this, let x, y, z be elements in A_0 such that $y \in Sx$ and $z \in Sy$. Then

$$d(y, Tx) = d(A,B) = d(z, Ty).$$

Since T is a generalized proximal contraction of the first kind,

$$\begin{aligned} d(z,y) &\leq \alpha d(y,x) + \beta d(y,z) + \beta d(x,y) + \gamma d(y,y) + \gamma d(x,z) \\ &\leq \alpha d(y,x) + \beta d(y,z) + \beta d(x,y) + \gamma d(x,y) + \gamma d(y,z). \end{aligned}$$

Hence

$$d(z,y) \leq cd(y,x).$$

So we have the claim.

Next, we show that the condition (2.1) in Theorem 4 holds. Let $x \in A_0$. Since $0 < b < 1$, we can choose $y \in Sx$ so that

$$bd(x,y) \leq d(x, Sx). \tag{2.3}$$

Let $z \in Sy$, then we obtain by the claim

$$d(y, Sy) \leq d(z,y) \leq cd(y,x). \tag{2.4}$$

Using (2.3) and (2.4), we obtain

$$d(y,Sy) + bd(x,y) \leq cd(x,y) + d(x,Sx).$$

Then

$$\frac{1}{b-c}d(y,Sy) + d(y,x) \leq \frac{1}{b-c}d(x,Sx).$$

Let $F : A_0 \to [0,\infty)$ be defined by $F(x) = \frac{1}{b-c}d(x,Sx)$ for each $x \in A_0$. So F satisfies the condition (2.1) in Theorem 4.

We show that the condition (2.2) in Theorem 4 holds. Let $\{x_n\}$ be a sequence in A_0 and $z \in X$ satisfying

$$\lim_{n\to\infty} d(x_n,z) = 0 \quad \text{and} \quad \lim_{n\to\infty} d(x_n,Sx_n) = 0.$$

Since A_0 is closed, we have $z \in A_0$ and $Tz \in T(A_0) \subset B_0$. Then there exists $u \in A_0$ such that

$$d(u,Tz) = d(A,B). \tag{2.5}$$

We choose a sequence $\{u_n\}$ in A_0 so that $u_n \in Sx_n$ and

$$d(x_n,u_n) < d(x_n,Sx_n) + \frac{1}{n}$$

for each $n \geq 1$. Hence, $\lim_{n\to\infty} d(x_n,u_n) = 0$. Since $u_n \in Sx_n$ for each $n \geq 0$,

$$d(u_n,Tx_n) = d(A,B). \tag{2.6}$$

Since $\lim_{n\to\infty} d(x_n,z) = 0$ and $\lim_{n\to\infty} d(x_n,u_n) = 0$, we get $\lim_{n\to\infty} u_n = z$. Using (2.5), (2.6), and the fact that T is a generalized proximal contraction of the first kind, we have, for each $n \geq 0$,

$$d(u,u_n) \leq \alpha d(z,x_n) + \beta d(z,u) + \beta d(x_n,u_n) + \gamma d(z,u_n) + \gamma d(x_n,u).$$

As $n \to \infty$, we get

$$d(u,z) \leq (\beta+\gamma)d(z,u).$$

So $z = u$ and hence $d(z,Tz) = d(A,B)$, that is, $z \in Sz$. Therefore, the condition (2.2) in Theorem 4 holds. Using Theorem 4, there exists $w \in A_0$ such that $w \in Sw$, that is,

$$d(w,Tw) = d(A,B).$$

To see the uniqueness, we assume that $d(\widehat{w},T\widehat{w}) = d(A,B)$ for some $\widehat{w} \in A$. Since T is a generalized proximal contraction of the first kind, we have

$$d(w,\widehat{w}) \leq (\alpha+2\gamma)d(w,\widehat{w}).$$

That is, $w = \widehat{w}$. So we have (a).

To see (b), let $\{x_n\}$ be a sequence in A_0 such that

$$d(x_{n+1}, Tx_n) = d(A, B) \quad \text{for all } n \geq 0.$$

Thus $x_{n+1} \in Sx_n$. By the claim, we get, for each $n \geq 0$,

$$d(x_{n+2}, x_{n+1}) \leq cd(x_{n+1}, x_n).$$

So $\{x_n\}$ is a Cauchy sequence and hence $\lim_{n\to\infty} x_n = x$ for some $x \in A_0$. Since T is a generalized proximal contraction of the first kind, we have

$$d(x_{n+1}, w) \leq \alpha d(x_n, w) + \beta d(x_n, x_{n+1}) + \beta d(w, w) + \gamma d(x_n, w) + \gamma d(w, x_{n+1})$$

for each $n \geq 0$. As $n \to \infty$, we get $d(x, w) \leq (\alpha + 2\gamma)d(x, w)$. That is, $x = w$. Hence, $\lim_{n\to\infty} x_n = w$. So we have (b). □

2.2 Results for a generalized proximal contraction of the second kind

The following result of Fernández-León [2] is also a consequence of our Theorem 4.

Theorem 6 ([2], Proposition 3.10) *Let (X, d) be a complete metric space. Let A and B be nonempty subsets of X such that A_0 is nonempty. Let $T : A \to B$ be a mapping such that $T(A_0) \subset B_0$. Let us assume the following conditions*:

- *$T(A_0)$ is closed*;
- *T is a generalized proximal contraction of the second kind.*

Then the following hold:

(a) *there exists $x \in A$ such that $d(x, Tx) = d(A, B)$*;
(b) *if there is $\widehat{x} \in A$ such that $d(\widehat{x}, T\widehat{x}) = d(A, B)$, then $T\widehat{x} = Tx$*;
(c) *if $\{x_n\}$ is a sequence in A_0 satisfying $d(x_{n+1}, Tx_n) = d(A, B)$ for each $n \geq 0$, then $\lim_{n\to\infty} Tx_n = Tx$.*

Proof For each $x \in T(A_0)$, we let

$$Sx = \bigl\{y : y = Tu \text{ where } u \in A_0 \text{ and } d(u, x) = d(A, B)\bigr\}.$$

It follows that $S : T(A_0) \to 2^{T(A_0)} \setminus \{\varnothing\}$. Since T is a generalized proximal contraction of the second kind, there are $\alpha, \beta, \gamma \geq 0$ with $\alpha + 2\beta + 2\gamma < 1$ such that $d(u, Tx) = d(A, B) = d(v, Ty)$ implies

$$\begin{aligned} d(Tu, Tv) \leq{}& \alpha d(Tx, Ty) + \beta d(Tx, Tu) + \beta d(Ty, Tv) \\ & + \gamma d(Tx, Tv) + \gamma d(Ty, Tu) \end{aligned}$$

for all $u, v, x, y \in A$. Put $c = \frac{\alpha+\beta+\gamma}{1-\beta-\gamma}$ and $b = \frac{c+1}{2}$. Then $0 \leq c < b < 1$.

Claim 1: for each $u, v, x, y \in T(A_0)$ if $u \in Sx$ and $v \in Sy$, then

$$\begin{aligned} d(u, v) \leq{}& \alpha d(x, y) + \beta d(x, u) + \beta d(y, v) \\ & + \gamma d(x, v) + \gamma d(y, u). \end{aligned}$$

To see this claim, let u, v, x, y be elements in $T(A_0)$ such that $u \in Sx$ and $v \in Sy$. So $u = T\widehat{u}$, $v = T\widehat{v}$, $x = T\widehat{x}$ and $y = T\widehat{y}$ for some $\widehat{u}, \widehat{v}, \widehat{x}, \widehat{y} \in A_0$ with

$$d(\widehat{u}, T\widehat{x}) = d(A,B) = d(\widehat{v}, T\widehat{y}).$$

Since T is a generalized proximal contraction of the second kind,

$$\begin{aligned} d(T\widehat{u}, T\widehat{v}) &\leq \alpha d(T\widehat{x}, T\widehat{y}) + \beta d(T\widehat{x}, T\widehat{u}) + \beta d(T\widehat{y}, T\widehat{v}) \\ &\quad + \gamma d(T\widehat{x}, T\widehat{v}) + \gamma d(T\widehat{y}, T\widehat{u}). \end{aligned}$$

That is,

$$d(u,v) \leq \alpha d(x,y) + \beta d(x,u) + \beta d(y,v) + \gamma d(x,v) + \gamma d(y,u).$$

So we have Claim 1.

Claim 2: for each $x,y,z \in T(A_0)$ if $y \in Sx$ and $z \in Sy$, then $d(z,y) \leq cd(x,y)$. To see this, let x, y, z be elements in $T(A_0)$ such that $y \in Sx$ and $z \in Sy$. Using Claim 1, we have

$$\begin{aligned} d(z,y) &\leq \alpha d(y,x) + \beta d(y,z) + \beta d(x,y) + \gamma d(y,y) + \gamma d(x,z) \\ &\leq \alpha d(y,x) + \beta d(y,z) + \beta d(x,y) + \gamma d(x,y) + \gamma d(y,z). \end{aligned}$$

So $d(z,y) \leq cd(x,y)$. That is, Claim 2 holds.

Now, we show that the condition (2.1) in Theorem 4 holds. Let $x \in T(A_0)$. Since $0 < b < 1$, there exists $y \in Sx$ such that

$$bd(x,y) \leq d(x,Sx). \tag{2.7}$$

Let $z \in Sy$, then we obtain by Claim 2

$$d(y,Sy) \leq d(z,y) \leq cd(x,y). \tag{2.8}$$

Using (2.7) and (2.8), we get

$$d(y,Sy) + bd(x,y) \leq cd(x,y) + d(x,Sx).$$

Then

$$\frac{1}{b-c} d(y,Sy) + d(x,y) \leq \frac{1}{b-c} d(x,Sx).$$

Let $F : T(A_0) \to [0,\infty)$ be defined by $F(x) = \frac{1}{b-c} d(x,Sx)$ for each $x \in T(A_0)$. So F satisfies the condition (2.1) in Theorem 4.

Next, we show that the condition (2.2) in Theorem 4 holds. Let $z \in X$ and let $\{x_n\}$ be a sequence in $T(A_0)$ such that

$$\lim_{n\to\infty} d(x_n,z) = 0 \quad \text{and} \quad \lim_{n\to\infty} d(x_n,Sx_n) = 0.$$

Since $T(A_0)$ is closed, $z \in T(A_0)$ and hence we can let

$$\widehat{z} \in Sz. \tag{2.9}$$

We show that $\widehat{z} = z$. Since $\lim_{n\to\infty} d(x_n, Sx_n) = 0$, we can choose a sequence $\{y_n\}$ in $T(A_0)$ so that

$$y_n \in Sx_n \tag{2.10}$$

for each $n \geq 0$ and

$$\lim_{n\to\infty} d(x_n, y_n) = 0.$$

Since $\lim_{n\to\infty} d(x_n, z) = 0$ and $\lim_{n\to\infty} d(x_n, y_n) = 0$, we obtain $\lim_{n\to\infty} y_n = z$. Using (2.9), (2.10), and Claim 1,

$$d(\widehat{z}, y_n) \leq \alpha d(z, x_n) + \beta d(z, \widehat{z}) + \beta d(x_n, y_n) + \gamma d(z, y_n) + \gamma d(x_n, \widehat{z}).$$

As $n \to \infty$, we get $d(\widehat{z}, z) \leq (\beta + \gamma) d(z, \widehat{z})$, that is, $\widehat{z} = z$. Hence, the condition (2.2) in Theorem 4 holds. Using Theorem 4, there exists $w \in T(A_0)$ such that $w \in Sw$, that is, there exists $w^* \in A_0$ such that $w = Tw^*$ and

$$d(w^*, Tw^*) = d(A, B).$$

So we have (a).

To see (b), let v be an element in A such that $d(v, Tv) = d(A, B)$. Since T is a generalized proximal contraction of the second kind,

$$\begin{aligned} d(Tv, Tw^*) \leq{}& \alpha d(Tv, Tw^*) + \beta d(Tv, Tv) + \beta d(Tw^*, Tw^*) \\ &+ \gamma d(Tv, Tw^*) + \gamma d(Tw^*, Tv). \end{aligned}$$

Then $d(Tv, Tw^*) \leq (\alpha + 2\gamma) d(Tv, Tw^*)$, which implies that $Tv = Tw^*$. So we have (b).

We show that (c) holds. Let $\{x_n\}$ be a sequence in A_0 such that $d(x_{n+1}, Tx_n) = d(A, B)$ for all $n \geq 0$. So we get $Tx_{n+1} \in STx_n$. By using Claim 2, we have, for all $n \geq 0$,

$$d(Tx_{n+2}, Tx_{n+1}) \leq c d(Tx_{n+1}, Tx_n).$$

Thus $\{Tx_n\}$ is a Cauchy sequence and hence $\lim_{n\to\infty} Tx_n = s$ for some $s \in X$. Since $w \in Sw$ and $Tx_{n+1} \in STx_n$, we have

$$\begin{aligned} d(w, Tx_{n+1}) \leq{}& \alpha d(w, Tx_n) + \beta d(w, w) + \beta d(Tx_n, Tx_{n+1}) \\ &+ \gamma d(w, Tx_{n+1}) + \gamma d(Tx_n, w). \end{aligned}$$

As $n \to \infty$, we have $d(w, s) \leq (\alpha + 2\gamma) d(w, s)$. So $w = s$. That is, $\lim_{n\to\infty} Tx_n = w$. So (c) holds. □

Remark 7 The conclusion (c) of Theorem 6 is not mentioned in [2], Proposition 3.10.

Definition 8 ([1]) Let (X, d) be a metric space. Let A and B be nonempty subsets of X. The set B is said to be *approximatively compact with respect to A* if every sequence $\{y_n\}$ of B satisfying the condition that $\lim_{n\to\infty} d(x, y_n) = d(x, B)$ for some x in A has a convergent subsequence.

We show that the following theorem of Basha and Shahzad [1] is also a consequence of our Theorem 4.

Theorem 9 ([1], Theorem 3.4) *Let (X, d) be a complete metric space. Let A and B be nonempty subsets of X such that A_0 is nonempty. Let $T : A \to B$ be a mapping such that $T(A_0) \subset B_0$. Let us assume the following conditions*:

- *A, B are closed*;
- *A is approximatively compact with respect to B*;
- *T is continuous*;
- *T is a generalized proximal contraction of the second kind.*

Then the following hold:

(a) *there is an element x in A such that $d(x, Tx) = d(A, B)$*;
(b) *if there exists $\widehat{x} \in A$ such that $d(\widehat{x}, T\widehat{x}) = d(A, B)$, then $T\widehat{x} = Tx$*;
(c) *if $\{x_n\}$ is a sequence in A_0 satisfying $d(x_{n+1}, Tx_n) = d(A, B)$ for each $n \geq 0$, then $\lim_{n\to\infty} Tx_n = Tx$.*

Proof We define the mappings $S : T(A_0) \to 2^{T(A_0)} \setminus \{\varnothing\}$ and $F : T(A_0) \to [0, \infty)$ as the ones in the proof of Theorem 6. It follows that the condition (2.1) in Theorem 4 holds.

Next, we show that the condition (2.2) in Theorem 4 holds. Let $\{x_n\}$ be a sequence in $T(A_0)$ and let $z \in X$. Assume that

$$\lim_{n\to\infty} d(x_n, z) = 0 \quad \text{and} \quad \lim_{n\to\infty} d(x_n, Sx_n) = 0. \tag{2.11}$$

Since $x_n \in T(A_0) \subset T(A) \subset B$ and B is closed, $z \in B$. We choose a sequence $\{y_n\}$ in $T(A_0)$ so that $y_n \in Sx_n$ for each $n \geq 0$ and

$$\lim_{n\to\infty} d(x_n, y_n) = 0. \tag{2.12}$$

Since $y_n \in Sx_n$ for each $n \geq 0$, we write $y_n = Tu_n$ for some $u_n \in A_0$ with

$$d(u_n, x_n) = d(A, B).$$

We have

$$d(A, B) \leq d(u_n, z) \leq d(u_n, x_n) + d(x_n, z) = d(A, B) + d(x_n, z).$$

So $\lim_{n\to\infty} d(u_n, z) = d(A, B)$. Since A is approximatively compact with respect to B, there is a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $u_{n_k} \to u$ for some $u \in A$. Since T is continuous, we get $Tu_{n_k} \to Tu$. Using (2.11) and (2.12), we get $y_n \to z$ and hence

$$Tu = \lim_{k\to\infty} Tu_{n_k} = \lim_{n\to\infty} Tu_n = \lim_{n\to\infty} y_n = z.$$

Therefore,

$$d(u, Tu) = d(u, z) = \lim_{k\to\infty} d(u_{n_k}, x_{n_k}) = d(A, B).$$

That is, $Tu \in STu$ or $z \in Sz \cap T(A_0)$. Therefore, the condition (2.2) in Theorem 4 holds. Using Theorem 4, there is $w \in T(A_0)$ such that $w \in Sw$, that is, $w = T\widehat{w}$ for some $\widehat{w} \in A_0$ with $d(\widehat{w}, T\widehat{w}) = d(A, B)$. So (a) holds. The rest of the conclusions follow from Theorem 6. □

For a generalized proximal contraction of the first kind, the closedness of A_0 is more general than the condition that B is approximatively compact with respect to A (see Proposition 3.3 of [2]). Hence Proposition 3.5 of [2] (see our Theorem 5) is a generalized version of Theorem 3.1 of [1]. However, this is not the case for a generalized proximal contraction of the second kind. The following example is applicable in Theorem 9 but not in Theorem 6. That is, there is a continuous generalized proximal contraction of the second kind $T : A \to B$ such that $T(A_0)$ is not closed but A is approximatively compact with respect to B.

Example 10 We consider the 2-dimensional Euclidean metric space $\mathbb{R}^2$. Let $A = \{(a, 0) : a \geq 0\}$ and $B = \{(b, 1) : b \geq 0\}$. We have $A_0 = A$ and $B_0 = B$. Let $T : A \to B$ be a mapping defined by, for each $(a, 0) \in A$,

$$T(a, 0) = \bigl(f(a), 1\bigr),$$

where

$$f(a) = \frac{1}{2} - \frac{1}{a+2}.$$

It is clear that, for each $a, b \geq 0$,

$$\bigl|f(a) - f(b)\bigr| \leq \frac{1}{4}|a - b|.$$

Note that $T(A_0) = T(A) = \{(x, 1) : x \in [0, \frac{1}{2})\}$ is *not* closed. It is clear that A is approximatively compact with respect to B and T is continuous. We show that T is a generalized proximal contraction of the second kind. In fact, let u, v, x, y be elements in A such that $d(u, Tx) = d(v, Ty) = d(A, B)$. We write $x = (a_1, 0)$ and $y = (a_2, 0)$ for some $a_1, a_2 \geq 0$. So $u = (f(a_1), 1)$ and $v = (f(a_2), 1)$. We obtain

$$d(Tu, Tv) = \bigl|f^2(a_1) - f^2(a_2)\bigr| \leq \frac{1}{4}\bigl|f(a_1) - f(a_2)\bigr| = \frac{1}{4} d(Tx, Ty).$$

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors confirm that the final manuscript has been read and approved by all authors. All authors contributed equally to the writing of this paper.

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