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The modified S-iteration process for nonexpansive mappings in CAT(κ) spaces

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Abstract

We establish Δ-convergence results of a sequence generated by the modified S-iteration process for two nonexpansive mappings in complete CAT(κ) spaces. Some numerical examples are also provided to compare with the Ishikawa-type iteration process. Our main result extends the corresponding results in the literature.

MSC: 47H09; 47H10

Keywords: Δ-convergence; S-iteration process; nonexpansive mapping; common fixed point; CAT(κ) space

1 Introduction

Let C be a nonempty subset of a metric space (X, d). A mapping $T : C \to C$ is said to be nonexpansive if

$$d(Tx, Ty) \leq d(x, y)$$

for all $x, y \in C$. We say that $x \in C$ is a fixed point of T if

$$Tx = x.$$

We denote the set of all fixed points of T by $\operatorname{Fix}(T)$; for more details see [1].

The concept of Δ-convergence in general metric spaces was introduced by Lim [2]. Kirk [3] has proved the existence of fixed point of nonexpansive mappings in CAT(0) spaces. Kirk and Panyanak [4] specialized this concept to CAT(0) spaces and showed that many Banach space results involving weak convergence have precise analogs in this setting. Dhompongsa and Panyanak [5] continued to work in this direction. Their results involved the Mann and Ishikawa iteration process involving one mapping. After that Khan and Abbas [6] studied the approximation of common fixed point by the Ishikawa-type iteration process involving two mappings in CAT(0) spaces.

The Mann iteration process [7] was defined by $x_0 \in C$ and

$$x_{n+1} = a_n Tx_n \oplus (1 - a_n)x_n, \quad n \geq 0, \tag{1.1}$$

where $\{a_n\}$ is a sequence in $(0, 1)$. He *et al.* [7] proved the convergence results in CAT(κ) spaces.

The Ishikawa iteration process [8] was defined by $x_0 \in C$ and

$$\begin{aligned} y_n &= b_n T x_n \oplus (1-b_n)x_n, \\ x_{n+1} &= a_n T y_n \oplus (1-a_n)x_n, \quad n \geq 0, \end{aligned} \tag{1.2}$$

where $\{a_n\}$ and $\{b_n\}$ are sequences in $(0,1)$. Jun [8] proved that the sequence $\{x_n\}$ generated by (1.2) Δ-converges to a fixed point of T in CAT(κ) spaces.

The S-iteration process [9] was defined by $x_0 \in C$ and

$$\begin{aligned} y_n &= b_n T x_n \oplus (1-b_n)x_n, \\ x_{n+1} &= a_n T y_n \oplus (1-a_n)T x_n, \quad n \geq 0, \end{aligned} \tag{1.3}$$

where $\{a_n\}$ and $\{b_n\}$ are sequences in $(0,1)$. This scheme has a better convergence rate than those of (1.1) and (1.2) for a contraction in metric space (see [9]).

In 2011, Khan and Abbas [6] studied the iteration (1.3) in CAT(0) spaces and proved the Δ-convergence. Khan and Abbas [6] also studied the following Ishikawa-type iteration process: $x_0 \in C$ and

$$\begin{aligned} y_n &= b_n T x_n \oplus (1-b_n)x_n, \\ x_{n+1} &= a_n S y_n \oplus (1-a_n)x_n, \quad n \geq 0, \end{aligned} \tag{1.4}$$

where $\{a_n\}$ and $\{b_n\}$ are sequences in $(0,1)$. This iteration was introduced by Das and Debata [10]. They proved some results on Δ-convergence in CAT(0) spaces for two nonexpansive mappings of the sequence defined by (1.4).

There have been, recently, many convergence and existence results established in CAT(0) and CAT(κ) spaces (see [11–19]).

Motivated by [6] and [9], in this paper, we study the following modified S-iteration process: $x_0 \in C$ and

$$\begin{aligned} y_n &= b_n T x_n \oplus (1-b_n)x_n, \\ x_{n+1} &= a_n S y_n \oplus (1-a_n)T x_n, \quad n \geq 0, \end{aligned} \tag{1.5}$$

where $\{a_n\}$ and $\{b_n\}$ are sequences in $(0,1)$. We prove some results on Δ-convergence for two nonexpansive mappings in CAT(κ) spaces with $\kappa \geq 0$ under suitable conditions. We finally provide some examples and numerical results to support our main result.

Remark 1.1 We note that this scheme reduces to the iteration process (1.3) when $S = T$. The iteration process (1.5) is quite different from (1.4).

2 Preliminaries and lemmas

In this section, we provide some basic concepts, definitions, and lemmas which will be used in the sequel and can be found in [20].

Let (X,d) be a metric space and $x, y \in X$ with $d(x,y) = l$. A geodesic path from x to y is an isometry $c : [0,l] \to X$ such that $c(0) = x$, $c(l) = y$. The image of a geodesic path is called geodesic segment. The space (X,d) is said to be a geodesic space if every two points of X

are joined by a geodesic, and X is a uniquely geodesic space if every two points of X are joined by only one geodesic segment. We write $(1-t)x \oplus ty$ for the unique point z in the geodesic segment joining x and y such that $d(x,z) = td(x,y)$ and $d(y,z) = (1-t)d(x,y)$ for $t \in [0,1]$. A subset E of X is said to be convex if E includes every geodesic segment joining any two of its points.

Let D be a positive number. A metric space (X,d) is called a D-geodesic space if any two points of X with the distance less than D are joined by a geodesic. If this holds in a convex set E, then E is said to be D-convex. For a constant κ, we denote M_κ by the 2-dimensional, complete, simply connected spaces of curvature κ.

In the following, we assume that $\kappa \geq 0$ and define the diameter D_κ of M_κ by $D_\kappa = \frac{\pi}{\sqrt{\kappa}}$ for $\kappa > 0$ and $D_\kappa = \infty$ for $\kappa = 0$. It is well known that any ball in X with radius less than $D_\kappa/2$ is convex [20]. A geodesic triangle $\Delta(x,y,z)$ in the metric space (X,d) consists of three points x, y, z in X (the vertices of Δ) and three geodesic segments between each pair of vertices. For $\Delta(x,y,z)$ in a geodesic space X satisfying

$$d(x,y) + d(y,z) + d(z,x) < 2D_\kappa,$$

there exist points $\bar{x},\bar{y},\bar{z} \in M_\kappa$ such that $d(x,y) = d_\kappa(\bar{x},\bar{y})$, $d(y,z) = d_\kappa(\bar{y},\bar{z})$, and $d(z,x) = d_\kappa(\bar{z},\bar{x})$ where d_κ is the metric of M_κ. We call the triangle having vertices $\bar{x},\bar{y},\bar{z} \in M_\kappa$ a comparison triangle of $\Delta(x,y,z)$. A geodesic triangle $\Delta(x,y,z)$ in X with $d(x,y) + d(y,z) + d(z,x) < 2D_\kappa$ is said to satisfy the CAT(κ) inequality if, for any $p,q \in \Delta(x,y,z)$ and for their comparison points $\bar{p},\bar{q} \in \bar{\Delta}(\bar{x},\bar{y},\bar{z})$, we have $d(p,q) \leq d_\kappa(\bar{p},\bar{q})$.

Definition 2.1 A metric space (X,d) is called a CAT(κ) space if it is D_κ-geodesic and any geodesic triangle $\Delta(x,y,z)$ in X with $d(x,y) + d(y,z) + d(z,x) < 2D_\kappa$ satisfies the CAT(κ) inequality.

Since the results in CAT(κ) spaces can be deduced from those in CAT(1) spaces, we now sufficiently state lemmas on CAT(1) spaces.

Lemma 2.2 [20] *Let (X,d) be a* CAT(1) *space and let K be a closed and π-convex subset of X. Then for each point $x \in X$ such that $d(x,K) < \pi/2$, there exists a unique point $y \in K$ such that $d(x,y) = d(x,K)$.*

Lemma 2.3 [21] *Let (X,d) be a* CAT(1) *space. Then there is a constant $M > 0$ such that*

$$d^2\big(x, ty \oplus (1-t)z\big) \leq td^2(x,y) + (1-t)d^2(x,z) - \frac{M}{2}t(1-t)d^2(y,z)$$

for any $t \in [0,1]$ and any point $x,y,z \in X$ such that $d(x,y) \leq \pi/4$, $d(x,z) \leq \pi/4$, and $d(y,z) \leq \pi/2$.

Let $\{x_n\}$ be a bounded sequence in X. For $x \in X$, we set

$$r\big(x, \{x_n\}\big) = \limsup_{n\to\infty} d(x,x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r\big(\{x_n\}\big) = \inf\big\{r\big(x, \{x_n\}\big) : x \in X\big\}$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A\big(\{x_n\}\big) = \big\{x \in X : r\big(\{x_n\}\big) = r\big(x, \{x_n\}\big)\big\}.$$

Definition 2.4 A sequence $\{x_n\}$ in X is said to Δ-converge to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$.

In this case we write $\Delta\text{-}\lim_{n\to\infty} x_n = x$ and call x the Δ-limit of $\{x_n\}$.

Definition 2.5 For a sequence $\{x_n\}$ in X, a point $x \in X$ is a Δ-cluster point of $\{x_n\}$ if there exists a subsequence of $\{x_n\}$ that Δ-converges to x.

Lemma 2.6 [7] *Let (X,d) be a complete* CAT(κ) *space and let $p \in X$. Suppose that a sequence $\{x_n\}$ in X Δ-converges to x such that $r(p,\{x_n\}) < D_\kappa/2$. Then*

$$d(x,p) \le \liminf_{n\to\infty} d(x_n,p).$$

Definition 2.7 Let (X,d) be a complete metric space and let K be a nonempty subset of X. Then a sequence $\{x_n\}$ in X is Fejér monotone with respect to K if

$$d(x_{n+1},q) \le d(x_n,q)$$

for all $n \ge 0$ and all $q \in K$.

Lemma 2.8 [7] *Let (X,d) be a complete* CAT(1) *space and let K be a nonempty subset of X. Suppose that the sequence $\{x_n\}$ in X is Fejér monotone with respect to K and the asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is less than $\pi/2$. If any Δ-cluster point x of $\{x_n\}$ belongs to K, then $\{x_n\}$ Δ-converges to a point in K.*

3 Main results

Lemma 3.1 *Let (X,d) be a complete* CAT(1) *space and let C be a nonempty, closed, and convex subset of X. Let T and S be two nonexpansive mappings of C such that $F := \mathrm{Fix}(T) \cap \mathrm{Fix}(S) \neq \emptyset$. Let $\{x_n\}$ be defined by* (1.5) *for $x_0 \in C$ such that $d(x_0,F) \le \pi/4$. Then there exists a unique point p in F such that $d(y_n,p) \le d(x_n,p) \le \pi/4$ for all $n \ge 0$.*

Proof By Theorem 3.4 in [22] and Lemma 2.2, there exists a unique point p in F such that $d(x_0,F) = d(x_0,p)$. From $d(Tx_0,p) \le d(x_0,p) \le \pi/4$ and $B_{\pi/4}[p]$ is convex, we have

$$d(y_0,p) = d\big(b_0 Tx_0 \oplus (1-b_0)x_0, p\big) \le d(x_0,p) \le \pi/4.$$

Suppose that $d(y_k,p) \le d(x_k,p) \le \pi/4$ for $k \ge 1$. Since $d(Sy_k,p) \le d(y_k,p) \le \pi/4$ and $B_{\pi/4}[p]$ is convex, we have

$$d(x_{k+1},p) = d\big(a_k Sy_k \oplus (1-a_k)Tx_k, p\big) \le d(x_k,p) \le \pi/4$$

and

$$d(y_{k+1},p) = d\big(b_{k+1} Tx_{k+1} \oplus (1-b_{k+1})x_{k+1}, p\big) \le d(x_{k+1},p) \le \pi/4.$$

It follows that $d(y_{k+1}, p) \leq d(x_{k+1}, p) \leq \pi/4$. By mathematical induction, hence $d(y_n, p) \leq d(x_n, p) \leq \pi/4$ for all $n \geq 0$. □

Lemma 3.2 *Let (X, d) be a complete* CAT(1) *space and let C be a nonempty, closed, and convex subset of X. Let T and S be two nonexpansive mappings of C such that $F := \mathrm{Fix}(T) \cap \mathrm{Fix}(S) \neq \emptyset$. Let $\{a_n\}$ and $\{b_n\}$ be such that $0 < a \leq a_n, b_n \leq b < 1$ for all $n \geq 0$ and for some a, b. If $\{x_n\}$ is defined by* (1.5) *for $x_0 \in C$ such that $d(x_0, F) \leq \pi/4$, then*

(i) $\lim_{n\to\infty} d(x_n, p)$ *exists;*

(ii) $\lim_{n\to\infty} d(Tx_n, x_n) = 0 = \lim_{n\to\infty} d(Sx_n, x_n)$.

Proof By Lemma 2.3 and Lemma 3.1, there exist $p \in F$ and $M > 0$ such that

$$\begin{aligned} d^2(x_{n+1}, p) &= d^2\big(a_n Sy_n \oplus (1-a_n)Tx_n, p\big) \\ &\leq a_n d^2(Sy_n, p) + (1-a_n)d^2(Tx_n, p) - \frac{M}{2}a_n(1-a_n)d^2(Sy_n, Tx_n) \\ &\leq a_n d^2(y_n, p) + (1-a_n)d^2(x_n, p) - \frac{M}{2}a_n(1-a_n)d^2(Sy_n, Tx_n) \qquad (3.1) \\ &\leq a_n d^2(y_n, p) + (1-a_n)d^2(x_n, p) \qquad (3.2) \end{aligned}$$

and

$$\begin{aligned} d^2(y_n, p) &= d^2\big(b_n Tx_n \oplus (1-b_n)x_n, p\big) \\ &\leq b_n d^2(Tx_n, p) + (1-b_n)d^2(x_n, p) - \frac{M}{2}b_n(1-b_n)d^2(Tx_n, x_n) \\ &\leq d^2(x_n, p) - \frac{M}{2}b_n(1-b_n)d^2(Tx_n, x_n) \\ &\leq d^2(x_n, p). \qquad (3.3) \end{aligned}$$

By (3.1) and (3.3), we have

$$\begin{aligned} d^2(x_{n+1}, p) &\leq d^2(x_n, p) - \frac{M}{2}a_n(1-a_n)d^2(Sy_n, Tx_n) \\ &\leq d^2(x_n, p). \end{aligned}$$

Hence

$$d(x_{n+1}, p) \leq d(x_n, p).$$

This shows that $\{d(x_n, p)\}$ is decreasing and this proves part (i). Let

$$\lim_{n\to\infty} d(x_n, p) = c. \qquad (3.4)$$

We next prove part (ii). From (3.2), we get

$$d^2(x_{n+1}, p) \leq a_n d^2(y_n, p) + (1-a_n)d^2(x_n, p),$$

from which it follows that

$$a_n d^2(x_n,p) \le d^2(x_n,p) + a_n d^2(y_n,p) - d^2(x_{n+1},p).$$

This implies that

$$d^2(x_n,p) \le d^2(y_n,p) + \frac{1}{a_n}\bigl[d^2(x_n,p) - d^2(x_{n+1},p)\bigr].$$

So

$$c^2 \le \liminf_{n\to\infty} d^2(y_n,p).$$

On the other hand, (3.3) gives

$$\limsup_{n\to\infty} d^2(y_n,p) \le c^2$$

so that

$$\lim_{n\to\infty} d(y_n,p) = c. \tag{3.5}$$

We see that

$$d^2(y_n,p) \le d^2(x_n,p) - \frac{M}{2} b_n(1-b_n) d^2(Tx_n,x_n),$$

thus

$$d^2(Tx_n,x_n) \le \frac{2}{b_n(1-b_n)M}\bigl[d^2(x_n,p) - d^2(y_n,p)\bigr].$$

Using (3.4) and (3.5), we can conclude that

$$\lim_{n\to\infty} d(Tx_n,x_n) = 0. \tag{3.6}$$

Next, we know that

$$d^2(x_{n+1},p) \le d^2(x_n,p) - \frac{M}{2} a_n(1-a_n) d^2(Sy_n,Tx_n),$$

from which it follows that

$$d^2(Sy_n,Tx_n) \le \frac{2}{a_n(1-a_n)M}\bigl[d^2(x_n,p) - d^2(x_{n+1},p)\bigr].$$

This yields

$$\lim_{n\to\infty} d(Sy_n,Tx_n) = 0. \tag{3.7}$$

Using (1.5), we obtain

$$d(y_n, x_n) = d\big(b_n Tx_n \oplus (1 - b_n)x_n, x_n\big) = b_n d(Tx_n, x_n).$$

This implies by (3.6),

$$\lim_{n\to\infty} d(y_n, x_n) = 0. \tag{3.8}$$

Since

$$d(Sy_n, x_n) \leq d(Sy_n, Tx_n) + d(Tx_n, x_n),$$

by (3.6) and (3.7),

$$\lim_{n\to\infty} d(Sy_n, x_n) = 0. \tag{3.9}$$

Finally, we see that

$$\begin{aligned} d(Sx_n, x_n) &\leq d(Sx_n, Sy_n) + d(Sy_n, x_n) \\ &\leq d(x_n, y_n) + d(Sy_n, x_n), \end{aligned}$$

hence, by (3.8) and (3.9), we get

$$\lim_{n\to\infty} d(Sx_n, x_n) = 0.$$

This completes the proof. □

Theorem 3.3 *Let (X, d) be complete a $\mathrm{CAT}(\kappa)$ space and let C be a nonempty, closed, and convex subset of X. Let T and S be two nonexpansive mappings of C such that $F := \mathrm{Fix}(T) \cap \mathrm{Fix}(S) \neq \emptyset$. Let $\{a_n\}$ and $\{b_n\}$ be such that $0 < a \leq a_n, b_n \leq b < 1$ for all $n \geq 0$ and for some a, b. If $\{x_n\}$ is defined by (1.5) for $x_0 \in C$ such that $d(x_0, F) < D_\kappa/4$, then $\{x_n\}$ Δ-converges to a point in F.*

Proof Without loss of generality, we assume that $\kappa = 1$. Set $F_0 := F \cap B_{\pi/2}(x_0)$. Let $q \in F_0$. Since $d(Tx_0, q) \leq d(x_0, q)$ and since the open ball $B_{\pi/2}(q)$ in C with radius $r < \pi/2$ is convex, we have

$$d(y_0, q) = d\big(b_0 Tx_0 \oplus (1 - b_0)x_0, q\big) \leq d(x_0, q).$$

Since $d(Sy_0, q) \leq d(y_0, q)$ and since the open ball $B_{\pi/2}(q)$ in C with radius $r < \pi/2$ is convex, we have

$$d(x_1, q) = d\big(a_0 Sy_0 \oplus (1 - a_0)Tx_0, q\big) \leq d(x_0, q).$$

By mathematical induction, we can show that

$$d(x_{n+1}, q) \leq d(x_n, q) \leq d(x_0, q)$$

for all $n \geq 0$. Hence $\{x_n\}$ is a Fejér monotone sequence with respect to F_0. Let $p \in F$ such that $d(x_0, p) \leq \pi/4$. Then $p \in F_0$. Also we have

$$d(x_{n+1}, p) \leq d(x_n, p) \leq d(x_0, p) < \pi/4 \tag{3.10}$$

for all $n \geq 0$. This shows that $r(\{x_n\}) < \pi/4$. By Lemma 2.8, let $x \in C$ be a Δ-cluster point of $\{x_n\}$. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which Δ-converges to x. From (3.10), we get

$$r\bigl(p, \{x_{n_k}\}\bigr) \leq d(x_0, p) < \pi/4.$$

By Lemma 2.6, we obtain

$$d(x, x_0) \leq d(x, p) + d(x_0, p) \leq \liminf_{k\to\infty} d(x_{n_k}, p) + d(x_0, p) < \pi/2.$$

This implies that $x \in B_{\pi/2}(x_0)$. By Lemma 3.2, we have

$$\begin{aligned} \limsup_{k\to\infty} d(Tx, x_{n_k}) &\leq \limsup_{k\to\infty} d(Tx, Tx_{n_k}) + \limsup_{k\to\infty} d(Tx_{n_k}, x_{n_k}) \\ &\leq \limsup_{k\to\infty} d(x, x_{n_k}) \end{aligned}$$

and

$$\begin{aligned} \limsup_{k\to\infty} d(Sx, x_{n_k}) &\leq \limsup_{k\to\infty} d(Sx, Sx_{n_k}) + \limsup_{k\to\infty} d(Sx_{n_k}, x_{n_k}) \\ &\leq \limsup_{k\to\infty} d(x, x_{n_k}). \end{aligned}$$

Hence $Tx, Sx \in A(\{x_{n_k}\})$ and $Tx = x = Sx$. Therefore $x \in F_0$. By Lemma 2.8, we thus complete the proof. □

We immediately obtain the following results in CAT(0) spaces.

Corollary 3.4 *Let (X, d) be a complete* CAT(0) *space and let C be a nonempty, closed, and convex subset of X. Let T and S be two nonexpansive mappings of C such that $F :=$* $\operatorname{Fix}(T) \cap \operatorname{Fix}(S) \neq \emptyset$. *Let $\{a_n\}$ and $\{b_n\}$ be such that $0 < a \leq a_n, b_n \leq b < 1$ for all $n \geq 0$ and for some a, b. If $\{x_n\}$ is defined by* (1.5), *then $\{x_n\}$ Δ-converges to a point in F.*

Remark 3.5 When $S = T$, we obtain Theorem 1 of Khan and Abbas [6].

Along a similar proof line, we can obtain the following result for the Ishikawa-type iteration process.

Theorem 3.6 *Let (X, d) be a complete* CAT(κ) *space and let C be a nonempty, closed, and convex subset of X. Let T and S be two nonexpansive mappings of C such that $F :=$* $\operatorname{Fix}(T) \cap \operatorname{Fix}(S) \neq \emptyset$. *Let $\{a_n\}$ and $\{b_n\}$ be such that $0 < a \leq a_n, b_n \leq b < 1$ for all $n \geq 0$ and for some a, b. If $\{x_n\}$ is defined by* (1.4) *for $x_0 \in C$ such that $d(x_0, F) < D_\kappa/4$, then $\{x_n\}$ Δ-converges to a point in F.*

Corollary 3.7 [6] *Let (X,d) be a complete* CAT(0) *space and let C be a nonempty, closed, and convex subset of X. Let T and S be two nonexpansive mappings of C such that $F := \mathrm{Fix}(T) \cap \mathrm{Fix}(S) \neq \emptyset$. Let $\{a_n\}$ and $\{b_n\}$ be such that $0 < a \leq a_n, b_n \leq b < 1$ for all $n \geq 0$ and for some a, b. If $\{x_n\}$ is defined by* (1.4), *then $\{x_n\}$ Δ-converges to a point in F.*

4 Numerical examples

In this section, we consider the m-sphere $\mathbb{S}^m$, which is a CAT(κ) space.

The m-sphere $\mathbb{S}^m$ is defined by

$$\left\{x = (x_1, \ldots, x_{m+1}) \in \mathbb{R}^{m+1} : \langle x, x\rangle = 1\right\},$$

where $\langle\cdot,\cdot\rangle$ denotes the Euclidean scalar product.

Next, the normalized geodesic $c : \mathbb{R} \to \mathbb{S}^m$ starting from $x \in \mathbb{S}^m$ is given by

$$c(t) = (\cos t)x + (\sin t)v, \quad \forall t \in \mathbb{R},$$

where $v \in T_x\mathbb{S}^m$ is the unit vector; while the distance d on $\mathbb{S}^m$ is

$$d(x,y) = \arccos\bigl(\langle x, y\rangle\bigr), \quad \forall x, y \in \mathbb{S}^m.$$

Then iteration process (1.4) has the form

$$\begin{aligned} y_n &= \bigl(\cos(1-b_n)r(x_n,x_n)\bigr)x_n + \bigl(\sin(1-b_n)r(x_n,x_n)\bigr)V(x_n,x_n), \\ x_{n+1} &= \bigl(\cos(1-a_n)\bar{r}(x_n,y_n)\bigr)x_n + \bigl(\sin(1-a_n)\bar{r}(x_n,y_n)\bigr)\bar{V}(x_n,y_n), \quad \forall n \geq 0; \end{aligned} \tag{4.1}$$

and iteration process (1.5) has the form

$$\begin{aligned} y_n &= \bigl(\cos(1-b_n)r(x_n,x_n)\bigr)x_n + \bigl(\sin(1-b_n)r(x_n,x_n)\bigr)V(x_n,x_n), \\ x_{n+1} &= \bigl(\cos(1-a_n)\bar{r}(Tx_n,y_n)\bigr)Tx_n + \bigl(\sin(1-a_n)\bar{r}(Tx_n,y_n)\bigr)\bar{V}(Tx_n,y_n), \quad \forall n \geq 0, \end{aligned} \tag{4.2}$$

where

$$\begin{aligned} &r(x,y) = \arccos\bigl(\langle x, Ty\rangle\bigr), \qquad \bar{r}(x,y) = \arccos\bigl(\langle x, Sy\rangle\bigr), \\ &V(x,y) = \frac{Ty - \langle x, Ty\rangle x}{\sqrt{1-\langle x, Ty\rangle^2}} \quad \text{and} \quad \bar{V}(x,y) = \frac{Sy - \langle x, Sy\rangle x}{\sqrt{1-\langle x, Sy\rangle^2}}, \quad \forall x, y \in \mathbb{R}^{m+1}. \end{aligned}$$

Example 4.1 Let $C = \mathbb{S}^3$ and let T and S be two nonexpansive mappings of C be defined by

$$Tx = (x_1, -x_2, -x_3, -x_4) \quad \text{and} \quad Sx = (x_1, -x_3, -x_2, -x_4).$$

For any $x = (x_1, x_2, x_3, x_4) \in \mathbb{S}^3$. Then $\mathrm{Fix}(T) = \{(1,0,0,0)\} = \mathrm{Fix}(S)$.

Choose $x_0 = (0.5, 0.5, 0.5, 0.5)$ and let $a_n = \frac{n}{20n+1}$ and $b_n = \frac{n}{10n+1}$. Then we obtain the numerical results in Table 1 and Figure 1.

We next consider the hyperbolic m-space $\mathbb{H}^m$.

Table 1 Convergence behavior of (4.1) and (4.2)

n	x_n is defined by (4.1)	x_n is defined by (4.2)
1	(0.64798180, 0.43974219, 0.43974219, 0.43974219)	(0.72059826, 0.40030744, 0.40030744, 0.40030744)
2	(0.75902424, 0.37589103, 0.37589103, 0.37589103)	(0.85117540, 0.30304039, 0.30304039, 0.30304039)
3	(0.83727864, 0.31568152, 0.31568152, 0.31568152)	(0.92240530, 0.22298615, 0.22298615, 0.22298615)
4	(0.89102247, 0.26209347, 0.26209347, 0.26209347)	(0.95999322, 0.16167149, 0.16167149, 0.16167149)
5	(0.92740383, 0.21596460, 0.21596460, 0.21596460)	(0.97950207, 0.11629804, 0.11629804, 0.11629804)
6	(0.95181217, 0.17706270, 0.17706270, 0.17706270)	(0.98953675, 0.08330070, 0.08330070, 0.08330070)
7	(0.96809244, 0.14468014, 0.14468014, 0.14468014)	(0.99467153, 0.05952183, 0.05952183, 0.05952183)
8	(0.97890884, 0.11795122, 0.11795122, 0.11795122)	(0.99729071, 0.04247057, 0.04247057, 0.04247057)
9	(0.98607580, 0.09601130, 0.09601130, 0.09601130)	(0.99862397, 0.03027739, 0.03027739, 0.03027739)
10	(0.99081568, 0.07806896, 0.07806896, 0.07806896)	(0.99930171, 0.02157235, 0.02157235, 0.02157235)
⋮	⋮	⋮
55	(1.00000000, 0.00000620, 0.00000620, 0.00000620)	(1.00000000, 0.00000000, 0.00000000, 0.00000000)

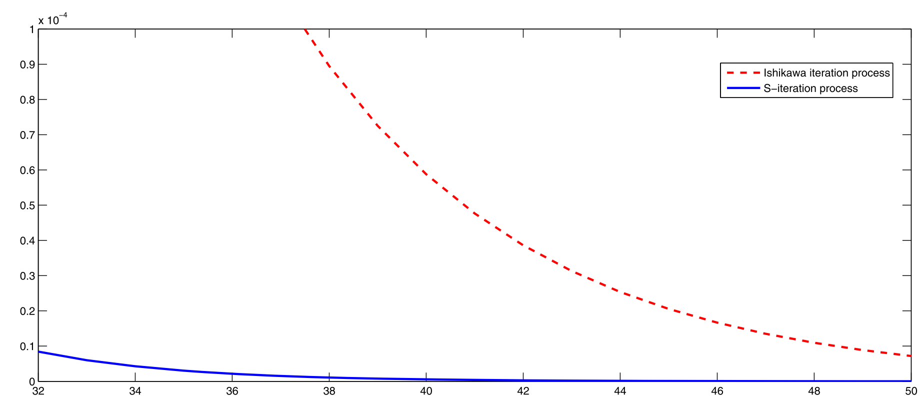

Figure 1 Comparison of errors in (4.1) and (4.2).

The hyperbolic m-space $\mathbb{H}^m$ is defined by

$$\bigl\{x = (x_1, \dots, x_{m+1}) \in \mathbb{R}^{m+1} : \langle x, x\rangle = -1, x_{m+1} \geq 1\bigr\},$$

where

$$\langle x, y\rangle = \sum_{i=1}^{m} x_i y_i - x_{m+1} y_{m+1}, \quad \forall x = (x_i), y = (y_i) \in \mathbb{R}^{m+1}.$$

Next, the normalized geodesic $c : \mathbb{R} \to \mathbb{H}^m$ starting from $x \in \mathbb{H}^m$ is given by

$$c(t) = (\cosh t)x + (\sinh t)v, \quad \forall t \in \mathbb{R},$$

where $v \in T_x\mathbb{H}^m$ is the unit vector; while the distance d on $\mathbb{H}^m$ is

$$d(x, y) = \operatorname{arccosh}\bigl(-\langle x, y\rangle\bigr), \quad \forall x, y \in \mathbb{H}^m.$$

Then iteration process (1.4) has the form

$$\begin{aligned} y_n &= \bigl(\cosh(1 - b_n) r(x_n, x_n)\bigr) x_n + \bigl(\sinh(1 - b_n) r(x_n, x_n)\bigr) V(x_n, x_n), \\ x_{n+1} &= \bigl(\cosh(1 - a_n) \bar{r}(x_n, y_n)\bigr) x_n + \bigl(\sinh(1 - a_n) \bar{r}(x_n, y_n)\bigr) \bar{V}(x_n, y_n), \quad \forall n \geq 0; \end{aligned} \tag{4.3}$$

Table 2 Convergence behavior of (4.3) and (4.4)

n	x_n is defined by (4.3)	x_n is defined by (4.4)
1	(1.77547237, 0.98692599, 1.18530561, 2.55563582)	(1.08924010, 0.51696725, 0.46166164, 1.63304335)
2	(0.86101141, 1.01967810, 0.66960079, 1.79706686)	(0.09467395, 0.45823859, 0.19274032, 1.12075626)
3	(0.50793301, 0.63730440, 0.65740263, 1.44787122)	(0.09214727, −0.02529962, 0.22462285, 1.02936224)
4	(0.45512909, 0.39762058, 0.47219289, 1.26024234)	(0.11301324, 0.05817553, −0.05672644, 1.00964067)
5	(0.35116401, 0.33063402, 0.31268711, 1.15343324)	(−0.05463911, 0.05458569, 0.04470058, 0.04470058)
6	(0.24499408, 0.26285320, 0.24775268, 1.09109821)	(0.03652253, −0.04228414, 0.02306040, 1.00182515)
7	(0.18906704, 0.19095564, 0.19822904, 1.05427945)	(0.00627063, 0.02939078, −0.02899485, 1.00087154)
8	(0.15056381, 0.14567561, 0.14817634, 1.03239870)	(−0.01800599, −0.00210393, 0.02266815, 1.00042115)
9	(0.11463049, 0.11504761, 0.11272592, 1.01935432)	(0.01662614, −0.00999700, −0.00565079, 1.00020413)
10	(0.08734712, 0.08852508, 0.08831015, 1.01156557)	(−0.00652282, 0.01156060, −0.00466467, 1.00009897)
⋮	⋮	⋮
50	(0.00000284, 0.00000284, 0.00000284, 1.00000000)	(0.00000000, 0.00000000, 0.00000000, 1.00000000)

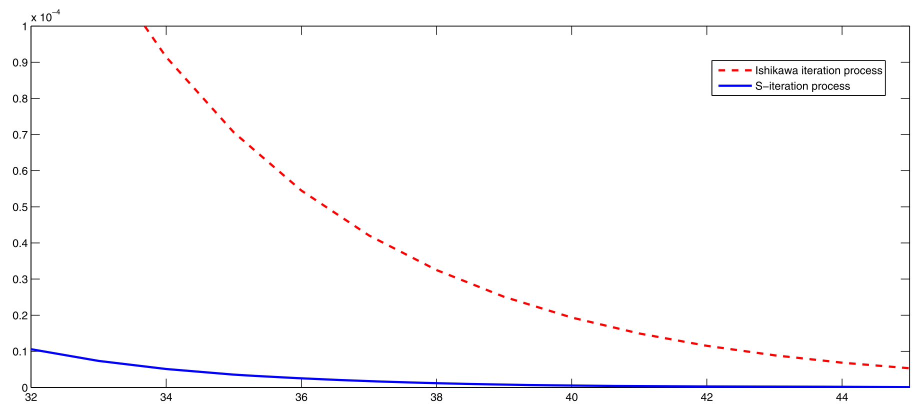

Figure 2 Comparison of errors in (4.3) and (4.4).

and iteration process (1.5) has the form

$$\begin{aligned} y_n &= \bigl(\cosh(1-b_n)r(x_n,x_n)\bigr)x_n + \bigl(\sinh(1-b_n)r(x_n,x_n)\bigr)V(x_n,x_n), \\ x_{n+1} &= \bigl(\cosh(1-a_n)\bar{r}(Tx_n,y_n)\bigr)Tx_n \\ &\quad + \bigl(\sinh(1-a_n)\bar{r}(Tx_n,y_n)\bigr)\bar{V}(Tx_n,y_n), \quad \forall n \geq 0, \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} &r(x,y) = \operatorname{arccosh}\bigl(-\langle x, Ty\rangle\bigr), \qquad \bar{r}(x,y) = \operatorname{arccosh}\bigl(-\langle x, Sy\rangle\bigr), \\ &V(x,y) = \frac{Ty + \langle x, Ty\rangle x}{\sqrt{\langle x, Ty\rangle^2 - 1}} \quad \text{and} \quad \bar{V}(x,y) = \frac{Sy + \langle x, Sy\rangle x}{\sqrt{\langle x, Sy\rangle^2 - 1}}, \quad \forall x, y \in \mathbb{R}^{m+1}. \end{aligned}$$

Example 4.2 Let $C = \mathbb{H}^3$ and let T and S be two nonexpansive mappings of C be defined by

$$Tx = (-x_1, -x_2, -x_3, x_4) \quad \text{and} \quad Sx = (-x_3, -x_1, -x_2, x_4)$$

for any $x = (x_1, x_2, x_3, x_4) \in \mathbb{H}^3$. Then $\operatorname{Fix}(T) = \{(0,0,0,1)\} = \operatorname{Fix}(S)$.

Choose $x_0 = (2, 2, 4, 5)$ and let $a_n = \frac{n}{5n+1}$ and $b_n = \frac{n}{7n+1}$. Then we obtain the numerical results in Table 2 and Figure 2.

From the numerical experience, we observe that the convergence rate of S-iteration process is much quicker than that of the Ishikawa iteration process.

Remark 4.3 The convergence behavior of Mann and Halpern iterations in Hadamard manifolds can be found in the work of Li *et al.* [18].

Competing interests
The authors declare that there is no conflict of interests regarding the publication of this article.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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