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Fixed point theorems and endpoint theorems for (α, ψ)-Meir-Keeler-Khan multivalued mappings

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Abstract

As an extension of the class of (α, ψ)-Meir-Keeler-Khan single-valued mappings defined by Redjel *et al.*, a new type of (α, ψ)-Meir-Keeler-Khan multivalued mappings is presented. Fixed point theorems and endpoints theorems are established on such mappings. Some main results by Redjel *et al.* and Khan *et al.* are extended and generalized.

MSC: Primary 47H09; 47H10; secondary 49M05

Keywords: (α, ψ)-Meir-Keeler-Khan multivalued mappings; approximate endpoint property; α-admissible function; α-continuous mapping

1 Introduction and preliminaries

In 1976, Khan [1] proved a fixed point theorem for metric spaces. Fisher [2] gave a revised version of this result.

Theorem 1.1 ([2]) *Let $T : X \to X$ be a mapping on a complete metric space (X, d) such that the following hypothesis holds*:

$$d(Tx, Ty) \le \mu \frac{d(x, Tx)d(x, Ty) + d(y, Ty)d(y, Tx)}{d(x, Ty) + d(y, Tx)}, \quad \mu \in [0, 1[,$$

if

$$d(x, Ty) + d(y, Tx) \ne 0,$$

and

$$d(Tx, Ty) = 0 \quad \text{if } d(x, Ty) + d(y, Tx) = 0.$$

Then T has a unique fixed point $\varsigma \in X$. Moreover, for every $x_0 \in X$, the sequence $\{T^n x_0\}$ converges to ς.

Samet *et al.* [3] introduced the notion of an α-*admissible* mapping.

Definition 1.2 ([3]) Let $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ be two mappings. The mapping T is said to be α-admissible if the following condition is satisfied:

$$\forall x, y \in X, \quad \alpha(x, y) \geq 1 \quad \Rightarrow \quad \alpha(Tx, Ty) \geq 1.$$

For some examples concerning the class of α-admissible mappings and other information on the subject, we refer to [3–5].

A (c)-comparison function ψ is a nondecreasing self-mapping on $[0, \infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for each $t > 0$, where ψ^n is the nth iteration of ψ. It is clear that $\psi(t) < t$ for all $t > 0$ and $\psi(0) = 0$ (see [5, 6]). We denote by Ω the family of all (c)-comparison functions,

Meir and Keeler [7] in 1969 established a fixed point theorem on a metric space (X, d). They studied the class of mappings satisfying the condition that for each $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that

$$\epsilon \leq d(x, y) < \epsilon + \delta(\epsilon) \quad \Rightarrow \quad d(Tx, Ty) < \epsilon$$

for any $x, y \in X$.

Latif *et al.* [5] defined the concept of (α, ψ)-Meir-Keeler self-mappings. Recently, Redjel *et al.* [8] introduced the concept of (α, ψ)-Meir-Keeler-Khan mappings.

Definition 1.3 ([8]) Let $T : X \to X$ be a self-mapping on a metric space (X, d). T is said to be an (α, ψ)-Meir-Keeler-Khan mapping if there exist $\psi \in \Omega$ and $\alpha : X \times X \to [0, \infty)$ such that for every $\epsilon > 0$, there exists $\delta(\epsilon)$ such that if

$$\epsilon \leq \psi\left(\frac{d(x, Tx)d(x, Ty) + d(y, Ty)d(y, Tx)}{d(x, Ty) + d(y, Tx)}\right) < \epsilon + \delta(\epsilon)$$

for any $x, y \in X$, then

$$\alpha(x, y)d(Tx, Ty) < \epsilon.$$

It is easily shown that if T is an (α, ψ)-Meir-Keeler-Khan mapping, then

$$\alpha(x, y)d(Tx, Ty) \leq \psi\left(\frac{d(x, Tx)d(x, Ty) + d(y, Ty)d(y, Tx)}{d(x, Ty) + d(y, Tx)}\right)$$

for any $x, y \in X$.

Concerning the class of α-admissible mappings, Redjel *et al.* [8] stated an existence theorem for fixed points of (α, ψ)-Meir-Keeler-Khan mappings with continuity assumption on the mapping.

Theorem 1.4 ([8]) *Let $T : X \to X$ be an (α, ψ)-Meir-Keeler-Khan mapping on the complete metric space (X, d). Suppose that the following hypotheses hold*:

(i) *T is an α-admissible mapping*;

(ii) *There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$*;

(iii) *T is continuous.*

Then there exists a fixed point of T in X.

In addition, they also established an existence theorem for fixed points of (α,ψ)-Meir-Keeler-Khan mappings without any continuity assumption on the mappings [8].

Theorem 1.5 ([8]) *Let $T: X \to X$ be an (α,ψ)-Meir-Keeler-Khan mapping on the complete metric space (X,d). Suppose that the following hypotheses hold*:

(i) *T is an α-admissible mapping*;
(ii) *There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$*;
(iii) *If $\{x_n\}$ is a sequence such that $\alpha(x_n, x_{n+1}) \geq 1$ for every $n \in \mathbb{N}$ and $x_n \xrightarrow{d} x^*$ as $n \to \infty$, then $\alpha(x_n, x^*) \geq 1$ for every $n \in \mathbb{N}$.*

Then there exists a fixed point of T in X.

To state the result for uniqueness of fixed point of (α,ψ)-Meir-Keeler-Khan mapping, some extra conditions [8] are added to Theorem 1.4 and Theorem 1.5; these conditions can be defined as follows:

(U1) $\alpha(x^*, x^{**}) \geq 1$ for any fixed points x^* and x^{**} of the mapping T.
(U2) There exists $z \in X$ such that $\alpha(x^*, z) \geq 1$ and $\alpha(x^{**}, z) \geq 1$ for any fixed points x^* and x^{**} of the mapping T.

Theorem 1.6 ([8]) *In the statement of Theorem* 1.4, *if the extra condition* (U1) *or* (U2) *is added to it, then the fixed point mentioned in the statement is unique*.

Theorem 1.7 ([8]) *In the statement of Theorem* 1.5, *if the extra condition* (U1) *or* (U2) *is added to it, then the fixed point mentioned in the statement is unique*.

Inspired and motivated by Redjel *et al.* [8], in Section 2, we introduce the new type of contractive multivalued mappings based on Meir-Keeler-Khan-type contractive condition. Via admissible mappings, we present the notion of *(α,ψ)-Meir-Keeler-Khan multivalued mapping*. We establish fixed point results for such mappings with continuity or α-continuity in the setting of complete metric spaces and α-complete metric spaces. In Section 3, some results of endpoints for (α,ψ)-Meir-Keeler-Khan multi-valued mapping are claimed. Our results extend and generalize the main results of Redjel *et al.* and Khan *et al.* in the literature [1, 2, 8]. To show the generality and effectiveness of main results, we provide some examples in the relevant sections of the article.

2 Main results

In 1969, the class of multivalued mappings on metric spaces is introduced by Nadler [9] as an extension of the class of Banach contraction mappings; henceforth, the investigations of fixed points of multivalued mappings have received much attention. We give some notation and recall some needed definitions. In the sequel, $\mathbb{N}$ denotes the set of all nonnegative integers, $\mathbb{R}^+$ denotes the set of all positive real numbers, $\mathcal{N}(X)$ denotes the family of nonempty subsets of X, $\mathcal{CL}(X)$ denotes the family of nonempty closed subsets of X, and $\mathcal{K}(X)$ denotes the family of nonempty compact subsets of X.

Let $T: X \to \mathcal{N}(X)$ be a multivalued mapping on a metric space (X,d). The point $x \in X$ is called a fixed point of T if $x \in Tx$. Set $\emptyset \neq A \subseteq X$ and $x \in X$. Defining the function

$$\operatorname{dist}: X \times \mathcal{N}(X) \to [0,\infty), \qquad \operatorname{dist}(x,A) = \inf\{d(x,y) : y \in A\}$$

for any $x \in X$, $A \in \mathcal{N}(X)$, $\mathrm{dist}(x, A)$ is called the distance from x to A. It is obvious that for fixed $A_0 \in \mathcal{K}(X)$, the function $\mathrm{dist}(x, A_0)$ is continuous at every $x \in X$.

For any $A, B \in \mathcal{CL}(X)$, the generalized Hausdorff distance H on the metric d is given by

$$H(A,B) = \begin{cases} \max\{\sup_{x\in A} \mathrm{dist}(x,B), \sup_{y\in B} \mathrm{dist}(y,A)\} & \text{if it exists,} \\ \infty & \text{otherwise.} \end{cases}$$

In 2013, following Samet's definition, Mohammadi *et al.* [10] extended the concept of an α-admissible single-valued mapping to the class of α-admissible multivalued mappings as follows.

Definition 2.1 ([10]) Let $\alpha : X \times X \to [0, \infty)$ and $T : X \to \mathcal{N}(X)$ be two mappings on a metric space (X, d). Then T is called an α-admissible mapping if for any $x \in X$ and $y \in Tx$ with $\alpha(x, y) \geq 1$, we have $\alpha(y, z) \geq 1$ for any $z \in Ty$.

Next, we introduce the class of (α, ψ)-Meir-Keeler-Khan multi-valued mappings. Some results on existence and uniqueness conditions for fixed points were established for such mappings via α-admissible meaning. Hereafter, all mappings $T : X \to \mathcal{K}(X)$ considered in the sequel of this paper satisfy

$$\forall x, y \in X, \quad x \neq y \quad \Rightarrow \quad \mathrm{dist}(x, Ty) + \mathrm{dist}(y, Tx) \neq 0. \tag{2.1}$$

Definition 2.2 Let $T : X \to \mathcal{K}(X)$ be a mapping on a metric space (X, d). Then T is called an *(α, ψ)-Meir-Keeler-Khan multivalued mapping* if there exist $\psi \in \Omega$ and $\alpha : X \times X \to [0, \infty)$ such that

$$H(Tx, Ty) \neq 0 \quad \Rightarrow \quad \alpha(x, y) H(Tx, Ty) \leq \psi\bigl(P(x, y)\bigr) \tag{2.2}$$

for any $x, y \in X$, where

$$P(x,y) = \frac{\mathrm{dist}(x, Tx)\,\mathrm{dist}(x, Ty) + \mathrm{dist}(y, Ty)\,\mathrm{dist}(y, Tx)}{\mathrm{dist}(x, Ty) + \mathrm{dist}(y, Tx)}.$$

First, we state an existence theorem for fixed points of (α, ψ)-Meir-Keeler-Khan multi-valued mappings.

Theorem 2.3 *Let $T : X \to \mathcal{K}(X)$ be an (α, ψ)-Meir-Keeler-Khan multivalued mapping on a metric space (X, d). Suppose that the following hypotheses hold*:

(i) *(X, d) is a complete metric space*;
(ii) *T is an α-admissible multi-valued mapping*;
(iii) *There exist x_0 and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$*;
(iv) *T is continuous.*

Then there exists a fixed point of T in X.

Proof We construct a sequence starting from x_0. If $x_0 \in Tx_0$, then x_0 is a fixed point. Suppose that $x_0 \notin Tx_0$. Because Tx_0 is a compact subset of X, then $d(x_0, Tx_0) > 0$. If $x_1 \in Tx_1$, then x_1 is a fixed point, and subsequently, this proof is complete. Assume that $x_1 \notin Tx_1$. Then it is clear that $\operatorname{dist}(x_1, Tx_1) > 0$ because Tx_1 is a compact subset of X. We have

$$\begin{aligned} H(Tx_0, Tx_1) &\le \alpha(x_0,x_1)H(Tx_0, Tx_1) \\ &\le \psi\left(\frac{\operatorname{dist}(x_0, Tx_0)\operatorname{dist}(x_0, Tx_1) + \operatorname{dist}(x_1, Tx_1)\operatorname{dist}(x_1, Tx_0)}{\operatorname{dist}(x_0, Tx_1) + \operatorname{dist}(x_1, Tx_0)}\right) \\ &= \psi\bigl(\operatorname{dist}(x_0, Tx_0)\bigr). \end{aligned} \tag{2.3}$$

Moreover, by the definition of the Hausdorff metric and the fact that $x_1 \in Tx_0$ we get

$$\operatorname{dist}(x_1, Tx_1) \le H(Tx_0, Tx_1) \le \psi\bigl(\operatorname{dist}(x_0, Tx_0)\bigr). \tag{2.4}$$

In addition, the compactness of Tx_1 implies that there exists $x_2 \in Tx_1$ such that

$$d(x_1,x_2) = \operatorname{dist}(x_1, Tx_1). \tag{2.5}$$

In combination with equations (2.4) and (2.5), we obtain that

$$d(x_1,x_2) \le \psi\bigl(\operatorname{dist}(x_0, Tx_0)\bigr). \tag{2.6}$$

We continue constructing the sequence similarly. If $x_2 \in Tx_2$, then this proof is done. Thus, we assume that $x_2 \notin Tx_2$. Because $\alpha(x_0,x_1) \ge 1$ and $x_1 \in Tx_0$, $x_2 \in Tx_1$, we have $\alpha(x_1,x_2) \ge 1$. Furthermore, using condition (2.2), we obtain that

$$\begin{aligned} H(Tx_1, Tx_2) &\le \alpha(x_1,x_2)H(Tx_1, Tx_2) \\ &\le \psi\left(\frac{\operatorname{dist}(x_1, Tx_1)\operatorname{dist}(x_1, Tx_2) + \operatorname{dist}(x_2, Tx_2)\operatorname{dist}(x_2, Tx_1)}{\operatorname{dist}(x_1, Tx_2) + \operatorname{dist}(x_2, Tx_1)}\right) \\ &= \psi\bigl(\operatorname{dist}(x_1, Tx_1)\bigr) \end{aligned} \tag{2.7}$$

and, subsequently,

$$\begin{aligned} \operatorname{dist}(x_2, Tx_2) &\le H(Tx_1, Tx_2) \\ &\le \psi\bigl(\operatorname{dist}(x_1, Tx_1)\bigr) \\ &= \psi\bigl(d(x_1,x_2)\bigr). \end{aligned} \tag{2.8}$$

Likewise, by the compactness of Tx_2, there exists $x_3 \in Tx_2$ such that

$$d(x_2,x_3) = \operatorname{dist}(x_2, Tx_2). \tag{2.9}$$

In combination with equations (2.6), (2.8), and (2.9), we obtain that

$$\begin{aligned} d(x_2,x_3) &\le \psi\bigl(\operatorname{dist}(x_1, Tx_1)\bigr) \\ &= \psi\bigl(d(x_1,x_2)\bigr) \\ &\le \psi^2\bigl(\operatorname{dist}(x_0, Tx_0)\bigr). \end{aligned} \tag{2.10}$$

By induction, we can obtain a sequence $\{x_n\}$ satisfying

$$x_{n+1} \in Tx_n, \qquad x_{n+1} \notin Tx_{n+1}, \qquad \alpha(x_n, x_{n+1}) \geq 1$$

and

$$d(x_n, x_{n+1}) \leq \psi^n\bigl(\operatorname{dist}(x_0, Tx_0)\bigr) \tag{2.11}$$

for all $n \in \mathbb{N}$.

Next task is to verify that $\{x_n\}$ is a Cauchy sequence. Regarding the properties of the function ψ, for any $\epsilon > 0$, there exists $n(\epsilon)$ such that

$$\sum_{k \geq n(\epsilon)}^{n-1} \psi^k\bigl(\operatorname{dist}(x_0, Tx_0)\bigr) < \epsilon. \tag{2.12}$$

Let $n > m > n(\epsilon)$. Applying the triangle inequality repeatedly, we get

$$\begin{aligned} d(x_m, x_n) &\leq \sum_{k=m}^{n-1} d(x_k, x_{k+1}) \\ &\leq \sum_{k=m}^{n-1} \psi^k\bigl(\operatorname{dist}(x_0, Tx_0)\bigr) \\ &\leq \sum_{k \geq n(\epsilon)}^{n-1} \psi^k\bigl(\operatorname{dist}(x_0, Tx_0)\bigr) \\ &< \epsilon, \end{aligned} \tag{2.13}$$

which means that $\{x_n\}$ is a Cauchy sequence in (X, d). By the completeness of (X, d) there exists $x^* \in X$ such that $x_k \xrightarrow{d} x^*$ as $k \to \infty$. Since T is continuous, we have that $Tx_k \xrightarrow{H} Tx^*$ as $k \to \infty$, and thus

$$\operatorname{dist}\bigl(x^*, Tx^*\bigr) = \lim_{k\to\infty} \operatorname{dist}\bigl(x_{k+1}, Tx^*\bigr) \leq \lim_{k\to\infty} H\bigl(Tx_k, Tx^*\bigr) = 0, \tag{2.14}$$

which shows that $x^* \in Tx^*$ because Tx^* is compact, and the proof is done. □

Remark 2.4 Observe that Theorem 1.4 (see also [8]) follows immediately from Theorem 2.3.

In 2014, Hussain *et al.* [11] introduced the concept of the α-completeness of metric spaces.

Definition 2.5 ([11]) Let $\alpha : X \times X \to [0, \infty)$ be a mapping on a metric space (X, d). The space (X, d) is said to be *α-complete* if each Cauchy sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ converges in X.

Recently, Kutbi and Sintunavarat [12] introduced the concept of an α-continuous multivalued mapping.

Definition 2.6 ([12]) Let $\alpha : X \times X \to [0,\infty)$ and $T : X \to \mathcal{C}L(X)$ be two given mappings on a metric space (X,d). The mapping T is called an *α-continuous multivalued mapping* if, for every sequence $\{x_n\}$ with $x_n \xrightarrow{d} x^*$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) \geq 1$ for each $n \in \mathbb{N}$, we have $Tx_n \xrightarrow{H} Tx^*$ as $n \to \infty$.

Remark 2.7 Notice that the concept of α-completeness of metric spaces is weaker than the concept of completeness and the concept of an α-continuous multivalued mapping is weaker than the concept of continuity in metric spaces.

Addressing to Remark 2.7, we provide two examples to illustrate it.

Example 2.8 Let $X = \mathbb{R}^2 \backslash \{0\}$, and let the metric $d : X^2 \to \mathbb{R}$ be defined by $d(x,y) = [(x_1 - x_2)^2 + (y_1 - y_2)^2]^{1/2}$ for any $x, y \in X$, where $x = (x_1, x_2)$, $y = (y_1, y_2)$. Let

$$Y = \{(x,y) \in \mathbb{R}^2 \backslash \{0\} : 1 \leq (x_1 - x_2)^2 + (y_1 - y_2)^2 \leq 4\}.$$

Define the mapping $\alpha : X \times X \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 2^{d(x,y)}, & x, y \in Y, \\ \frac{1}{3} & \text{otherwise.} \end{cases}$$

Note that (X,d) is just an α-complete metric space, not a complete metric space. Indeed, if $\{x_n\} \subset X$ is a Cauchy sequence with $\alpha(x_n, x_{n+1}) \geq 1$ for each $n \in \mathbb{N}$, then $x_n \in Y$ for each $n \in \mathbb{N}$. Since Y is a closed subset of X, it follows that (Y,d) is a complete metric space, and so there exists $x^* \in Y$ such that $x_n \xrightarrow{d} x^*$ as $n \to \infty$.

Example 2.9 Let $X = (\mathbb{R}^+ \cup \{0\}) \times (\mathbb{R}^+ \cup \{0\})$, and let the metric $d : X^2 \to \mathbb{R}$ be defined by $d(x,y) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$ for any $x, y \in X$, where $x = (x_1, x_2)$, $y = (y_1, y_2)$. Let $A = [0,1] \times [0,1] \subset X$. Define the mapping $\alpha : X \times X \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} \frac{1}{d(x,y)+1} + \frac{1}{2}, & x \in A, \\ \frac{1}{2} & \text{otherwise} \end{cases}$$

and define mapping $T : X \to \mathcal{C}L(X)$ by

$$Tx = \begin{cases} \{\lambda x\}, & x \in A, \\ \{x\}, & x \in X \backslash A, \end{cases}$$

where $\lambda \in [5,10]$. It is clear that T is not a continuous multivalued mapping from X into $\mathcal{C}L(X)$ on H, but we can verify that T is an α-continuous multivalued mapping from X into $\mathcal{C}L(X)$ on H. In fact, if $\{x_n\} \subset X$ is a sequence defined by $x_n = (1 + \frac{1}{n+1}, 1 + \frac{1}{n+1})$ for every $n \in \mathbb{N}$, then $x_n \in X \backslash A$ for every $n \in \mathbb{N}$. Note that $x_n \xrightarrow{d} (1,1)$ as $n \to \infty$; however, $Tx_n = (1 + \frac{1}{n+1}, 1 + \frac{1}{n+1}) \xrightarrow{H} \{(1,1)\} \neq \{(\lambda,\lambda)\} = T(1,1)$ as $n \to \infty$. If $\{x_n\}$ is a sequence with $\alpha(x_n, x_{n+1}) \geq 1$ for every $n \in \mathbb{N}$ and $x_n \xrightarrow{d} x$ as $n \to \infty$, then $x_n, x \in A$ for every $n \in \mathbb{N}$, and, subsequently, $Tx_n = \{\lambda x_n\} \xrightarrow{H} \{\lambda x\} = Tx$ as $n \to \infty$, that is, T is an α-continuous multivalued mapping from $\mathcal{C}L(X)$ into H.

Remark 2.10 It is easy to observe from the proof of Theorem 2.3 that if we weaken conditions (i) and (iv) of theorem to α-completeness and α-continuity, respectively, then the conclusion still holds.

We state the following theorem with the α-completeness assumption of a metric space (X,d) and the α-continuity assumption of the mapping instead of the completeness assumption and α-continuity assumption.

Theorem 2.11 *Let $T : X \to \mathcal{K}(X)$ be an (α,ψ)-Meir-Keeler-Khan multivalued mapping on a metric space (X,d). Suppose that the following hypotheses hold*:

(i) *(X,d) is an α-complete metric space*;
(ii) *T is an α-admissible multivalued mapping*;
(iii) *There exist x_0 and $x_1 \in Tx_0$ such that $\alpha(x_0,x_1) \geq 1$*;
(iv) *T is an α-continuous multivalued mapping.*

Then there exists a fixed point of T in X.

Proof See the proof of Theorem 2.3. □

Remark 2.12 As an application of Theorem 2.11 and Remark 2.7, we find Redjel's theorem (see Theorem 1.4) in [8].

Example 2.13 Let $X = [0,10)$ with the metric $d(x,y) = |x - y|$ for any $x,y \in X$. Define the mapping $\alpha : X^2 \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 1, & x,y \in [0,1], \\ 0 & \text{otherwise.} \end{cases}$$

Define the mapping $T : X \to \mathcal{CL}(X)$ by $Tx = [0,\frac{x}{3}]$ for $0 \leq x \leq 1$ and $Tx = [0,2x - \frac{5}{3}]$ for $1 < x < 10$. Then it is easy to check that T is α-admissible and X is not complete.

$$H(Tx,Ty) = \begin{cases} \frac{|x-y|}{3}, & x,y \in [0,1], \\ |y - \frac{x}{3}|, & x \in [0,1], y \in (1,10), \\ 2|x - y|, & x,y \in (1,10). \end{cases}$$

We can prove that T is not a continuous multivalued mapping on $(\mathcal{CL}(X),H)$, but T is a α-continuous multivalued mapping on $(\mathcal{CL}(X),H)$. Indeed, we have $H(Tx,T1) = \frac{|x-1|}{3} \to 0$ when $x \in [0,1]$ and $x \to 1$. $H(Tx,T1) = |x - \frac{1}{3}| \to \frac{2}{3}$ when $x \to 1$ and $x \in (1,10)$, that is, T is not continuous at $1 \in [0,10)$. If $\{x_n\} \subset X$ is a sequence with $\alpha(x_n,x_{n+1}) \geq 1$ for each $n \in \mathbb{N}$ and $x_n \xrightarrow{d} x$ as $n \to \infty$, then $x_n, x \in [0,1]$ for all $n \in \mathbb{N}$, and, subsequently, $Tx_n = [0,\frac{x_n}{3}] \xrightarrow{H} [0,\frac{x}{3}] = Tx$ as $n \to \infty$, that is, T is an α-continuous multivalued mapping on $(\mathcal{CL}(X),H)$.

Let $\psi(t) = \frac{3}{4}t$ for all $t \geq 0$. Because $\alpha(x,y) = 1$ whenever $x,y \in [0,1]$ and $\alpha(x,y) = 0$ whenever $x \notin [0,1]$ or $y \notin [0,1]$, it is clear that

$$\alpha(x,y)H(Tx,Ty) \leq \psi\left(\frac{\operatorname{dist}(x,Tx)\operatorname{dist}(x,Ty) + \operatorname{dist}(y,Ty)\operatorname{dist}(y,Tx)}{\operatorname{dist}(x,Ty) + \operatorname{dist}(y,Tx)}\right)$$

whenever $x \notin [0,1]$ or $y \notin [0,1]$.

In the sequel, we only consider $x, y \in [0,1]$; for $x, y \in [0,1]$, we calculate that

$$\alpha(x,y)H(Tx,Ty) = \frac{|x-y|}{3}$$

and

$$\begin{aligned}
&\psi\left(\frac{\operatorname{dist}(x,Tx)\operatorname{dist}(x,Ty)+\operatorname{dist}(y,Ty)\operatorname{dist}(y,Tx)}{\operatorname{dist}(x,Ty)+\operatorname{dist}(y,Tx)}\right)\\
&\quad=\frac{3}{4}\frac{\frac{2x}{3}\operatorname{dist}(x,Ty)+\frac{2y}{3}\operatorname{dist}(y,Tx)}{\operatorname{dist}(x,Ty)+\operatorname{dist}(y,Tx)}. \qquad (2.15)
\end{aligned}$$

Since $x \neq y$, the inequalities $x \leq \frac{y}{3}$ and $y \leq \frac{x}{3}$ cannot be simultaneously true; otherwise, $x = y = 0$ from $x \leq \frac{x}{9}$ or $y \leq \frac{y}{9}$.

If $x > \frac{y}{3}$ and $y \leq \frac{x}{3}$, then $\operatorname{dist}(y,Tx) = 0$, and hence

$$\begin{aligned}
&\psi\left(\frac{\operatorname{dist}(x,Tx)\operatorname{dist}(x,Ty)+\operatorname{dist}(y,Ty)\operatorname{dist}(y,Tx)}{\operatorname{dist}(x,Ty)+\operatorname{dist}(y,Tx)}\right)\\
&\quad=\frac{3}{4}\frac{\frac{2x}{3}\operatorname{dist}(x,Ty)+\frac{2y}{3}\operatorname{dist}(y,Tx)}{\operatorname{dist}(x,Ty)+\operatorname{dist}(y,Tx)}\\
&\quad=\frac{x}{2}. \qquad (2.16)
\end{aligned}$$

Note that if $x > \frac{y}{3}$, $y \leq \frac{x}{3}$, then $\frac{x-y}{3} \leq \frac{x}{2}$. Hence,

$$\alpha(x,y)H(Tx,Ty) \leq \psi\left(\frac{\operatorname{dist}(x,Tx)\operatorname{dist}(x,Ty)+\operatorname{dist}(y,Ty)\operatorname{dist}(y,Tx)}{\operatorname{dist}(x,Ty)+\operatorname{dist}(y,Tx)}\right). \qquad (2.17)$$

Similarly, we can obtain that when $y > \frac{x}{3}$, $x \leq \frac{y}{3}$, $x, y \in [0,1]$, the following inequality holds:

$$\alpha(x,y)H(Tx,Ty) \leq \psi\left(\frac{\operatorname{dist}(x,Tx)\operatorname{dist}(x,Ty)+\operatorname{dist}(y,Ty)\operatorname{dist}(y,Tx)}{\operatorname{dist}(x,Ty)+\operatorname{dist}(y,Tx)}\right). \qquad (2.18)$$

If $x > \frac{y}{3}$, $y > \frac{x}{3}$, $x, y \in [0,1]$, then

$$\begin{aligned}
&\psi\left(\frac{\operatorname{dist}(x,Tx)\operatorname{dist}(x,Ty)+\operatorname{dist}(y,Ty)\operatorname{dist}(y,Tx)}{\operatorname{dist}(x,Ty)+\operatorname{dist}(y,Tx)}\right)\\
&\quad=\frac{3}{4}\frac{\frac{2x}{3}(x-\frac{y}{3})+\frac{2y}{3}(y-\frac{x}{3})}{(x-\frac{y}{3})+(y-\frac{x}{3})}\\
&\quad=\frac{1}{2}\frac{x(x-\frac{y}{3})+y(y-\frac{x}{3})}{(x-\frac{y}{3})+(y-\frac{x}{3})}\\
&\quad=\frac{3(x^2+y^2)-2xy}{4(x+y)}. \qquad (2.19)
\end{aligned}$$

Notice that

$$\frac{|x-y|}{3} < \frac{3(x^2+y^2)-2xy}{4(x+y)}$$

whenever $x > y$ or $y > x$.

Moreover, there exist $x_0 = \frac{1}{5} \in [0,1]$ and $x_1 = \frac{1}{16} \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$. Thus, condition (iii) in Theorem 2.11 holds. Therefore, by Theorem 2.11 it follows that there exists a fixed point of T in X. In this case, T has infinitely many fixed points such as 0 and 2.

Theorem 2.14 *Let $T : X \to \mathcal{K}(X)$ be an (α, ψ)-Meir-Keeler-Khan multivalued mapping on a metric space (X, d). Suppose that the following hypotheses hold*:

(i) *(X, d) is an α-complete metric space*;

(ii) *T is an α-admissible multivalued mapping*;

(iii) *There exist x_0 and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$*;

(iv) *If $\{x_n\}$ is a sequence with $x_n \xrightarrow{d} x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) \geq 1$, then $\alpha(x_n, x) \geq 1$ for each $n \in \mathbb{N}$.*

Then there exists a fixed point of T in X.

Proof Following the proof of Theorem 2.3, we obtain a Cauchy sequence $\{x_n\}$ with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ that converges to some $x^* \in X$. Applying condition (iv), we have $\alpha(x_n, x^*) \geq 1$ for all $n \in \mathbb{N}$. Next, assume that $\operatorname{dist}(x^*, Tx^*) \neq 0$. Then, for each $n \in \mathbb{N}$, we can derive

$$\begin{aligned}
&\operatorname{dist}\bigl(x_{n+1}, Tx^*\bigr) \\
&\quad \leq H\bigl(Tx_n, Tx^*\bigr) \\
&\quad \leq \alpha\bigl(x_n, x^*\bigr) H\bigl(Tx_n, Tx^*\bigr) \\
&\quad \leq \psi\left(\frac{\operatorname{dist}(x_n, Tx_n)\operatorname{dist}(x_n, Tx^*) + d(x^*, Tx^*)d(x^*, Tx_n)}{\operatorname{dist}(x_n, Tx^*) + \operatorname{dist}(x^*, Tx_n)}\right) \\
&\quad \leq \psi\left(\frac{\operatorname{dist}(x_n, Tx_n)\operatorname{dist}(x_n, Tx^*) + \operatorname{dist}(x^*, Tx^*)d(x^*, x_{n+1})}{\operatorname{dist}(x_n, Tx^*) + \operatorname{dist}(x^*, Tx_n)}\right).
\end{aligned} \tag{2.20}$$

Since $\psi(t) \leq t$, $t \in [0, \infty)$, and $\psi(t) = t$ if and only if $t = 0$, we thus have

$$\begin{aligned}
\operatorname{dist}\bigl(x_{n+1}, Tx^*\bigr) &\leq \psi\left(\frac{\operatorname{dist}(x_n, Tx_n)\operatorname{dist}(x_n, Tx^*) + \operatorname{dist}(x^*, Tx^*)d(x^*, x_{n+1})}{\operatorname{dist}(x_n, Tx^*) + \operatorname{dist}(x^*, Tx_n)}\right) \\
&\leq \frac{\operatorname{dist}(x_n, Tx_n)\operatorname{dist}(x_n, Tx^*) + \operatorname{dist}(x^*, Tx^*)d(x^*, x_{n+1})}{\operatorname{dist}(x_n, Tx^*) + \operatorname{dist}(x^*, Tx_n)}.
\end{aligned} \tag{2.21}$$

Letting $n \to \infty$ on the two sides of this inequality, we get

$$\begin{aligned}
\operatorname{dist}\bigl(x^*, Tx^*\bigr) &\leq \frac{\lim_{n\to\infty} \operatorname{dist}(x_n, Tx_n)\operatorname{dist}(x_n, Tx^*)}{\lim_{n\to\infty}[\operatorname{dist}(x_n, Tx^*) + \operatorname{dist}(x^*, Tx_n)]} \\
&\leq \frac{\lim_{n\to\infty} \operatorname{dist}(x_n, Tx_n)\lim_{n\to\infty}\operatorname{dist}(x_n, Tx^*)}{\lim_{n\to\infty}\operatorname{dist}(x_n, Tx^*)} \\
&= \frac{\lim_{n\to\infty} \operatorname{dist}(x_n, Tx_n)\operatorname{dist}(x^*, Tx^*)}{\operatorname{dist}(x^*, Tx^*)} \\
&= \lim_{n\to\infty} \operatorname{dist}(x_n, Tx_n) \\
&\leq \lim_{n\to\infty} d(x_n, x_{n+1}) \\
&= 0,
\end{aligned} \tag{2.22}$$

which contradicts to $\operatorname{dist}(x^*, Tx^*) \neq 0$. As a consequence, $\operatorname{dist}(x^*, Tx^*) = 0$ implies that $x^* \in Tx^*$ because Tx^* is a compact subset of X, and we complete this proof. □

Example 2.15 Reconsidering Example 2.13, we can find a sequence $\{x_n = \frac{1}{n+1}\}_{n\in\mathbb{N}}$ with $x_n \xrightarrow{d} 0$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) \geq 1$, $\alpha(x_n, 0) = 1 \geq 1$ for all $n \in \mathbb{N}$. Thus, conditions (i)-(iv) in Theorem 2.14 hold. Therefore, by Theorem 2.14, 0 is a fixed point of T .

Remark 2.16 As an application of Theorem 2.14 and Remark 2.7, Redjel's conclusion (see Theorem 1.5) in [8] is a direct result of Theorem 2.14 and Remark 2.7.

Uniqueness of α-admissible mappings usually requires some extra conditions on the mapping itself or on the space on which the mapping is defined. These conditions can be defined as follows:

(H1) $\alpha(x^*, x^{**}) \geq 1$ for any fixed points x^* and x^{**} of T.

(H2) There exists $z \in X$ with $\alpha(x^*, z) \geq 1$, $\alpha(x^{**}, z) \geq 1$ and $\alpha(x^*, y) \geq 1$, $\alpha(x^{**}, y) \geq 1$ for any $y \in Tz$ and any fixed points x^*, x^{**} of T.

(H3) There exists $z \in Tx^* \cap Tx^{**}$ such that $\alpha(x^*, z) \geq 1$ and $\alpha(x^{**}, z) \geq 1$ for any fixed points x^*, x^{**} of T.

Theorem 2.17 *In the statements of Theorem* 2.11 *and Theorem* 2.14, *if the extra condition* (H1) *is added to them, then the fixed point mentioned in these two statements is unique.*

Proof Following the proof of Theorem 2.3 (resp. Theorem 2.14), there exists a fixed point x^* under the conditions of Theorem 2.11 (resp. Theorem 2.14). Assume that the mapping T has another fixed point x^{**} and $x^* \neq x^{**}$. Using condition (H1), we get $\alpha(x^*, x^{**}) \geq 1$, and hence

$$\begin{aligned} &H(Tx^*, Tx^{**}) \\ &\quad \leq \alpha(x^*, x^{**}) H(Tx^*, Tx^{**}) \\ &\quad \leq \psi\left(\frac{\operatorname{dist}(x^*, Tx^*)\operatorname{dist}(x^*, Tx^{**}) + \operatorname{dist}(x^{**}, Tx^{**})\operatorname{dist}(x^{**}, Tx^*)}{\operatorname{dist}(x^*, Tx^{**}) + \operatorname{dist}(x^{**}, Tx^*)}\right). \end{aligned} \tag{2.23}$$

Notice that $x^* \in Tx^*$, $x^{**} \in Tx^{**}$, and $\psi(0) = 0$; therefore,

$$\begin{aligned} &H(Tx^*, Tx^{**}) \\ &\quad \leq \psi\left(\frac{\operatorname{dist}(x^*, Tx^*)\operatorname{dist}(x^*, Tx^{**}) + \operatorname{dist}(x^{**}, Tx^{**})d(x^{**}, Tx^*)}{\operatorname{dist}(x^*, Tx^{**}) + d(x^{**}, Tx^*)}\right) \\ &\quad = \psi(0) \\ &\quad = 0, \end{aligned} \tag{2.24}$$

and hence $H(Tx^*, Tx^{**}) = 0$, which implies that $Tx^* = Tx^{**}$. In addition,

$$\begin{aligned} d(x^*, x^{**}) &= H(\{x^*\}, \{x^{**}\}) \\ &\leq H(Tx^*, Tx^{**}) \end{aligned}$$

$$= H(Tx^*, Tx^*)$$
$$= 0, \tag{2.25}$$

and thus $d(x^*, x^{**}) = 0$, which implies $x^* = x^{**}$. □

Theorem 2.18 *In the statement of Theorem* 2.11, *if the extra condition* (H2) *is added to it, then the fixed point mentioned in the statement is unique.*

Proof As shown in the proof of Theorem 2.3, there exists a fixed point x^* under the hypotheses of Theorem 2.11. Assume that T has another fixed point x^{**} and $x^* \neq x^{**}$. By condition (H2), an element $z \in X$ satisfying

$$\alpha(x^*, z) \geq 1, \qquad \alpha(x^{**}, z) \geq 1, \qquad \alpha(x^*, y) \geq 1, \qquad \alpha(x^{**}, y) \geq 1$$

can be found in X for any $y \in Tz$.

Defining the sequence $\{z_n\}$ by $z_0 = z$, $z_{n+1} \in Tz_n$ for all $n \in \mathbb{N}$, we get

$$\alpha(x^*, z_n) \geq 1 \quad \text{and} \quad \alpha(x^{**}, z_n) \geq 1$$

for each $n \in \mathbb{N}$. Hence,

$$\begin{aligned} d(x^*, z_{n+2}) &= H(\{x^*\}, \{z_{n+2}\}) \\ &\leq H(Tx^*, Tz_{n+1}) \\ &\leq \alpha(x^*, z_{n+1}) H(Tx^*, Tz_{n+1}) \\ &\leq \psi\left(\frac{\operatorname{dist}(x^*, Tx^*)\operatorname{dist}(x^*, Tz_{n+1}) + d(z_{n+1}, Tz_{n+1})\operatorname{dist}(z_{n+1}, Tx^*)}{\operatorname{dist}(x^*, Tz_{n+1}) + \operatorname{dist}(z_{n+1}, Tx^*)}\right) \\ &= \psi\left(\frac{\operatorname{dist}(z_{n+1}, Tz_{n+1})\operatorname{dist}(z_{n+1}, Tx^*)}{\operatorname{dist}(x^*, Tz_{n+1}) + \operatorname{dist}(z_{n+1}, Tx^*)}\right). \end{aligned}$$

Note that $\psi(t) \leq t$ and

$$\begin{aligned} &\operatorname{dist}(z_{n+1}, Tx^*) \leq \operatorname{dist}(x^*, Tz_{n+1}) + \operatorname{dist}(z_{n+1}, Tx^*), \\ &d(x^*, z_{n+2}) = \psi\left(\frac{\operatorname{dist}(z_{n+1}, Tz_{n+1})\operatorname{dist}(z_{n+1}, Tx^*)}{\operatorname{dist}(x^*, Tz_{n+1}) + \operatorname{dist}(z_{n+1}, Tx^*)}\right) \\ &\qquad \leq \psi(\operatorname{dist}(z_{n+1}, Tz_{n+1})). \end{aligned} \tag{2.26}$$

On the other hand,

$$\begin{aligned} \operatorname{dist}(z_{n+1}, Tz_{n+1}) &= H(Tz_n, Tz_{n+1}) \\ &\leq H(Tz_n, Tz_{n+1}) \\ &\leq \alpha(z_n, z_{n+1}) H(Tz_n, Tz_{n+1}) \\ &\leq \psi\left(\frac{\operatorname{dist}(z_n, Tz_n)\operatorname{dist}(z_n, Tz_{n+1}) + d(z_{n+1}, Tz_{n+1})\operatorname{dist}(z_{n+1}, Tz_n)}{\operatorname{dist}(z_n, Tz_{n+1}) + \operatorname{dist}(z_{n+1}, Tz_n)}\right) \end{aligned}$$

$$= \psi \left(\frac{\operatorname{dist}(z_n, Tz_n) \operatorname{dist}(z_n, Tz_{n+1})}{\operatorname{dist}(z_n, Tz_{n+1}) + \operatorname{dist}(z_{n+1}, Tz_n)} \right)$$
$$\le \psi \big(\operatorname{dist}(z_n, Tz_n)\big). \tag{2.27}$$

Iteratively, this inequality implies

$$\operatorname{dist}(z_{n+1}, Tz_{n+1}) \le \psi^{n+1}\big(d(z_0, Tz_0)\big)$$

for each $n \in \mathbb{N}$. In combination with (2.26) and (2.27), we have

$$d\big(x^*, z_{n+2}\big) \le \psi^{n+2}\big(d(z_0, Tz_0)\big).$$

Letting $n \to \infty$, we obtain

$$\lim_{n\to\infty} d\big(x^*, z_n\big) = 0.$$

Similarly, we get

$$\lim_{n\to\infty} d\big(x^{**}, z_n\big) = 0.$$

Immediately, $x^* = x^{**}$ due to the uniqueness of the limit, and this completes the proof. □

Theorem 2.19 *In the statement of Theorem* 2.14, *if the extra condition* (H3) *is added to it, then the fixed point mentioned in the statement is unique.*

Proof The existence of a fixed point is proved in Theorem 2.14. To prove the uniqueness, let x^* and x^{**} be any two fixed points of T with $x^* \neq x^{**}$ under the conditions of Theorem 2.14. By condition (H3), there exists $z \in Tx^* \cap Tx^{**}$ such that

$$\alpha\big(x^*, z\big) \ge 1 \quad \text{and} \quad \alpha\big(x^{**}, z\big) \ge 1.$$

From the proof of Theorem 2.3 and Remark 2.10, we can construct a sequence $\{z_n\}$ by $z_0 = x^*$, $z_1 = z$, $z_{n+1} \in Tz_n$ for any $n \in \mathbb{N}$ such that z_n converges to a fixed point ζ of T as $n \to \infty$. Define the sequence $\{w_n\}$ by $w_0 = x^{**}$, $w_i = z_i$ for $i \in \mathbb{N} \setminus \{0\}$. Because T is α-admissible, we have

$$\alpha(z_n, z_{n+1}) \ge 1 \quad \text{and} \quad \alpha(w_n, w_{n+1}) \ge 1$$

for any $n \in \mathbb{N}$. Note that the difference between the sequences $\{z_n\}$ and $\{w_n\}$ only lies in the first terms, so that $w_n \xrightarrow{d} \zeta$ as $n \to \infty$. Applying condition (iv) of Theorem 2.14, we get

$$\alpha(z_n, \zeta) \ge 1 \quad \text{and} \quad \alpha(w_n, \zeta) \ge 1$$

for all $n \in \mathbb{N}$. In particular, $\alpha(z_0, \zeta) = \alpha(x^*, \zeta) \ge 1$ and $\alpha(w_0, \zeta) = \alpha(x^{**}, \zeta) \ge 1$.

If $x^* \neq \zeta$, then

$$\begin{aligned} d(x^*, \zeta) &= H(\{x^*\}, \{\zeta\}) \\ &\leq H(Tx^*, T\zeta) \\ &\leq \alpha(x^*, \zeta) H(Tx^*, T\zeta) \\ &\leq \psi\left(\frac{\operatorname{dist}(x^*, Tx^*)\operatorname{dist}(x^*, T\zeta) + d(\zeta, T\zeta)\operatorname{dist}(\zeta, Tx^*)}{\operatorname{dist}(x^*, T\zeta) + \operatorname{dist}(\zeta, Tx^*)}\right) \\ &= \psi(0) \\ &= 0. \end{aligned} \tag{2.28}$$

Thus, $x^* = \zeta$, which contradicts $x^* \neq \zeta$. So $x^* = \zeta$. Similarly, we get $x^{**} = \zeta$, and therefore $x^* = x^{**}$. This completes the proof. □

3 Endpoint property

Let $T : X \to \mathcal{N}(X)$ be a multivalued mapping on a metric space (X, d). An element $x \in X$ is called an endpoint of T if $Tx = \{x\}$. It is obvious that an endpoint of T is a fixed point of T. In recent years, some problems on the existence and uniqueness of endpoints have been studied extensively (see, *e.g.*, [13–16]).

Let $T : X \to \mathcal{N}(X)$ be a multivalued mapping on a metric space (X, d). We say that T has the approximate endpoint property if

$$\inf_{x \in X} \sup_{y \in Tx} d(x, y) = 0$$

or, equivalently, if there exists a sequence $\{x_n\}$ such that $\lim_{n\to\infty} H(\{x_n\}, Tx_n) = 0$. Concerning a single-valued map $f : X \to X$, f has the approximate endpoint property if and only if f has the approximate fixed point property, that is,

$$\inf_{x \in X} d(x, f(x)) = 0,$$

or, equivalently, if there exists a sequence $\{x_n\}$ such that $\lim_{n\to\infty} d(x_n, f(x_n)) = 0$.

Theorem 3.1 *Let $T : X \to \mathcal{K}(X)$ be an (α, ψ)-Meir-Keeler-Khan multivalued mapping on a metric space (X, d). Suppose that the following conditions hold*:

(i) *(X, d) is an α-complete metric space*;

(ii) *T is an α-admissible multi-valued mapping*;

(iii) *There exist u and $v \in Tu$ such that $\alpha(u, v) \geq 1$*;

(iv) *If $\{x_n\}$ is a sequence of X with $\lim_{n\to\infty} H(\{x_n\}, Tx_n) = 0$, then $u, v \in \{x_n\}$, $\alpha(x_n, x_{n+1}) \geq 1$, and $x_{n+1} \in Tx_n$ for all $n \in \mathbb{N}$; moreover, if $x_n \xrightarrow{d} y^*$ as $n \to \infty$, then $\alpha(x_n, y^*) \geq 1$ for every $n \in \mathbb{N}$*;

(v) *For all endpoints y^* and y^{**} of the mapping T, we have $\alpha(y^*, y^{**}) \geq 1$.*

Then T has a unique endpoint if and only if T has the approximate endpoint property.

Proof It is obvious that if T has an endpoint, then T has the approximate endpoint property. Conversely, suppose that T has the approximate endpoint property, that is, we can find a sequence $\{x_n\} \subset X$ with $\lim_{n\to\infty} H(\{x_n\}, Tx_n) = 0$. Hence, there exist $m_0, n_0 \in \mathbb{N}$, $m_0 > n_0$, such that $u = x_{n_0}$, $v = x_{m_0}$. Construct a subsequence $\{y_n\}$ of $\{x_n\}$, letting

$$y_0 = u, \qquad y_1 = v, \qquad y_2 = x_{m_0+1}, \qquad \ldots, \qquad y_n = x_{m_0+n-1}, \qquad \ldots.$$

It is obvious that $\lim_{n\to\infty} H(\{y_n\}, Ty_n) = 0$ and $\alpha(y_n, y_{n+1}) \geq 1$ for all $n \in \mathbb{N}$.

As proved in Theorem 2.3, we can deduce that the sequence $\{y_n\} \subset X$ is a Cauchy sequence. Note that T is α-complete; thus, there exists $y^* \in X$ such that $y_n \xrightarrow{d} y^*$, and it follows from (iv) that $\alpha(y_n, y^*) \geq 1$ for all $n \in \mathbb{N}$. Additionally,

$$\begin{aligned} H(y^*, Ty^*) - H(\{y_n\}, Ty_n) &\leq H(Ty_n, Ty^*) \\ &\leq \alpha(y_n, y^*) H(Ty_n, Ty^*) \\ &\leq \psi\bigl(M(y_n, y^*)\bigr) \\ &\leq M(y_n, y^*) \\ &\leq H(\{y_n\}, Ty_n) + H(\{y^*\}, Ty^*), \end{aligned} \tag{3.1}$$

where

$$M(y_n, y^*) = \frac{H(\{y_n\}, Ty_n)\operatorname{dist}(y_n, Ty^*) + H(\{y^*\}, Ty^*)\operatorname{dist}(y^*, Ty_n)}{\operatorname{dist}(y_n, Ty^*) + \operatorname{dist}(y^*, Ty_n)}.$$

This shows that $\lim_{n\to\infty} M(y_n, y^*) = H(\{y^*\}, Ty^*)$, and we thus have

$$\limsup_{n\to\infty} \psi\bigl(M(y_n, y^*)\bigr) = \psi\bigl(H(\{y^*\}, Ty^*)\bigr). \tag{3.2}$$

By (3.1) and (3.2) we conclude that

$$H(\{y^*\}, Ty^*) \leq \psi\bigl(H(\{y^*\}, Ty^*)\bigr).$$

Therefore,

$$H(\{y^*\}, Ty^*) = 0,$$

which means that $\{y^*\} = Ty^*$, that is, y^* is an endpoint of X.

Assume that there exists another endpoint $y^{**} \in X$. Then

$$\begin{aligned} H(\{y^*\}, \{y^{**}\}) &= H(Ty^*, Ty^{**}) \\ &\leq \alpha(y^*, y^{**}) H(Ty^*, Ty^{**}) \\ &\leq \psi\left(\frac{\operatorname{dist}(y^*, Ty^*)\operatorname{dist}(y^*, Ty^{**}) + \operatorname{dist}(y^{**}, Ty^{**})\operatorname{dist}(y^{**}, Ty^*)}{\operatorname{dist}(y^*, Ty^{**}) + \operatorname{dist}(y^{**}, Ty^*)}\right). \end{aligned} \tag{3.3}$$

Notice that $y^* \in Ty^*$, $y^{**} \in Ty^{**}$, and $\psi(0) = 0$; therefore,

$$\begin{aligned} &H(\{y^*\}, \{y^{**}\}) \\ &\quad\leq \psi\left(\frac{\operatorname{dist}(y^*, Ty^*)\operatorname{dist}(y^*, Ty^{**}) + \operatorname{dist}(y^{**}, Ty^{**})\operatorname{dist}(y^{**}, Ty^*)}{\operatorname{dist}(y^*, Ty^{**}) + \operatorname{dist}(y^{**}, Ty^*)}\right) \end{aligned}$$

$$
\begin{aligned}
&= \psi(0) \\
&= 0, \qquad (3.4)
\end{aligned}
$$

and thus $\{y^*\} = \{y^{**}\}$, which implies $y^* = y^{**}$. □

The following corollary on a single-valued mapping f is a direct consequence of Theorem 3.1.

Corollary 3.2 *Let $f : X \to X$ be an (α, ψ)-Meir-Keeler-Khan mapping on a metric space (X, d). Suppose that the following conditions hold*:

(i) *(X, d) is an α-complete metric space*;
(ii) *f is an α-admissible mapping*;
(iii) *There exist u and $v \in f(u)$ such that $\alpha(u, v) \geq 1$*;
(iv) *If there exists a sequence $\{x_n\}$ of X with $d(x_n, f(x_n)) = 0$, then $u, v \in \{x_n\}$ and $\alpha(x_n, x_{n+1}) \geq 1$; moreover, if $x_n \xrightarrow{d} x^*$ as $n \to \infty$, then $\alpha(x_n, x^*) \geq 1$ for all $n \in \mathbb{N}$*;
(v) *$\alpha(x^*, x^{**}) \geq 1$ for all the fixed points x^*, x^{**} of the mapping f.*

Then f has a unique fixed point if and only if f has the approximate fixed point property.

Proof Let $fx = \{f(x)\}$ and apply Theorem 3.1. □

Theorem 3.3 *Let $f : X \to X$ be an (α, ψ)-Meir-Keeler-Khan mapping on a metric space (X, d). Suppose that the following conditions hold*:

(i′) *(X, d) is an α-complete metric space*;
(ii′) *f is an α-admissible mapping*;
(iii′) *There exists $x_0 \in X$ such that $\alpha(x_0, f(x_0)) \geq 1$*;
(iv′) *If $\{x_n\}$ is a sequence with $x_n \xrightarrow{d} x^*$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) \geq 1$, then $\alpha(x_n, x^*) \geq 1$ for every $n \in \mathbb{N}$.*

Then condition (iii) *in Corollary* 3.2 *holds, and f has the approximate endpoint property.*

Proof It is clear that conditions (iii′) and (iv′) imply condition (iv) of Corollary 3.2. In addition, following (iii′), there exists $x_0 \in X$ such that $\alpha(x_0, f(x_0)) \geq 1$. We define the sequence $\{x_n\}$ in X by $x_{n+1} = f(x_n)$ for all $n \in \mathbb{N}$. If $x_{n_0+1} = f(x_{n_0})$ for some n_0, then $x_n = x_{n_0}$ for all $n \geq n_0$. Hence, $d(x_n, f(x_n)) = d(x_{n_0}, x_{n_0}) = 0$ for $n \geq n_0$, that is, $d(x_n, f(x_n)) \to 0$ as $n \to \infty$. This means that f has the approximate endpoint property. We assume that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$. The fact that f is α-admissible implies that

$$
\alpha\big(x_0, f(x_0)\big) = \alpha(x_0, x_1) \geq 1 \quad \Rightarrow \quad \alpha\big(x_1, f(x_1)\big) = \alpha(x_1, x_2) \geq 1.
$$

By induction we deduce that

$$
\alpha(x_n, x_{n+1}) \geq 1
$$

for all $n \in \mathbb{N}$. Moreover, for $n \geq 1$, $n \in \mathbb{N}$, we deduce that

$$
\begin{aligned}
d(x_n, x_{n+1}) &= d\big(f(x_{n-1}), f(x_n)\big) \\
&\leq \alpha(x_{n-1}, x_n) d\big(f(x_{n-1}), f(x_n)\big)
\end{aligned}
$$

$$\begin{aligned}
&\leq \psi\left(\frac{d(x_{n-1},f(x_{n-1}))d(x_{n-1},f(x_n)) + d(x_n,f(x_n))d(x_n,f(x_{n-1}))}{d(x_{n-1},f(x_n)) + d(x_n,f(x_{n-1}))}\right)\\
&= \psi\left(\frac{d(x_{n-1},x_n)d(x_{n-1},f(x_n)) + d(x_n,f(x_n))d(x_n,x_n)}{d(x_{n-1},f(x_n)) + d(x_n,x_n)}\right)\\
&= \psi\left(\frac{d(x_{n-1},x_n)d(x_{n-1},f(x_n))}{d(x_{n-1},f(x_n))}\right)\\
&= \psi\bigl(d(x_{n-1},x_n)\bigr). \qquad (3.5)
\end{aligned}$$

Iteratively, this inequality implies

$$d(x_n,x_{n+1}) \leq \psi^n\bigl(d(x_0,x_1)\bigr)$$

for all $n \in \mathbb{N}$.

Letting $n \to \infty$, we obtain

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = d\bigl(x_n,f(x_n)\bigr) = 0,$$

which implies that f has the approximate endpoint property. □

Remark 3.4 As an application of Corollary 3.2 and Theorem 3.3, the result of Redjel's theorem (adding condition (U1) to the statement of Theorem 1.5, the uniqueness of the fixed point can be obtained. Ref. Theorem 1.7) is a direct result of Corollary 3.2 and Theorem 3.3.

In Theorem 3.1, let (X,d) be complete, and let $\alpha(x,y) = 1$ for any $x,y \in X$. Then conditions (i)-(v) hold clearly, and thus as an application of Theorem 3.1, we have the following corollary.

Corollary 3.5 *Let $T : X \to \mathcal{K}(X)$ be a mapping on a complete metric space (X,d). Suppose that, for any $x,y \in X$,*

$$H(Tx,Ty) \leq \mu P(x,y)$$

for some $0 \leq \mu < 1$, where

$$P(x,y) = \frac{\operatorname{dist}(x,Tx)\operatorname{dist}(x,Ty) + \operatorname{dist}(y,Ty)\operatorname{dist}(y,Tx)}{\operatorname{dist}(x,Ty) + \operatorname{dist}(y,Tx)}.$$

Then T has a unique endpoint if and only if T has the approximate endpoint property.

Proof Apply Theorem 3.1 for $\alpha(x,y) = 1$ and $\psi(t) = \mu t$. □

Especially, we will mention the case of $\mu = 1$ in Corollary 3.5. When X is a reflexive Banach space, using the notion of normal structure set, the existence and uniqueness of fixed points is established by Redjel and Dehici [17].

Remark 3.6 It is easy to observe that the Khan theorem (see Theorem 1.1) by Fisher is a direct result of Theorem 3.1, Corollary 3.2, Theorem 3.3, and Corollary 3.5.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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