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Some generalizations of Darbo's theorem and applications to fractional integral equations

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Abstract

In this paper, some generalizations of Darbo's fixed point theorem are presented. An existence result for a class of fractional integral equations is given as an application of the obtained results.

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1 Introduction and preliminaries

Let E be a Banach space over $\mathbb{R}$ (or $\mathbb{C}$) with respect to a certain norm $\|\cdot\|$. For any subsets X and Y of E, we have the following notations:

$\overline{X}$ denotes the closure of X;

$\mathrm{conv}(X)$ denotes the convex hull of X;

$P(X)$ denotes the set of nonempty subsets of X;

$X + Y$ and λX ($\lambda \in \mathbb{R}$) stand for algebraic operations on sets X and Y.

We denote by $\mathcal{B}_E$ the family of all nonempty bounded subsets of E. Finally, if X is a nonempty subset of E and $T : X \to X$ is a given operator, we denote by $\mathrm{Fix}(T)$ the set of fixed points of T, that is,

$$\mathrm{Fix}(T) = \{x \in X : Tx = x\}.$$

Banaś and Goebel [1] introduced the following axiomatic definition of the concept of a measure of noncompactness.

Definition 1.1 Let $\sigma : \mathcal{B}_E \to [0, \infty)$ be a given mapping. We say that σ is a BG-measure of noncompactness (in the sense of Banaś and Gobel) on E if the following conditions are satisfied:

(i) For every $X \in \mathcal{B}_E$, $\sigma(X) = 0$ iff X is precompact.

(ii) For every pair $(X, Y) \in \mathcal{B}_E \times \mathcal{B}_E$, we have

$$X \subseteq Y \implies \sigma(X) \leq \sigma(Y).$$

(iii) For every $X \in \mathcal{B}_E$, we have

$$\sigma(\overline{X}) = \sigma(X) = \sigma\big(\operatorname{conv}(X)\big).$$

(iv) For every pair $(X, Y) \in \mathcal{B}_E \times \mathcal{B}_R$ and $\lambda \in (0, 1)$, we have

$$\sigma\big(\lambda X + (1 - \lambda)Y\big) \le \lambda \sigma(X) + (1 - \lambda)\sigma(Y).$$

(v) If $\{X_n\} \subseteq \mathcal{B}_E$ is a decreasing sequence (w.r.t. $\subseteq$) of closed sets such that $\sigma(X_n) \to 0$ as $n \to \infty$, then $X_\infty := \bigcap_{n=1}^{\infty} X_n$ is nonempty.

Let $\mathcal{X}$ be a nonempty, bounded, closed, and convex subset of the Banach space E. We denote by $\mathcal{D}_{\mathcal{X}}$ the set of self-mappings $D : \mathcal{X} \to \mathcal{X}$ satisfying the following conditions:

(i) D is a continuous mapping.
(ii) There exist $\sigma : \mathcal{B}_E \to [0, \infty)$, a BG-measure of noncompactness on E, and a constant $k \in (0, 1)$ such that

$$\sigma(DW) \le k\sigma(W), \quad W \in P(\mathcal{X}).$$

The following result is known as Darbo's fixed point theorem (see [1, 2]).

Theorem 1.2 *Let $D : \mathcal{X} \to \mathcal{X}$ be a mapping that belongs to $\mathcal{D}_{\mathcal{X}}$. Then D has at least one fixed point. Moreover, the set $\operatorname{Fix}(D)$ is precompact.*

Many generalizations and extensions of Darbo's fixed point theorem can be found in the literature (see, for example, [3–9] and the references therein). Using the BG-measure of noncompactness, Aghajani *et al.* [4] obtained the following generalization of Darbo's theorem. Let $\mathcal{F}_{\mathcal{X}}$ be the set of self-mappings $D : \mathcal{X} \to \mathcal{X}$ satisfying the following conditions:

(i) D is a continuous mapping.
(ii) There exists $\sigma : \mathcal{B}_E \to [0, \infty)$, a BG-measure of noncompactness on E, such that for all $\varepsilon > 0$, there exists some $\delta_\varepsilon > 0$ for which

$$W \in P(\mathcal{X}), \quad \varepsilon \le \sigma(W) < \varepsilon + \delta_\varepsilon \quad \implies \quad \sigma(DW) < \varepsilon.$$

Theorem 1.3 (Aghajani *et al.* [4]) *Let $D : \mathcal{X} \to \mathcal{X}$ be a mapping that belongs to $\mathcal{F}_{\mathcal{X}}$. Then D has at least one fixed point.*

Observe that $\mathcal{D}_{\mathcal{X}} \subseteq \mathcal{F}_{\mathcal{X}}$. In fact, let $D : \mathcal{X} \to \mathcal{X}$ be a given mapping that belongs to $\mathcal{D}_{\mathcal{X}}$. Let $\varepsilon > 0$. From the definition of $\mathcal{D}_{\mathcal{X}}$, there is some $k \in (0, 1)$ such that

$$\sigma(DW) \le k\sigma(W),$$

for any nonempty subset W of $\mathcal{X}$. Let $\delta_\varepsilon = (\frac{1}{k} - 1)\varepsilon$. Then for any nonempty subset W of $\mathcal{X}$, we have

$$\varepsilon \le \sigma(W) < \varepsilon + \delta_\varepsilon = \frac{\varepsilon}{k} \quad \implies \quad \sigma(DW) \le k\sigma(W) < \varepsilon,$$

so $D \in \mathcal{F}_{\mathcal{X}}$.

In [6], Dhage introduced the following axiomatic definition of the measure of noncompactness.

Definition 1.4 Let $\sigma : \mathcal{B}_E \to [0, \infty)$ be a given mapping. We say that σ is a D-measure of noncompactness (in the sense of Dhage) on E if the following conditions are satisfied:

(i) For every $X \in \mathcal{B}_E$, $\sigma(X) = 0$ iff X is precompact.

(ii) For every pair $(X, Y) \in \mathcal{B}_E \times \mathcal{B}_E$, we have

$$X \subseteq Y \quad \Longrightarrow \quad \sigma(X) \leq \sigma(Y).$$

(iii) For every $X \in \mathcal{B}_E$, we have

$$\sigma(\overline{X}) = \sigma(X) = \sigma\big(\mathrm{conv}(X)\big).$$

(iv) If $\{X_n\} \subseteq \mathcal{B}_E$ is a decreasing sequence (w.r.t. $\subseteq$) such that $\sigma(X_n) \to 0$ as $n \to \infty$, then the $X_\infty := \bigcap_{n=1}^{\infty} \overline{X_n}$ is nonempty.

Observe that if $\sigma : \mathcal{B}_E \to [0, \infty)$ is a BG-measure of noncompactess on E, then σ is a D-measure of noncompactess on E.

In this paper, using the axiomatic definition of the measure of noncompactness given by Dhage, we obtain new generalizations of Theorem 1.2. Finally, an existence result for a certain class of fractional integral equations will be given as an application.

2 Main results

Let $\mathcal{X}$ be a nonempty, bounded, closed, and convex subset of a Banach space E. We continue to use the same notations presented in the previous section of this paper.

Let $\mathcal{F}'_{\mathcal{X}}$ be the set of self-mappings $D : \mathcal{X} \to \mathcal{X}$ satisfying the following conditions:

(i) D is a continuous mapping.

(ii) There exists $\sigma : \mathcal{B}_E \to [0, \infty)$, a D-measure of noncompactness on E, such that for all $\varepsilon > 0$, there exists some $\delta_\varepsilon > 0$ for which

$$W \in P(\mathcal{X}), \quad \varepsilon \leq \sigma(W) < \varepsilon + \delta_\varepsilon \quad \Longrightarrow \quad \sigma(DW) < \varepsilon.$$

We have the following result.

Theorem 2.1 *Let $D : \mathcal{X} \to \mathcal{X}$ be a mapping that belongs to $\mathcal{F}'_{\mathcal{X}}$. Then D has at least one fixed point.*

The result of Theorem 2.1 can be obtained using the same arguments of the proof of Theorem 1.3 in [4]. By Theorem 2.1, we want just to mention that Theorem 1.3 is still valid for any D-measure of noncompactness.

Let $\mathcal{G}_{\mathcal{X}}$ be the set of mappings $D : \mathcal{X} \to \mathcal{X}$ such that

(i) D is continuous.

(ii) There exists a function $\omega : [0, \infty) \to [0, \infty)$ such that

(ω_1) $\omega(t) = 0$ iff $t = 0$;

(ω_2) ω is nondecreasing and right continuous;

(ω_3) for every $\varepsilon > 0$, there exists $\gamma_\varepsilon > 0$ such that

$$W \in P(\mathcal{X}), \quad \varepsilon \leq \omega\big(\sigma(W)\big) < \varepsilon + \gamma_\varepsilon \quad \Longrightarrow \quad \omega\big(\sigma(DW)\big) < \varepsilon,$$

where $\sigma : \mathcal{B}_E \to [0, \infty)$ is a D-measure of noncompactness.

The following lemma can be proved using a similar argument as in the proof of Theorem 2.6 in [4].

Lemma 2.2 *We have*

$$\mathcal{G}_\mathcal{X} \subseteq \mathcal{F}'_\mathcal{X}.$$

Using Theorem 2.1 and Lemma 2.2, we obtain the following result.

Corollary 2.3 *Let $D : \mathcal{X} \to \mathcal{X}$ be a mapping that belongs to $\mathcal{G}_\mathcal{X}$. Then D has at least one fixed point.*

Let Φ be the set of functions $\varphi : [0, \infty) \to [0, \infty)$ satisfying the conditions:

(Φ_1) $\varphi \in L^1_{\mathrm{loc}}[0, \infty)$;
(Φ_2) for every $\xi > 0$, we have

$$\int_0^\xi \varphi(s)\, ds > 0.$$

Let $\mathcal{H}_\mathcal{X}$ be the set of mappings $D : \mathcal{X} \to \mathcal{X}$ such that

($\mathcal{H}_1$) D is continuous;
($\mathcal{H}_2$) for every $\varepsilon > 0$, there exists some $\gamma_\varepsilon > 0$ such that

$$W \in P(\mathcal{X}), \quad \varepsilon \leq \int_0^{\sigma(W)} \varphi(s)\, ds < \varepsilon + \gamma_\varepsilon \quad \Longrightarrow \quad \int_0^{\sigma(DW)} \varphi(s)\, ds < \varepsilon,$$

where $\varphi \in \Phi$ and $\sigma : \mathcal{B}_E \to [0, \infty)$ is a D-measure of noncompactness.

Lemma 2.4 *We have*

$$\mathcal{H}_\mathcal{X} \subseteq \mathcal{G}_\mathcal{X}.$$

Proof Take

$$\omega(t) = \int_0^t \varphi(s)\, ds, \quad t \geq 0,$$

we obtain the desired result. □

Using Corollary 2.3 and Lemma 2.4, we obtain the following result.

Corollary 2.5 *Let $D : \mathcal{X} \to \mathcal{X}$ be a mapping that belongs to $\mathcal{H}_\mathcal{X}$. Then D has at least one fixed point.*

Let $\mathcal{I}_{\mathcal{X}}$ be the set of mappings $D : \mathcal{X} \to \mathcal{X}$ such that

($\mathcal{I}_1$) D is continuous;
($\mathcal{I}_2$) there exists some $\varphi \in \Phi$ such that

$$\int_0^{\sigma(DW)} \varphi(s)\, ds \le k \int_0^{\sigma(W)} \varphi(s)\, ds, \quad W \in P(\mathcal{X}),$$

where $k \in (0,1)$ is a constant and $\sigma : \mathcal{B}_E \to [0, \infty)$ is a D-measure of noncompactness.

Lemma 2.6 *We have*

$$\mathcal{I}_{\mathcal{X}} \subseteq \mathcal{H}_{\mathcal{X}}.$$

Proof Let $D : \mathcal{X} \to \mathcal{X}$ be a mapping that belongs to $\mathcal{I}_{\mathcal{X}}$. Let $\varepsilon > 0$ be fixed. Let $\gamma_\varepsilon = (\frac{1}{k} - 1)\varepsilon$. Take $W \in P(\mathcal{X})$ such that

$$\varepsilon \le \int_0^{\sigma(W)} \varphi(s)\, ds < \varepsilon + \gamma_\varepsilon = \frac{\varepsilon}{k}.$$

From ($\mathcal{I}_2$), we obtain

$$\int_0^{\sigma(DW)} \varphi(s)\, ds \le k \int_0^{\sigma(W)} \varphi(s)\, ds < k \frac{\varepsilon}{k} = \varepsilon,$$

so $D \in \mathcal{H}_{\mathcal{X}}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Using Corollary 2.5 and Lemma 2.6, we obtain the following result.

Corollary 2.7 *Let $D : \mathcal{X} \to \mathcal{X}$ be a mapping that belongs to $\mathcal{I}_{\mathcal{X}}$. Then D has at least one fixed point.*

Remark 2.8 Take $\varphi(t) = 1$, $t \ge 0$ in Corollary 2.7, we obtain Theorem 1.2.

Let $\mathcal{J}_{\mathcal{X}}$ be the set of mappings $D : \mathcal{X} \to \mathcal{X}$ such that

($\mathcal{J}_1$) D is continuous;
($\mathcal{J}_2$) there exists a function $\eta : (0, \infty) \to \mathbb{R}$ such that

(η_1) for each sequence $\{\alpha_n\} \subset (0, \infty)$, we have

$$\lim_{n \to \infty} \eta(\alpha_n) = -\infty \quad \Longrightarrow \quad \lim_{n \to \infty} \alpha_n = 0;$$

(η_2) there exists $\tau > 0$ such that

$$W \in P(\mathcal{X}), \quad \sigma(W)\sigma(DW) > 0 \quad \Longrightarrow \quad \tau + \eta\big(\sigma(DW)\big) \le \eta\big(\sigma(W)\big),$$

where $\sigma : \mathcal{B}_E \to [0, \infty)$ is a D-measure of noncompactness.

Theorem 2.9 *Let $D : \mathcal{X} \to \mathcal{X}$ be a mapping that belongs to $\mathcal{J}_{\mathcal{X}}$. Then D has at least one fixed point.*

Proof Consider the sequence $\{\mathcal{X}_n\}$ of subsets of E defined by

$$\begin{cases} \mathcal{X}_0 := \mathcal{X}, \\ \mathcal{X}_{n+1} := \overline{\text{conv}}(D\mathcal{X}_n), \quad n = 0, 1, 2, \ldots. \end{cases} \tag{2.1}$$

By induction, we observe easily that

$$\mathcal{X}_{n+1} \subseteq \mathcal{X}_n, \quad n = 0, 1, 2, \ldots. \tag{2.2}$$

If for some N, we have $\sigma(\mathcal{X}_N) = 0$, then by the property (i) of the D-measure of noncompactness, $\mathcal{X}_N$ is compact. Since $D(\mathcal{X}_N) \subseteq \mathcal{X}_N$ (from (2.2)), Schauder's fixed point theorem applied to the self-mapping $D : \mathcal{X}_N \to \mathcal{X}_N$ gives the desired result. So, without loss of the generality, we may assume that

$$\sigma(\mathcal{X}_n) > 0, \quad n = 0, 1, 2, \ldots.$$

For $n = 0$, since $\sigma(\mathcal{X}_0) > 0$ and $\sigma(D\mathcal{X}_0) = \sigma(\mathcal{X}_1) > 0$, from the property (η_2) we have

$$\tau + \eta\big(\sigma(D\mathcal{X}_0)\big) \le \eta\big(\sigma(\mathcal{X}_0)\big),$$

which yields

$$\eta\big(\sigma(\mathcal{X}_1)\big) \le \eta\big(\sigma(\mathcal{X}_0)\big) - \tau.$$

Similarly, for $n = 1$, we have

$$\eta\big(\sigma(\mathcal{X}_2)\big) \le \eta\big(\sigma(\mathcal{X}_1)\big) - \tau \le \eta\big(\sigma(\mathcal{X}_0)\big) - 2\tau.$$

By induction, we obtain

$$\eta\big(\sigma(\mathcal{X}_n)\big) \le \eta\big(\sigma(\mathcal{X}_0)\big) - n\tau, \quad n = 0, 1, 2, \ldots.$$

Since

$$\lim_{n \to \infty} \eta\big(\sigma(\mathcal{X}_0)\big) - n\tau = -\infty,$$

we deduce that

$$\lim_{n \to \infty} \eta\big(\sigma(\mathcal{X}_n)\big) = -\infty,$$

so from the property (η_1) we have

$$\lim_{n \to \infty} \sigma(\mathcal{X}_n) = 0. \tag{2.3}$$

From the property (iv) of the D-measure of noncompactness, the set $M := \bigcap_{n=1}^{\infty} \mathcal{X}_n$ is nonempty. Moreover, for every $p = 0, 1, 2, \ldots$, we have

$$M \subseteq \mathcal{X}_p, \tag{2.4}$$

which implies from (2.2) that

$$DM \subseteq D\mathcal{X}_p \subseteq \mathcal{X}_{p+1} \subseteq \mathcal{X}_p, \quad p = 0, 1, 2, \ldots.$$

Then $D : M \to M$ is well defined. On the other hand, from (2.4) and the property (ii) of the D-measure of noncompactness, we have

$$\sigma(M) \leq \sigma(\mathcal{X}_p), \quad p = 0, 1, 2, \ldots.$$

Passing to the limit as $p \to \infty$ and using (2.3), we obtain

$$\sigma(M) = 0,$$

which implies from the property (i) of the D-measure of noncompactness that $\overline{M} = M$ is compact. Applying Schauder's fixed point theorem to the mapping $D : M \to M$, we obtain the desired result. $\qquad\square$

Remark 2.10 Observe that $\mathcal{D}_{\mathcal{X}} \subseteq \mathcal{J}_{\mathcal{X}}$. In fact, if $D : \mathcal{X} \to \mathcal{X}$ belongs to $\mathcal{D}_{\mathcal{X}}$, that is,

$$\sigma(DW) \leq k\sigma(W), \quad W \in P(\mathcal{X}),$$

then

$$W \in P(X), \quad \sigma(W)\sigma(DW) > 0 \quad \implies \quad \ln \sigma(DW) - \ln k \leq \ln \sigma(W).$$

Then $D \in \mathcal{J}_{\mathcal{X}}$ with $\eta(t) = \ln t$, $t > 0$. Therefore, Theorem 2.9 is a generalization of Theorem 1.2.

Let $\mathcal{K}_{\mathcal{X}}$ be the set of mappings $D : \mathcal{X} \to \mathcal{X}$ such that

$(\mathcal{K}_1)$ D is continuous;
$(\mathcal{K}_2)$ there exists a function $\theta : (0, \infty) \to (1, \infty)$ such that

(θ_1) for each sequence $\{u_n\} \subset (0, \infty)$, we have

$$\lim_{n \to \infty} \theta(u_n) = 1 \quad \implies \quad \lim_{n \to \infty} u_n = 0;$$

(θ_2) there exist $k \in (0, 1)$ and a D-measure of noncompactness $\sigma : \mathcal{B}_E \to [0, \infty)$ such that

$$W \in P(\mathcal{X}), \quad \sigma(W)\sigma(DW) > 0 \quad \implies \quad \theta(\sigma(DW)) \leq [\theta(\sigma(W))]^k.$$

We have the following result.

Theorem 2.11 *Let $D : \mathcal{X} \to \mathcal{X}$ be a mapping that belongs to $\mathcal{K}_{\mathcal{X}}$. Then D has at least one fixed point.*

Proof Consider the sequence $\{\mathcal{X}_n\}$ of subsets of E defined by (2.1). As in the proof of Theorem 2.9, without loss of the generality, we may assume that

$$\sigma(\mathcal{X}_n) > 0, \quad n = 0, 1, 2, \ldots.$$

For $n = 0$, since $\sigma(\mathcal{X}_0) > 0$ and $\sigma(D\mathcal{X}_0) = \sigma(\mathcal{X}_1) > 0$, we have

$$\theta\big(\sigma(D\mathcal{X}_0)\big) \le \big[\theta\big(\sigma(\mathcal{X}_0)\big)\big]^k,$$

that is,

$$\theta\big(\sigma(\mathcal{X}_1)\big) \le \big[\theta\big(\sigma(\mathcal{X}_0)\big)\big]^k.$$

Again, for $n = 1$, since $\sigma(\mathcal{X}_1) > 0$ and $\sigma(D\mathcal{X}_1) = \sigma(\mathcal{X}_2) > 0$, we have

$$\theta\big(\sigma(D\mathcal{X}_1)\big) \le \big[\theta\big(\sigma(\mathcal{X}_1)\big)\big]^k,$$

that is,

$$\theta\big(\sigma(\mathcal{X}_2)\big) \le \big[\theta\big(\sigma(\mathcal{X}_1)\big)\big]^k,$$

so

$$\theta\big(\sigma(\mathcal{X}_2)\big) \le \big[\theta\big(\sigma(\mathcal{X}_0)\big)\big]^{k^2}.$$

Therefore, by induction, we get

$$1 < \theta\big(\sigma(\mathcal{X}_n)\big) \le \big[\theta\big(\sigma(\mathcal{X}_0)\big)\big]^{k^n}, \quad n = 0, 1, 2, \ldots.$$

Passing to the limit as $n \to \infty$, we obtain

$$\lim_{n \to \infty} \theta\big(\sigma(\mathcal{X}_n)\big) = 1,$$

so from the property (θ_1) we have

$$\lim_{n \to \infty} \sigma(\mathcal{X}_n) = 0.$$

The rest of the proof is similar to that in the proof of Theorem 2.9. □

Corollary 2.12 *Let $D : \mathcal{X} \to \mathcal{X}$ be a continuous mapping. Suppose that there exist a constant $k \in (0, 1)$ and a D-measure of noncompactness $\sigma : \mathcal{B}_E \to [0, \infty)$ such that*

$$2 - \frac{2}{\pi} \arctan\left(\frac{1}{\sqrt{\sigma(DW)}}\right) \le \left[2 - \frac{2}{\pi} \arctan\left(\frac{1}{\sqrt{\sigma(W)}}\right)\right]^k,$$

for any $W \in P(\mathcal{X})$ with $\sigma(W)\sigma(DW) > 0$. Then D has at least one fixed point.

Proof Taking

$$\theta(t) = 2 - \frac{2}{\pi} \arctan\left(\frac{1}{\sqrt{t}}\right), \quad t > 0,$$

in Theorem 2.11, we obtain the desired result. □

Remark 2.13 Observe that $\mathcal{D}_{\mathcal{X}} \subseteq \mathcal{K}_{\mathcal{X}}$. In fact, if $D : \mathcal{X} \to \mathcal{X}$ belongs to $\mathcal{D}_{\mathcal{X}}$, that is,

$$\sigma(DW) \le k\sigma(W), \quad W \in P(\mathcal{X}),$$

then

$$W \in P(\mathcal{X}), \quad \sigma(W)\sigma(DW) > 0 \quad \Longrightarrow \quad e^{\sigma(DW)} \le \left[e^{\sigma(W)}\right]^{k}.$$

Therefore $D \in \mathcal{K}_{\mathcal{X}}$ with $\theta(t) = e^t$.

Let $\mathcal{L}_{\mathcal{X}}$ be the set of mappings $D : \mathcal{X} \to \mathcal{X}$ such that

($\mathcal{L}_1$) D is continuous;
($\mathcal{L}_2$) there exists a function $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ such that

    (ζ_1) $\zeta(z_1, z_2) < z_2 - z_1$, for all $z_1, z_2 > 0$;
    (ζ_2) if $\{u_n\}$ and $\{v_n\}$ are two sequences in $(0, \infty)$ such that $\lim_{n \to \infty} u_n = \lim_{n \to \infty} v_n = \ell > 0$, then

$$\limsup_{n \to \infty} \zeta(u_n, v_n) < 0,$$

    (ζ_3)

$$\zeta\big(\sigma(DW), \sigma(W)\big) \ge 0, \quad W \in P(\mathcal{X}),$$

where $\sigma : \mathcal{B}_E \to [0, \infty)$ is a D-measure of noncompactness.

Theorem 2.14 *Let $D : \mathcal{X} \to \mathcal{X}$ be a mapping that belongs to $\mathcal{L}_{\mathcal{X}}$. Then D has at least one fixed point.*

Proof Consider the sequence $\{\mathcal{X}_n\}$ of subsets of E defined by (2.1). From the property (ζ_3), we have

$$\zeta\big(\sigma(\mathcal{X}_{n+1}), \sigma(\mathcal{X}_n)\big) \ge 0, \quad n = 0, 1, 2, \ldots. \tag{2.5}$$

As before, without loss of the generality, we may assume that

$$\sigma(\mathcal{X}_n) > 0, \quad n = 0, 1, 2, \ldots. \tag{2.6}$$

From the property (ζ_1), (2.5) and (2.6), we get

$$\sigma(\mathcal{X}_n) \ge \sigma(\mathcal{X}_{n+1}), \quad n = 0, 1, 2, \ldots.$$

Then there is some $r \geq 0$ such that

$$\lim_{n \to \infty} \sigma(\mathcal{X}_n) = r.$$

If $r > 0$, then from the property (ζ_2), we have

$$\limsup_{n \to \infty} \zeta\big(\sigma(\mathcal{X}_{n+1}), \sigma(\mathcal{X}_n)\big) < 0,$$

which contradicts (2.5). As consequence, we have

$$\lim_{n \to \infty} \sigma(\mathcal{X}_n) = 0.$$

The rest of the proof is similar to the proof of Theorem 2.9.   □

Remark 2.15 Taking

$$\zeta(z_1, z_2) = k z_2 - z_1,$$

where $k \in (0,1)$ is a constant, we obtain Theorem 1.2.

Corollary 2.16 *Let $D : \mathcal{X} \to \mathcal{X}$ be a continuous mapping such that*

$$\sigma(DW) \leq \sigma(W) - \Phi\big(\sigma(W)\big), \quad W \in P(\mathcal{X}),$$

where $\Phi : [0, \infty) \to [0, \infty)$ is a lower semi-continuous function with $\Phi^{-1}(0) = \{0\}$ and $\sigma : \mathcal{B}_E \to [0, \infty)$ is a D-measure of noncompactness. Then D has at least one fixed point.

Proof Taking

$$\zeta(z_1, z_2) = z_2 - \Phi(z_2) - z_1$$

in Theorem 2.14, we obtain the desired result.   □

Corollary 2.17 *Let $D : \mathcal{X} \to \mathcal{X}$ be a continuous mapping such that*

$$\sigma(DW) \leq \psi\big(\sigma(W)\big), \quad W \in P(\mathcal{X}),$$

where $\psi : [0, \infty) \to [0, \infty)$ is an upper semi-continuous function with $\psi(t) < t$ for all $t > 0$ and $\sigma : \mathcal{B}_E \to [0, \infty)$ is a D-measure of noncompactness. Then D has at least one fixed point.

Proof Taking

$$\zeta(z_1, z_2) = \psi(z_2) - z_1$$

in Theorem 2.14, we obtain the desired result.   □

3 An existence result for a fractional integral equation

The measure of noncompactness argument is a useful tool in Nonlinear Analysis. In particular, such argument can be used to obtain existence results for various classes of integral equations. For more details on the applications of the measure of noncompactness concept, we refer the reader to [1, 3, 4, 6, 8–15] and the references therein.

In this section, we discuss the existence of solutions to the fractional integral equation

$$y(t) = \frac{f(t, y(t))}{\Gamma(\alpha)} \int_0^t \frac{g'(s)u(s, y(s))}{(g(t) - g(s))^{1-\alpha}} \, ds, \quad t \in [0, T], \tag{3.1}$$

where $T > 0$, $\alpha \in (0, 1)$, $u, f : [0, T] \times \mathbb{R} \to \mathbb{R}$ and $g : [0, T] \to \mathbb{R}$.

We suppose that the following conditions are satisfied.

(i) The function $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is continuous.

(ii) There exists an upper semi-continuous function $\psi : [0, \infty) \to [0, \infty)$ such that $\psi(0) = 0$, $\psi(t) < t$ for all $t > 0$, ψ is nondecreasing, and

$$\left| f(t, x) - f(t, y) \right| \leq \psi\left(|x - y|\right), \quad (t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}.$$

(iii) The function $u : [0, \infty) \to [0, \infty)$ is continuous and there exists a nondecreasing function $\omega : [0, \infty) \to [0, \infty)$ such that

$$\left| u(t, z) \right| \leq \omega\left(|z|\right), \quad (t, z) \in [0, T] \times \mathbb{R}.$$

(iv) The function $g : [0, T] \to \mathbb{R}$ is C^1 and nondecreasing.

(v) There exists $r_0 > 0$ such that

$$\left(\psi(r_0) + F\right)\omega(r_0)\left(g(T) - g(0)\right)^\alpha \leq r_0 \Gamma(\alpha + 1)$$

and

$$\frac{\omega(r_0)}{\Gamma(\alpha + 1)}\left(g(T) - g(0)\right)^\alpha \leq 1,$$

where $F = \max\{|f(t, 0) : t \in [0, T]\}$.

Let $E = C([0, T]; \mathbb{R})$ be the set of real continuous functions defined in $[0, T]$. The set E endowed with the norm

$$\|z\| = \max\left\{\left|z(t)\right| : t \in [0, T]\right\}, \quad z \in E,$$

is a Banach space. Let W be a nonempty and bounded subset of E. Let us define the mapping $\gamma : W \times [0, \infty) \to [0, \infty)$ by

$$\gamma(z, \rho) = \sup\left\{\left|z(a) - z(b)\right| : a, b \in [0, T], |a - b| \leq \rho\right\}, \quad z \in W, \rho \geq 0.$$

Set

$$\gamma(W, \rho) = \sup\left\{\gamma(z, \rho) : z \in W\right\}, \quad \rho \geq 0.$$

Let $\mathcal{B}_E$ be the set of all nonempty bounded subsets of E. Then the mapping

$$\sigma : \mathcal{B}_E \to [0, \infty)$$

defined by

$$\sigma(W) = \lim_{\rho \to 0^+} \gamma(W, \rho), \quad W \in \mathcal{B}_E,$$

is a BG-measure of noncompactness (then it is a D-measure of noncompactness) on the space E (see [1]).

We have the following existence result.

Theorem 3.1 *Under the assumptions* (i)-(v), *equation* (3.1) *has at least one solution* $y^* \in E$. *Moreover, we have* $\|y^*\| \leq r_0$.

Proof Let us consider the operator D defined on E by

$$(Dy)(t) = \frac{f(t, y(t))}{\Gamma(\alpha)} \int_0^t \frac{g'(s)u(s, y(s))}{(g(t) - g(s))^{1-\alpha}} \, ds, \quad (y, t) \in E \times [0, T]. \tag{3.2}$$

At first, we show that the operator D maps E into itself. Set

$$(Hy)(t) = \int_0^t \frac{g'(s)u(s, y(s))}{(g(t) - g(s))^{1-\alpha}} \, ds, \quad (y, t) \in E \times [0, T]. \tag{3.3}$$

From the assumption (i), we have just to show that H maps E into itself. In order to prove this fact, let us fix some $y \in E$. Observe that $Hy : [0, T] \to \mathbb{R}$ is a well-defined function. In fact, using the assumptions (iii) and (iv), for all $t \in [0, T]$ we have

$$\begin{aligned}
\left| (Hy)(t) \right| &\leq \int_0^t \frac{g'(s)|u(s, y(s))|}{(g(t) - g(s))^{1-\alpha}} \, ds \\
&\leq \omega(\|y\|) \int_0^t \frac{g'(s)}{(g(t) - g(s))^{1-\alpha}} \, ds \\
&= \frac{\omega(\|y\|)}{\alpha} \big(g(t) - g(0)\big)^\alpha,
\end{aligned}$$

that is,

$$\left| (Hy)(t) \right| \leq \frac{\omega(\|y\|)}{\alpha} \big(g(t) - g(0)\big)^\alpha < \infty, \quad t \in [0, T]. \tag{3.4}$$

Let us prove the continuity of Hy at 0. To do this, let $\{t_n\}$ be a sequence in $[0, T]$ such that $t_n \to 0^+$ as $n \to \infty$. From (3.4), for all n we have

$$\left| (Hy)(t_n) \right| \leq \frac{\omega(\|y\|)}{\alpha} \big(g(t_n) - g(0)\big)^\alpha.$$

Passing to the limit as $n \to \infty$ and using the continuity of g at 0, we obtain

$$\lim_{n \to \infty} (Hy)(t_n) = 0 = (Hy)(0).$$

Then Hy is continuous at 0.

Now, let $t \in (0, T]$ be fixed and $\{t_n\}$ be a sequence in $(0, T]$ such that $t_n \to t$ as $n \to \infty$. Without restriction of the generality, we may assume that $t_n \geq t$ for n large enough. For every n, we have

$$\left|(Hy)(t_n) - (Hy)(t)\right| = \left| \int_0^{t_n} \frac{g'(s)u(s, y(s))}{(g(t_n) - g(s))^{1-\alpha}} \, ds - \int_0^t \frac{g'(s)u(s, y(s))}{(g(t) - g(s))^{1-\alpha}} \, ds \right|.$$

For n large enough, we can write

$$\begin{aligned}
\left|(Hy)(t_n) - (Hy)(t)\right| &\leq \left| \int_0^t \left(\frac{g'(s)u(s, y(s))}{(g(t_n) - g(s))^{1-\alpha}} - \frac{g'(s)u(s, y(s))}{(g(t) - g(s))^{1-\alpha}} \right) d\tau \right| \\
&\quad + \left| \int_t^{t_n} \frac{g'(s)u(s, y(s))}{(g(t_n) - g(s))^{1-\alpha}} \, ds \right| \\
&\leq \omega(\|y\|) \int_0^t \left(\frac{g'(s)}{(g(t) - g(s))^{1-\alpha}} - \frac{g'(s)}{(g(t_n) - g(s))^{1-\alpha}} \right) ds \\
&\quad + \omega(\|y\|) \int_t^{t_n} \frac{g'(s)}{(g(t_n) - g(s))^{1-\alpha}} \, ds \\
&= \frac{\omega(\|y\|)}{\alpha} \left((g(t) - g(0))^\alpha + (g(t_n) - g(t))^\alpha - (g(t_n) - g(0))^\alpha \right) \\
&\quad + \frac{\omega(\|y\|)}{\alpha} (g(t_n) - g(t))^\alpha.
\end{aligned}$$

Since g is continuous in $[0, T]$, we have

$$\lim_{n \to \infty} \frac{\omega(\|y\|)}{\alpha} \left((g(t) - g(0))^\alpha + (g(t_n) - g(t))^\alpha - (g(t_n) - g(0))^\alpha \right) + \frac{\omega(\|y\|)}{\alpha} (g(t_n) - g(t))^\alpha = 0,$$

which yields $\lim_{n \to \infty} |(Hy)(t_n) - (Hy)(t)| = 0$. Then Hy is continuous at t. As consequence, $Hy \in E$, for all $y \in E$, and $D : E \to E$ is well defined.

On the other hand, using the assumptions (ii) and (iii), for an arbitrarily fixed $y \in E$ and $t \in [0, T]$, we have

$$\begin{aligned}
\left|(Dy)(t)\right| &\leq \frac{|f(t, y(t))|}{\Gamma(\alpha)} \int_0^t \frac{g'(s)|u(s, y(s))|}{(g(t) - g(s))^{1-\alpha}} \, ds \\
&\leq \frac{|f(t, y(t)) - f(t, 0)| + |f(t, 0)|}{\Gamma(\alpha)} \int_0^t \frac{g'(s)\omega(|y(s)|)}{(g(t) - g(s))^{1-\alpha}} \, ds \\
&\leq \frac{(\psi(|y(t)|) + F)\omega(\|y\|)}{\Gamma(\alpha + 1)} (g(t) - g(0))^\alpha \\
&\leq \frac{(\psi(\|y\|) + F)\omega(\|y\|)}{\Gamma(\alpha + 1)} (g(T) - g(0))^\alpha.
\end{aligned}$$

Then

$$\|Dy\| \leq \frac{(\psi(\|y\|) + F)\omega(\|y\|)}{\Gamma(\alpha + 1)} (g(T) - g(0))^\alpha, \quad y \in E.$$

Using the above inequality, the fact that the functions $\psi, \omega : [0, \infty) \to [0, \infty)$ are nondecreasing, and the assumption (v), we infer that the operator D maps $\overline{B(0, r_0)}$ into itself,

where

$$\overline{B(0, r_0)} = \big\{ z \in E : \|z\| \le r_0 \big\}.$$

Now, we claim that the operator $D : \overline{B(0, r_0)} \to \overline{B(0, r_0)}$ is continuous. From (3.2), we can write D in the form

$$Dy = \frac{1}{\Gamma(\alpha)} Gy \cdot Hy, \quad y \in E,$$

where

$$(Gy)(t) = f\big(t, y(t)\big), \quad (y, t) \in E \times [0, T],$$

and Hy is defined by (3.3). In order to prove our claim, it is sufficient to show that the operators G and H are continuous on $\overline{B(0, r_0)}$. First of all, we show that G is a continuous operator on $\overline{B(0, r_0)}$. To do this, we take a sequence $\{y_n\} \subset \overline{B(0, r_0)}$ and $y \in \overline{B(0, r_0)}$ such that $\|y_n - y\| \to 0$ as $n \to \infty$, and we have to prove that $\|Gy_n - Gy\| \to 0$ as $n \to \infty$. In fact, for all $t \in [0, T]$, using the condition (ii), we have

$$
\begin{aligned}
\big|(Gy_n)(t) - (Gy)(t)\big| &= \big|f\big(t, y_n(t)\big) - f\big(t, y(t)\big)\big| \\
&\le \psi\big(\big|y_n(t) - y(t)\big|\big) \\
&\le \psi\big(\|y_n - y\|\big) \\
&\le \|y_n - y\|.
\end{aligned}
$$

Thus we have

$$\|Gy_n - Gy\| \le \|y_n - y\|, \quad \text{for all } n.$$

Passing to the limit as $n \to \infty$ in the above inequality, we obtain

$$\lim_{n \to \infty} \|Gy_n - Gy\| = 0.$$

This proves that G is a continuous operator on $\overline{B(0, r_0)}$. Next, we show that H is a continuous operator on $\overline{B(0, r_0)}$. To do this, we fix a real number $\varepsilon > 0$ and we take arbitrary functions $x, y \in \overline{B(0, r_0)}$ such that $\|x - y\| < \varepsilon$. For all $t \in [0, T]$, we have

$$
\begin{aligned}
\big|(Hx)(t) - (Hy)(t)\big| &\le \int_0^t \frac{g'(s)|u(s, x(s)) - u(s, y(s))|}{(g(t) - g(s))^{1-\alpha}} \, ds \\
&\le u(r_0, \varepsilon) \int_0^t \frac{g'(s)}{(g(t) - g(s))^{1-\alpha}} \, ds \\
&\le \frac{u(r_0, \varepsilon)}{\alpha} \big(g(T) - g(0)\big)^\alpha,
\end{aligned}
$$

where

$$u(r_0, \varepsilon) = \sup\big\{ \big|u(\tau, v) - u(\tau, w)\big| : \tau \in [0, T], v, w \in [-r_0, r_0], |v - w| < \varepsilon \big\}.$$

Therefore,

$$\|Hx - Hy\| \le \frac{u(r_0, \varepsilon)}{\alpha} \big(g(T) - g(0)\big)^\alpha.$$

Since u is uniformly continuous on the compact $[0, T] \times [-r_0, r_0]$, we have $u(r_0, \varepsilon) \to 0$ as $\varepsilon \to 0^+$ and, therefore, the last inequality gives us

$$\lim_{\varepsilon \to 0^+} \|Hx - Hy\| = 0.$$

Then H is continuous on $\overline{B(0, r_0)}$ and D maps continuously the set $\overline{B(0, r_0)}$ into itself.

Further, let W be a nonempty subset of $\overline{B(0, r_0)}$. Let $\rho > 0$ be fixed, $y \in W$, and $t_1, t_2 \in [0, T]$ be such that $|t_1 - t_2| \le \rho$. Without restriction of the generality, we may assume that $t_1 \ge t_2$. We have

$$
\begin{aligned}
&\big|(Dy)(t_1) - (Dy)(t_2)\big| \\
&\le \left| \frac{f(t_1, y(t_1))}{\Gamma(\alpha)} \int_0^{t_1} \frac{g'(s)u(s, y(s))}{(g(t_1) - g(s))^{1-\alpha}}\, ds - \frac{f(t_2, y(t_2))}{\Gamma(\alpha)} \int_0^{t_2} \frac{g'(s)u(s, y(s))}{(g(t_2) - g(s))^{1-\alpha}}\, ds \right| \\
&\le \left| \frac{f(t_1, y(t_1))}{\Gamma(\alpha)} \int_0^{t_1} \frac{g'(s)u(s, y(s))}{(g(t_1) - g(s))^{1-\alpha}}\, ds - \frac{f(t_2, y(t_2))}{\Gamma(\alpha)} \int_0^{t_1} \frac{g'(s)u(s, y(s))}{(g(t_1) - g(s))^{1-\alpha}}\, ds \right| \\
&\quad + \frac{|f(t_2, y(t_2))|}{\Gamma(\alpha)} \left| \int_0^{t_1} \frac{g'(s)u(s, y(s))}{(g(t_1) - g(s))^{1-\alpha}}\, ds - \int_0^{t_2} \frac{g'(s)u(s, y(s))}{(g(t_2) - g(s))^{1-\alpha}}\, ds \right| \\
&\le \frac{|f(t_1, y(t_1)) - f(t_1, y(t_2))| + |f(t_1, y(t_2)) - f(t_2, y(t_2))|}{\Gamma(\alpha)} \int_0^{t_1} \frac{g'(s)|u(s, y(s))|}{(g(t_1) - g(s))^{1-\alpha}}\, ds \\
&\quad + \frac{|f(t_2, y(t_2)) - f(t_2, 0)| + |f(t_2, 0)|}{\Gamma(\alpha)} \left| \int_0^{t_1} \frac{g'(s)u(s, y(s))}{(g(t_1) - g(s))^{1-\alpha}}\, ds \right. \\
&\quad \left. - \int_0^{t_2} \frac{g'(s)u(s, y(s))}{(g(t_1) - g(s))^{1-\alpha}}\, ds \right| \\
&\quad + \frac{|f(t_2, y(t_2)) - f(t_2, 0)| + |f(t_2, 0)|}{\Gamma(\alpha)} \left| \int_0^{t_2} \frac{g'(s)u(s, y(s))}{(g(t_1) - g(s))^{1-\alpha}}\, ds \right. \\
&\quad \left. - \int_0^{t_2} \frac{g'(s)u(s, y(s))}{(g(t_2) - g(s))^{1-\alpha}}\, ds \right| \\
&\le \frac{\psi(|y(t_1) - y(t_2)|) + \omega_f(r_0, \rho)}{\Gamma(\alpha + 1)} \omega(r_0)\big(g(t_1) - g(0)\big)^\alpha \\
&\quad + \frac{(\psi(r_0) + F)\omega(r_0)}{\Gamma(\alpha)} \int_{t_2}^{t_1} \frac{g'(s)}{(g(t_1) - g(s))^{1-\alpha}}\, ds \\
&\quad + \frac{(\psi(r_0) + F)\omega(r_0)}{\Gamma(\alpha + 1)} \Big(\big(g(t_2) - g(0)\big)^\alpha + \big(g(t_1) - g(t_2)\big)^\alpha - \big(g(t_1) - g(0)\big)^\alpha \Big) \\
&\le \frac{\psi(\gamma(y, \rho)) + \omega_f(r_0, \rho)}{\Gamma(\alpha + 1)} \omega(r_0)\big(g(T) - g(0)\big)^\alpha + \frac{2(\psi(r_0) + F)\omega(r_0)}{\Gamma(\alpha + 1)} \big(g(t_1) - g(t_2)\big)^\alpha \\
&\le \frac{\psi(\gamma(y, \rho)) + \omega_f(r_0, \rho)}{\Gamma(\alpha + 1)} \omega(r_0)\big(g(T) - g(0)\big)^\alpha + \frac{2(\psi(r_0) + F)\omega(r_0)}{\Gamma(\alpha + 1)} \big[\gamma(g, \rho)\big]^\alpha,
\end{aligned}
$$

where

$$\omega_f(r_0, \rho) = \sup\{|f(t, u) - f(s, u)| : u \in [-r_0, r_0], t, s \in [0, T], |t - s| \le \rho\}.$$

Therefore,

$$\gamma(DW, \rho) \le \frac{\psi(\gamma(W, \rho)) + \omega_f(r_0, \rho)}{\Gamma(\alpha + 1)} \omega(r_0)\big(g(T) - g(0)\big)^\alpha + \frac{2(\psi(r_0) + F)\omega(r_0)}{\Gamma(\alpha + 1)}\big[\omega(g, \rho)\big]^\alpha.$$

Passing to the limit superior as $\rho \to 0^+$ and using the fact that ψ is upper semi-continuous, we obtain

$$\sigma(DW) \le \frac{\psi(\sigma(W))}{\Gamma(\alpha + 1)} \omega(r_0)\big(g(T) - g(0)\big)^\alpha.$$

Then, from the assumption (v), we obtain

$$\sigma(DW) \le \psi\big(\sigma(W)\big).$$

As a consequence, for any nonempty subsets W of $\overline{B(0, r_0)}$, we have

$$\zeta\big(\sigma(DW), \sigma(W)\big) \ge 0,$$

where $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ is defined by

$$\zeta(z_1, z_2) = \psi(z_2) - z_1, \quad (z_1, z_2) \in [0, \infty) \times [0, \infty).$$

Under the assumptions on the function ψ, the operator $D : \overline{B(0, r_0)} \to \overline{B(0, r_0)}$ belongs to the family of operators $\mathcal{L}_{\mathcal{X}}$, where $\mathcal{X} = \overline{B(0, r_0)}$. Then by Theorem 2.14, we deduce that D has at least one fixed point $y^* \in \overline{B(0, r_0)}$, which is a solution to equation (3.1). $\qquad \square$

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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