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Iterative algorithms for infinite accretive mappings and applications to *p*-Laplacian-like differential systems

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Abstract

Some new iterative algorithms with errors for approximating common zero point of an infinite family of m-accretive mappings in a real Banach space are presented. A path convergence theorem and some new weak and strong convergence theorems are proved by means of some new techniques, which extend the corresponding works by some authors. As applications, an infinite p-Laplacian-like differential system is investigated, from which we construct an infinite family of m-accretive mappings and discuss the connections between the equilibrium solution of the differential systems and the zero point of the m-accretive mappings.

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1 Introduction

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* denote the dual space of E. We use '$\to$' and '$\rightharpoonup$' (or 'w-lim') to denote strong and weak convergence, respectively. We denote the value of $f \in E^*$ at $x \in E$ by $\langle x, f\rangle$.

Define a function $\rho_E : [0,+\infty) \to [0,+\infty)$ called the modulus of smoothness of E as follows:

$$\rho_E(t) = \sup\left\{\frac{\|x+y\| + \|x-y\|}{2} - 1 : x, y \in E, \|x\| = 1, \|y\| \le t\right\}.$$

A Banach space E is said to be uniformly smooth if $\frac{\rho_E(t)}{t} \to 0$, as $t \to 0$.

A Banach space E is said to be strictly convex if and only if $\|x\| = \|y\| = \|(1-\lambda)x + \lambda y\|$ for $x, y \in E$ and $0 < \lambda < 1$ implies that $x = y$. A Banach space E is said to be uniformly convex if for any $\varepsilon \in (0,2]$ there exists $\delta > 0$ such that

$$\|x\| = \|y\| = 1, \qquad \|x-y\| \ge \varepsilon \quad \Rightarrow \quad \left\|\frac{x+y}{2}\right\| \le 1-\delta.$$

It is well known that a uniformly convex Banach space is reflexive and strictly convex.

An operator $B : E \to E^*$ is said to be monotone if $\langle u - v, Bu - Bv \rangle \geq 0$, for all $u, v \in D(B)$. The monotone operator B is said to be maximal monotone if the graph of B, $G(B)$, is not contained properly in any other monotone subset of $E \times E^*$.

A single-valued mapping $F : D(F) = E \to E^*$ is said to be hemi-continuous [1] if $w\text{-}\lim_{t\to 0} F(x + ty) = Fx$, for any $x, y \in E$. A single-valued mapping $F : D(F) = E \to E^*$ is said to be demi-continuous [1] if $w\text{-}\lim_{n\to\infty} Fx_n = Fx$, for any sequence $\{x_n\}$ strongly convergent to x in E.

Following from [1] or [2], the function h is said to be a proper convex function on E if h is defined from E onto $(-\infty, +\infty]$, h is not identically $+\infty$ such that $h((1 - \lambda)x + \lambda y) \leq (1 - \lambda)h(x) + \lambda h(y)$, whenever $x, y \in E$ and $0 \leq \lambda \leq 1$. h is said to be strictly convex if $h((1 - \lambda)x + \lambda y) < (1 - \lambda)h(x) + \lambda h(y)$, for all $0 < \lambda < 1$ and $x, y \in E$ with $x \neq y$, $h(x) < +\infty$ and $h(y) < +\infty$. The function $h : E \to (-\infty, +\infty]$ is said to be lower-semi-continuous on E if $\liminf_{y\to x} h(y) \geq h(x)$, for any $x \in E$.

A continuous strictly increasing function $\varphi : [0, +\infty) \to [0, +\infty)$ is called a gauge function [2] if $\varphi(0) = 0$ and $\varphi(t) \to \infty$, as $t \to \infty$. The duality mapping $J_\varphi : E \to 2^{E^*}$ associated with the gauge function φ is defined by [2]

$$J_\varphi(x) = \bigl\{f \in E^* : \langle x, f\rangle = \|x\|\varphi\bigl(\|x\|\bigr), \|f\| = \varphi\bigl(\|x\|\bigr)\bigr\}, \quad x \in E.$$

It can be seen from [2] that the duality mapping J_φ has the following properties:

(i) $J_\varphi(-x) = -J_\varphi(x)$ and $J_\varphi(kx) = \frac{\varphi(\|kx\|)}{\varphi(\|x\|)} J_\varphi(x)$, for $\forall x \in E$ and $k > 0$;

(ii) if E^* is uniformly convex, then J_φ is uniformly continuous on each bounded subset in E;

(iii) the reflexivity of E and strict convexity of E^* imply that J_φ is single-valued, monotone and demi-continuous.

In the case $\varphi(t) \equiv t$, we call J_φ the normalized duality mapping, which is usually denoted by J.

For the gauge function φ, the function $\Phi : [0, +\infty) \to [0, +\infty)$ defined by

$$\Phi(t) = \int_0^t \varphi(s)\, ds \tag{1.1}$$

is a continuous convex strictly increasing function on $[0, +\infty)$.

Following the result in [3], a Banach space E is said to have a weakly continuous duality mapping if there is a gauge φ for which the duality mapping $J_\varphi(x)$ is single-valued and weak-to-weak* sequentially continuous (*i.e.*, if $\{x_n\}$ is a sequence in E weakly convergent to a point x, then the sequence $J_\varphi(x_n)$ converges weakly* to $J_\varphi(x)$). It is well known that l^p has a weakly continuous duality mapping with a gauge function $\varphi(t) = t^{p-1}$ for all $1 < p < +\infty$.

Let C be a nonempty, closed and convex subset of E and Q be a mapping of E onto C. Then Q is said to be sunny [4] if $Q(Q(x) + t(x - Q(x))) = Q(x)$, for all $x \in E$ and $t \geq 0$.

A mapping Q of E into E is said to be a retraction [4] if $Q^2 = Q$. If a mapping Q is a retraction, then $Q(z) = z$ for every $z \in R(Q)$, where $R(Q)$ is the range of Q.

A mapping $f : C \to C$ is called a contraction with contractive constant $k \in (0, 1)$ if $\|f(x) - f(y)\| \leq k\|x - y\|$, for $\forall x, y \in C$.

A mapping $T : C \to C$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$, for $\forall x, y \in C$. We use $\operatorname{Fix}(T)$ to denote the fixed point set of T. That is, $\operatorname{Fix}(T) := \{x \in C : Tx = x\}$. A mapping

$T: E \supset D(T) \to R(T) \subset E$ is said to be demi-closed at p if whenever $\{x_n\}$ is a sequence in $D(T)$ such that $x_n \rightharpoonup x \in D(T)$ and $Tx_n \to p$ then $Tx = p$.

A subset C of E is said to be a sunny nonexpansive retract of E [5, 6] if there exists a sunny nonexpansive retraction of E onto C and it is called a nonexpansive retract of E if there exists a nonexpansive retraction of E onto C.

A mapping $A: D(A) \subset E \to E$ is said to be accretive if $\|x_1 - x_2\| \leq \|x_1 - x_2 + r(y_1 - y_2)\|$, for $\forall x_i \in D(A)$, $y_i \in Ax_i$, $i = 1, 2$, and $r > 0$. For the accretive mapping A, we use $N(A)$ to denote the set of zero points of it; that is, $N(A) := \{x \in D(A) : Ax = 0\}$. If A is accretive, then we can define, for each $r > 0$, a nonexpansive single-valued mapping $J_r^A : R(I + rA) \to D(A)$ by $J_r^A := (I + rA)^{-1}$, which is called the resolvent of A [1]. We also know that, for an accretive mapping A, $N(A) = \mathrm{Fix}(J_r^A)$. An accretive mapping A is said to be m-accretive if $R(I + \lambda A) = E$, for $\forall \lambda > 0$.

It is well known that if A is an accretive mapping, then the solutions of the problem $0 \in Ax$ correspond to the equilibrium points of some evolution equations. Hence, the problem of finding a solution $x \in E$ with $0 \in Ax$ has been studied by many researchers (see [7–15] and the references therein).

One classical method for studying the problem $0 \in Ax$, where A is an m-accretive mapping, is the following so-called proximal method (*cf.* [7]), presented in a Hilbert space:

$$x_0 \in H, \quad x_{n+1} \approx J_{r_n}^A x_n, \quad n \geq 0, \tag{1.2}$$

where $J_{r_n}^A := (I + r_n A)^{-1}$. It was shown that the sequence generated by (1.2) converges weakly or strongly to a zero point of A under some conditions.

An explicit iterative process to approximate fixed point of a nonexpansive mapping $T: C \to C$ was introduced in 1967 by Halpern [16] in the frame of Hilbert spaces:

$$u \in C, x_0 \in C, \quad x_{n+1} = \alpha_n u + (1 - \alpha_n) T x_n, \quad n \geq 0, \tag{1.3}$$

where $\{\alpha_n\} \subset [0, 1]$.

In 2007, based on (1.2) and (1.3), Qin and Su [9] presented the following iterative algorithm:

$$\begin{cases} x_1 \in C \quad \text{chosen arbitrarily}, \\ y_n = \beta_n x_n + (1 - \beta_n) J_{r_n}^A x_n, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n) y_n. \end{cases} \tag{1.4}$$

They showed that $\{x_n\}$ generated by (1.4) converges strongly to a zero point of an m-accretive mapping A.

Motivated by iterative algorithms (1.2) and (1.3), Zegeye and Shahzad extended their discussion to the case of finite m-accretive mappings $\{A_i\}_{i=1}^l$. They presented in [17] the following iterative algorithm:

$$x_0 \in C, \quad x_{n+1} = \alpha_n u + (1 - \alpha_n) S_r x_n, \quad n \geq 0, \tag{1.5}$$

where $S_r = a_0 I + a_1 J_{A_1} + a_2 J_{A_2} + \cdots + a_l J_{A_l}$ with $J_{A_i} = (I + A_i)^{-1}$ and $\sum_{i=0}^l a_i = 1$. If $\bigcap_{i=1}^l N(A_i) \neq \emptyset$, they proved that $\{x_n\}$ generated by (1.5) converges strongly to the common zero point of A_i $(i = 1, 2, \ldots, l)$ under some conditions.

The work in [17] was then extended to the following one presented by Hu and Liu in [18]:

$$x_0 \in C, \quad x_{n+1} = \alpha_n u + \beta_n x_n + \vartheta_n S_{r_n} x_n, \quad n \geq 0, \tag{1.6}$$

where $S_{r_n} = a_0 I + a_1 J_{r_n}^{A_1} + a_2 J_{r_n}^{A_2} + \cdots + a_l J_{r_n}^{A_l}$ with $J_{r_n}^{A_i} = (I + r_n A_i)^{-1}$ and $\sum_{i=0}^{l} a_i = 1$. $\{\alpha_n\}, \{\beta_n\}, \{\vartheta_n\} \subset (0,1)$ and $\alpha_n + \beta_n + \vartheta_n = 1$. If $\bigcap_{i=1}^{l} N(A_i) \neq \emptyset$, they proved that $\{x_n\}$ converges strongly to the common point in $N(A_i)$ $(i = 1, 2, \ldots, l)$ under some conditions.

In 2009, Yao *et al.* presented the following iterative algorithm in the frame of Hilbert space in [19]:

$$\begin{cases} x_1 \in C, \\ y_n = P_C[(1-\alpha_n)x_n], \\ x_{n+1} = (1-\beta_n)x_n + \beta_n T y_n, \quad n \geq 1, \end{cases} \tag{1.7}$$

where $T : C \to C$ is nonexpansive with $\mathrm{Fix}(T) \neq \emptyset$. Suppose $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences in $(0,1)$ satisfying

(a) $\sum_{n=1}^{\infty} \alpha_n = +\infty$ and $\lim_{n\to\infty} \alpha_n = 0$;
(b) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then $\{x_n\}$ constructed by (1.7) converges strongly to a point in $\mathrm{Fix}(T)$.

Motivated by the work in [17] and [19], Shehu and Ezeora [5] presented the following iterative algorithm and the discussion is undertaken in the frame of a real uniformly smooth and uniformly convex Banach space:

$$\begin{cases} x_1 \in C, \\ y_n = Q_C[(1-\alpha_n)x_n], \\ x_{n+1} = (1-\beta_n)x_n + \beta_n S_N y_n, \quad n \geq 1. \end{cases} \tag{1.8}$$

Here Q_C is the sunny nonexpansive retraction of E onto C. $A_i : C \to E$ is m-accretive mapping with $\bigcap_{i=1}^{N} N(A_i) \neq \emptyset$. $S_N := a_0 I + a_1 J_{A_1} + a_2 J_{A_2} + \cdots + a_N J_{A_N}$ with $J_{A_i} = (I + A_i)^{-1}$, for $i = 1, 2, \ldots, N$. $0 < a_k < 1$, for $k = 0, 1, 2, \ldots, N$, and $\sum_{k=0}^{N} a_k = 1$. $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$. Then $\{x_n\}$ converges strongly to the common zero point of A_i, where $i = 1, 2, \ldots, N$.

In 2014, by modifying iterative algorithm (1.8) and employing new techniques, Wei and Tan [20] presented and studied the following three-step iterative algorithm:

$$\begin{cases} x_1 \in C, \\ u_n = Q_C[(1-\alpha_n)(x_n + e_n)], \\ v_n = (1-\beta_n)x_n + \beta_n S_n u_n, \\ x_{n+1} = \gamma_n x_n + (1-\gamma_n) S_n v_n, \quad n \geq 1, \end{cases} \tag{1.9}$$

where Q_C is the sunny nonexpansive retraction of E onto C, $\{e_n\} \subset E$ is the error sequence and $\{A_i\}_{i=1}^{N}$ is a finite family of m-accretive mappings. $S_n := a_0 I + a_1 J_{r_{n,1}}^{A_1} + a_2 J_{r_{n,2}}^{A_2} + \cdots + a_N J_{r_{n,N}}^{A_N}$, $J_{r_{n,i}}^{A_i} = (I + r_{n,i} A_i)^{-1}$, for $i = 1, 2, \ldots, N$, $\sum_{k=0}^{N} a_k = 1$, $0 < a_k < 1$, for $k = 0, 1, 2, \ldots, N$. And, some strong convergence theorems to approximate common zero point of A_i $(i = 1, 2, \ldots, N)$ are obtained.

In 2015, Wang and Zhang [21] extended the discussion of the finite family of m-accretive mappings $\{A_i\}_{i=1}^{N}$ to that of infinite family of m-accretive mappings $\{A_i\}_{i=1}^{\infty}$. They presented the following two-step iterative algorithms with errors $\{e_n\} \subset E$:

$$\begin{cases} x_1 \in \bigcap_{i=1}^{\infty} \overline{D(A_i)} \quad \text{chosen arbitrarily}, \\ y_n = \alpha'_n x_n + \beta'_n \sum_{i=1}^{\infty} \delta_{n,i} J_{r_i} x_n + \gamma'_n e_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n, \quad n \geq 1, \end{cases} \tag{1.10}$$

where f is a contraction on $\bigcap_{i=1}^{\infty} \overline{D(A_i)}$. For $i = 1, 2, \ldots, J_{r_i} = (I + r_i A_i)^{-1}$. $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, and $\{\gamma'_n\}$ are real number sequences in $(0, 1)$. Some strong convergence theorems to approximate common zero point of A_i $(i = 1, 2, \ldots)$ are obtained under some conditions.

Inspired by the work in [5, 9, 17–21], we shall design a four-step iterative algorithm with errors in a Banach space in Section 3. Some weak and strong convergence theorems for approximating common zero point of an infinite family of m-accretive mappings are obtained. Some new proof techniques can be found. In Section 4, we shall present an example of infinite p-Laplacian-like differential systems, to highlight the significance of the studies on iterative construction for zero points of accretive mappings in applied mathematics and engineering. We demonstrate the applications of the main results in Section 3.

Our main contributions are:

(i) a new four-step iterative algorithm is designed by combining the ideas of famous iterative algorithms such as proximal methods, Halpern methods, convex combination methods, and viscosity methods;

(ii) three sequences constructed in the new iterative algorithm are proved to be weakly or strongly to the common zero point of an infinite family of m-accretive mappings;

(iii) the characteristic of the weakly convergent point of the new iterative algorithm is pointed out;

(iv) under the new assumptions, a path convergence theorem for nonexpansive mappings is proved;

(v) some new techniques are employed, for example, the tool $\|\cdot\|^2$ for estimating the convergence of the iterative sequence $\{x_n\}$ in most of the existing related work is partly replaced by function Φ defined by (1.1);

(vi) the discussion is undertaken in the frame of a Banach space, which is more general than that in Hilbert space; the assumption that 'the normalized duality mapping J is weakly sequentially continuous' in most of the existing related work is weakened to 'J_φ is weakly sequentially continuous for a given gauge function φ';

(vii) compared to (1.10), J_{r_i} is replaced by $J_{r_{n,i}}$; compared to (1.9), a contraction f is considered; compared to the work in [5, 9, 17–20], an infinite family of m-accretive mappings is discussed;

(viii) in Section 4, the applications of the main results in Section 3 on approximating the equilibrium solution of the nonlinear p-Laplacian-like differential systems are demonstrated.

2 Preliminaries

Now, we list some results we need in the sequel.

Lemma 2.1 (see [22]) *Let E be a real strictly convex Banach space and let C be a nonempty closed and convex subset of E. Let $T_m : C \to C$ be a nonexpansive mapping for each*

$m \geq 1$. *Let* $\{a_m\}$ *be a real number sequence in* $(0,1)$ *such that* $\sum_{m=1}^{\infty} a_m = 1$. *Suppose that* $\bigcap_{m=1}^{\infty} \mathrm{Fix}(T_m) \neq \emptyset$. *Then the mapping* $\sum_{m=1}^{\infty} a_m T_m$ *is nonexpansive with* $\mathrm{Fix}(\sum_{m=1}^{\infty} a_m T_m) = \bigcap_{m=1}^{\infty} \mathrm{Fix}(T_m)$.

Lemma 2.2 (see [23]) *Assume that a real Banach space E has a weakly continuous duality mapping* J_φ *with a gauge* φ. *Then* Φ *defined by* (1.1) *has the following properties.*

(i) $\Phi(\|x + y\|) \leq \Phi(\|x\|) + \langle y, J_\varphi(x + y)\rangle$, $\forall x, y \in E$.

(ii) *Assume that a sequence* $\{x_n\}$ *in E converges weakly to a point* $x \in E$. *Then*

$$\limsup_{n\to\infty} \Phi\big(\|x_n - y\|\big) = \limsup_{n\to\infty} \Phi\big(\|x_n - x\|\big) + \Phi\big(\|y - x\|\big), \quad \forall y \in E.$$

Lemma 2.3 (see [24]) *Let* $\{a_n\}$ *and* $\{c_n\}$ *be two sequences of nonnegative real numbers satisfying*

$$a_{n+1} \leq (1 - t_n)a_n + b_n + c_n, \quad \forall n \geq 1,$$

where $\{t_n\} \subset (0,1)$ *and* $\{b_n\}$ *is a number sequence. Assume that* (i) $\sum_{n=1}^{\infty} t_n = +\infty$, (ii) *either* $\limsup_{n\to\infty} \frac{b_n}{t_n} \leq 0$ *and* $\sum_{n=1}^{\infty} c_n < +\infty$. *Then* $\lim_{n\to\infty} a_n = 0$.

Lemma 2.4 (see [20]) *Let* $\{x_n\}$ *and* $\{y_n\}$ *be two bounded sequences in a Banach space E such that* $x_{n+1} = \beta_n x_n + (1 - \beta_n)y_n$, *for* $n \geq 1$. *Suppose* $\{\beta_n\} \subset (0,1)$ *satisfying* $0 < \liminf_{n\to+\infty} \beta_n \leq \limsup_{n\to+\infty} \beta_n < 1$. *If* $\limsup_{n\to+\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$, *then* $\lim_{n\to+\infty} \|y_n - x_n\| = 0$.

Lemma 2.5 (see [25]) *Let E be a real uniformly convex Banach space and let C be a nonempty, closed, and convex subset of E. Let* $T : C \to C$ *be a nonexpansive mapping such that* $\mathrm{Fix}(T) \neq \emptyset$, *then* $I - T$ *is demi-closed at zero.*

Lemma 2.6 (see [1]) *Let E be a Banach space and let A be an m-accretive mapping. For* $\lambda > 0$, $\mu > 0$, *and* $x \in E$, *we have*

$$J_\lambda x = J_\mu\left(\frac{\mu}{\lambda}x + \left(1 - \frac{\mu}{\lambda}\right)J_\lambda x\right),$$

where $J_\lambda = (I + \lambda A)^{-1}$ *and* $J_\mu = (I + \mu A)^{-1}$.

Lemma 2.7 (see [20]) *Let E be a real uniformly smooth and uniformly convex Banach space. Let C be a nonempty, closed, and convex sunny nonexpansive retract of E, and let* Q_C *be the sunny nonexpansive retraction of E onto C. Let* $T : C \to C$ *be a nonexpansive mapping with* $\mathrm{Fix}(T) \neq \emptyset$. *If for each* $t \in (0,1)$, *define* $T_t : C \to C$ *by*

$$T_t x := TQ_C\big[(1 - t)x\big].$$

Then T_t *is a contraction and has a fixed point* z_t, *which satisfies* $\|z_t - Tz_t\| \to 0$, *as* $t \to 0$.

3 Weak and strong convergence theorems

Lemma 3.1 *Let E be a real strictly convex Banach space and C be a nonempty, closed, and convex subset of E. Let* $A_i : C \to E$ *be m-accretive mappings, where* $i = 1, 2, \ldots$. *Suppose* $D :=$

$\bigcap_{i=1}^{\infty} N(A_i) \neq \emptyset$ and $\{r_{n,i}\} \subset (0,+\infty)$ for $i = 1,2,\ldots$. If $\{a_i\}_{i=0}^{\infty} \subset (0,1)$ satisfies $\sum_{i=0}^{\infty} a_i = 1$. Then $(a_0 I + \sum_{i=1}^{\infty} a_i J_{r_{n,i}}^{A_i}) : E \to E$ is nonexpansive and

$$\operatorname{Fix}\left(a_0 I + \sum_{i=1}^{\infty} a_i J_{r_{n,i}}^{A_i}\right) = D,$$

where $J_{r_{n,i}}^{A_i} = (I + r_{n,i}A_i)^{-1}$ for $n \geq 1$ and $i = 1,2,\ldots$.

Proof If we set $T_1 = I$ and $T_{i+1} = J_{r_{n,i}}^{A_i}$, then $(a_0 I + \sum_{i=1}^{\infty} a_i J_{r_{n,i}}^{A_i}) = \sum_{i=0}^{\infty} a_i T_{i+1}$. Since both I and $J_{r_{n,i}}^{A_i}$ are nonexpansive, Lemma 2.1 implies that $(a_0 I + \sum_{i=1}^{\infty} a_i J_{r_{n,i}}^{A_i})$ is nonexpansive and

$$\operatorname{Fix}\left(a_0 I + \sum_{i=1}^{\infty} a_i J_{r_{n,i}}^{A_i}\right) = \bigcap_{i=1}^{\infty} \operatorname{Fix}\left(J_{r_{n,i}}^{A_i}\right) = \bigcap_{i=1}^{\infty} N(A_i) = D.$$

This completes the proof. □

Theorem 3.1 *Let E be a real strictly convex Banach space which has a weakly continuous duality mapping J_φ. Let C be a nonempty, closed, and convex sunny nonexpansive retract of E, and Q_C be the sunny nonexpansive retraction of E onto C. Let $f : C \to C$ be a contraction with contractive constant $k \in (0,1)$. Let $A_i : C \to E$ be m-accretive mappings, for $i = 1,2,\ldots$. Let $D := \bigcap_{i=1}^{\infty} N(A_i) \neq \emptyset$. Suppose $\{\alpha_n\}, \{\beta_n\}, \{\mu_n\}, \{\gamma_n\}, \{\delta_n\}, \{\zeta_n\} \subset (0,1)$, and $\{r_{n,i}\} \subset (0,+\infty)$ for $i = 1,2,\ldots$. Suppose $\{a_i\}_{i=0}^{\infty} \subset (0,1)$ with $\sum_{i=0}^{\infty} a_i = 1$ and $\{e_n\} \subset E$ is the error sequence. Let $\{x_n\}$ be generated by the following iterative algorithm*:

$$\begin{cases} x_1 \in C, \\ u_n = Q_C[(1-\alpha_n)(x_n + e_n)], \\ v_n = (1-\beta_n)x_n + \beta_n(a_0 I + \sum_{i=1}^{\infty} a_i J_{r_{n,i}}^{A_i})u_n, \\ w_n = \mu_n f(x_n) + \gamma_n x_n + \delta_n v_n, \\ x_{n+1} = (1-\zeta_n)w_n + \zeta_n x_n, \quad n \geq 1. \end{cases} \tag{3.1}$$

Further suppose that the following conditions are satisfied:

(i) $\alpha_n \to 0$, $\delta_n \to 0$, $\mu_n \to 0$, *as* $n \to \infty$;
(ii) $\mu_n + \gamma_n + \delta_n \equiv 1$, $n \geq 1$;
(iii) $0 < \liminf_{n\to+\infty} \beta_n \leq \limsup_{n\to+\infty} \beta_n < 1$ *and*
$0 < \liminf_{n\to+\infty} \zeta_n \leq \limsup_{n\to+\infty} \zeta_n < 1$;
(iv) $\sum_{n=1}^{\infty} |r_{n+1,i} - r_{n,i}| < +\infty$ *and* $r_{n,i} \geq \varepsilon > 0$, *for* $n \geq 1$ *and* $i = 1,2,\ldots$;
(v) $\gamma_{n+1} - \gamma_n \to 0$, $\beta_{n+1} - \beta_n \to 0$, *as* $n \to \infty$;
(vi) $\sum_{n=1}^{\infty} \|e_n\| < +\infty$.

Then the three sequences $\{x_n\}$, $\{u_n\}$, and $\{w_n\}$ converge weakly to the unique element $q_0 \in D$, which satisfies, for $\forall y \in D$,

$$\limsup_{n\to\infty} \Phi\left(\|x_n - q_0\|\right) = \min_{y\in D} \limsup_{n\to\infty} \Phi\left(\|x_n - y\|\right). \tag{3.2}$$

Proof We shall split the proof into five steps.

Step 1. $\{x_n\}$, $\{u_n\}$, $\{\sum_{i=1}^{\infty} a_i J_{r_{n,i}}^{A_i} u_n\}$, $\{v_n\}$, and $\{f(x_n)\}$ are all bounded.

We shall first show that $\forall p \in D$,

$$\|x_{n+1} - p\| \le M_1 + \sum_{i=1}^{n} \|e_i\|, \tag{3.3}$$

where $M_1 = \max\{\|x_1 - p\|, \frac{\|f(p)-p\|}{1-k}, \|p\|\}$.

By using Lemma 3.1 and the induction method, we see that, for $n = 1$, $\forall p \in D$,

$$\begin{aligned}
\|x_2 - p\| &\le (1-\zeta_1)\|w_1 - p\| + \zeta_1\|x_1 - p\| \\
&\le (1-\zeta_1)\big[\mu_1\big\|f(x_1) - p\big\| + \gamma_1\|x_1 - p\| + \delta_1\|v_1 - p\|\big] + \zeta_1\|x_1 - p\| \\
&\le (1-\zeta_1)\Bigg[k\mu_1\|x_1 - p\| + \mu_1\big\|f(p) - p\big\| + \gamma_1\|x_1 - p\| \\
&\quad + \delta_1(1-\beta_1)\|x_1 - p\| + \delta_1\beta_1\Bigg\|\Bigg(a_0 I + \sum_{i=1}^{\infty} a_i J_{r_{1,i}}^{A_i}\Bigg)u_1 - p\Bigg\|\Bigg] + \zeta_1\|x_1 - p\| \\
&\le (1-\zeta_1)\big[k\mu_1 + \gamma_1 + \delta_1(1-\beta_1)\big]\|x_1 - p\| + (1-\zeta_1)\mu_1\big\|f(p) - p\big\| + \zeta_1\|x_1 - p\| \\
&\quad + (1-\zeta_1)\delta_1\beta_1\|u_1 - p\| \\
&\le (1-\zeta_1)\big[k\mu_1 + \gamma_1 + \delta_1(1-\beta_1)\big]\|x_1 - p\| + (1-\zeta_1)\mu_1\big\|f(p) - p\big\| + \zeta_1\|x_1 - p\| \\
&\quad + (1-\zeta_1)\delta_1\beta_1\big\|(1-\alpha_1)(x_1 + e_1) - p\big\| \\
&\le \big[1 - (1-k)\mu_1(1-\zeta_1) - \alpha_1\beta_1\delta_1(1-\zeta_1)\big]\|x_1 - p\| + (1-\zeta_1)(1-\alpha_1)\beta_1\delta_1\|e_1\| \\
&\quad + (1-\zeta_1)\alpha_1\beta_1\delta_1\|p\| + (1-\zeta_1)\mu_1(1-k)\frac{\|f(p) - p\|}{1-k} \\
&\le M_1 + \|e_1\|.
\end{aligned}$$

Suppose that (3.3) is true for $n = k$. Then, for $n = k + 1$,

$$\begin{aligned}
\|x_{k+2} - p\| &\le (1-\zeta_{k+1})\|w_{k+1} - p\| + \zeta_{k+1}\|x_{k+1} - p\| \\
&\le (1-\zeta_{k+1})\big[\mu_{k+1}\big\|f(x_{k+1}) - p\big\| + \gamma_{k+1}\|x_{k+1} - p\| + \delta_{k+1}\|v_{k+1} - p\|\big] \\
&\quad + \zeta_{k+1}\|x_{k+1} - p\| \\
&\le (1-\zeta_{k+1})\Bigg[k\mu_{k+1}\|x_{k+1} - p\| + \mu_{k+1}\big\|f(p) - p\big\| + \gamma_{k+1}\|x_{k+1} - p\| \\
&\quad + \delta_{k+1}(1-\beta_{k+1})\|x_{k+1} - p\| + \delta_{k+1}\beta_{k+1}\Bigg\|\Bigg(a_0 I + \sum_{i=1}^{\infty} a_i J_{r_{k+1,i}}^{A_i}\Bigg)u_{k+1} - p\Bigg\|\Bigg] \\
&\quad + \zeta_{k+1}\|x_{k+1} - p\| \\
&\le (1-\zeta_{k+1})\big[k\mu_{k+1} + \gamma_{k+1} + \delta_{k+1}(1-\beta_{k+1})\big]\|x_{k+1} - p\| \\
&\quad + (1-\zeta_{k+1})\mu_{k+1}\big\|f(p) - p\big\| + \zeta_{k+1}\|x_{k+1} - p\| + (1-\zeta_{k+1})\delta_{k+1}\beta_{k+1}\|u_{k+1} - p\| \\
&\le \big\{(1-\zeta_{k+1})\big[k\mu_{k+1} + \gamma_{k+1} + \delta_{k+1}(1-\beta_{k+1})\big] + \zeta_{k+1}\big\}\|x_{k+1} - p\| \\
&\quad + (1-\zeta_{k+1})\mu_{k+1}\big\|f(p) - p\big\| \\
&\quad + (1-\zeta_{k+1})\delta_{k+1}\beta_{k+1}\big\|(1-\alpha_{k+1})(x_{k+1} + e_{k+1}) - p\big\|
\end{aligned}$$

$$\begin{aligned}
&\le \big[1-(1-k)\mu_{k+1}(1-\zeta_{k+1})-\alpha_{k+1}\beta_{k+1}\delta_{k+1}(1-\zeta_{k+1})\big]\|x_{k+1}-p\|+\|e_{k+1}\| \\
&\quad + (1-\zeta_{k+1})\alpha_{k+1}\beta_{k+1}\delta_{k+1}\|p\| + (1-\zeta_{k+1})\mu_{k+1}(1-k)\frac{\|f(p)-p\|}{1-k} \\
&\le M_1 + \sum_{i=1}^{k+1}\|e_i\|.
\end{aligned}$$

Thus (3.3) is true for all $n \in N^+$. Since $\sum_{n=1}^{\infty}\|e_n\| < +\infty$, (3.3) ensures that $\{x_n\}$ is bounded.

For $\forall p \in D$, from $\|u_n - p\| \le \|(1-\alpha_n)(x_n+e_n)-p\| \le \|x_n\|+\|e_n\|+\|p\|$, we see that $\{u_n\}$ is bounded.

Since $\|\sum_{i=1}^{\infty} a_i J_{r_{n,i}}^{A_i} u_n\| \le \|\sum_{i=1}^{\infty} a_i J_{r_{n,i}}^{A_i} u_n - \sum_{i=1}^{\infty} a_i J_{r_{n,i}}^{A_i} p\| + (1-a_0)\|p\| \le \|u_n - p\| + \|p\|$, $\{\sum_{i=1}^{\infty} a_i J_{r_{n,i}}^{A_i} u_n\}$ is bounded. Since both $\{\sum_{i=1}^{\infty} a_i J_{r_{n,i}}^{A_i} u_n\}$ and $\{x_n\}$ are bounded, $\{v_n\}$ is bounded. From the definition of a contraction, we can easily see that $\{f(x_n)\}$ is bounded.

Set $M_2 = \sup\{\|u_n\|, \|J_{r_{n,i}}^{A_i} u_n\|, \|\sum_{i=1}^{\infty} a_i J_{r_{n,i}}^{A_i} u_n\|, \|f(x_n)\|, \|v_n\|, \|x_n\|, : n \ge 1, i \ge 1\}$.

Step 2. $\lim_{n\to\infty}\|x_n - w_n\| = 0$ and $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$.

In fact, if $r_{n,i} \le r_{n+1,i}$, then, using Lemma 2.6,

$$\begin{aligned}
&\left\|\sum_{i=1}^{\infty} a_i J_{r_{n+1,i}}^{A_i} u_{n+1} - \sum_{i=1}^{\infty} a_i J_{r_{n,i}}^{A_i} u_n\right\| \\
&\le \sum_{i=1}^{\infty} a_i \left\|J_{r_{n+1,i}}^{A_i} u_{n+1} - J_{r_{n,i}}^{A_i} u_n\right\| \\
&= \sum_{i=1}^{\infty} a_i \left\|J_{r_{n,i}}^{A_i}\left(\frac{r_{n,i}}{r_{n+1,i}}u_{n+1} + \left(1-\frac{r_{n,i}}{r_{n+1,i}}\right)J_{r_{n+1,i}}^{A_i} u_{n+1}\right) - J_{r_{n,i}}^{A_i} u_n\right\| \\
&\le \sum_{i=1}^{\infty} a_i \left\|\frac{r_{n,i}}{r_{n+1,i}}u_{n+1} + \left(1-\frac{r_{n,i}}{r_{n+1,i}}\right)J_{r_{n+1,i}}^{A_i} u_{n+1} - u_n\right\| \\
&\le (1-a_0)\|u_{n+1}-u_n\| + \sum_{i=1}^{\infty} a_i\left(1-\frac{r_{n,i}}{r_{n+1,i}}\right)\left\|J_{r_{n+1,i}}^{A_i} u_{n+1} - u_n\right\| \\
&\le (1-a_0)\|u_{n+1}-u_n\| + \frac{2M_2}{\varepsilon}\sum_{i=1}^{\infty}(r_{n+1,i}-r_{n,i}).
\end{aligned} \tag{3.4}$$

If $r_{n+1,i} \le r_{n,i}$, then imitating the proof of (3.4), we have

$$\begin{aligned}
&\left\|\sum_{i=1}^{\infty} a_i J_{r_{n+1,i}}^{A_i} u_{n+1} - \sum_{i=1}^{\infty} a_i J_{r_{n,i}}^{A_i} u_n\right\| \\
&\le (1-a_0)\|u_{n+1}-u_n\| + \frac{2M_2}{\varepsilon}\sum_{i=1}^{\infty}(r_{n,i}-r_{n+1,i}).
\end{aligned} \tag{3.5}$$

Combining (3.4) and (3.5), we have

$$\begin{aligned}
&\left\|\sum_{i=1}^{\infty} a_i J_{r_{n+1,i}}^{A_i} u_{n+1} - \sum_{i=1}^{\infty} a_i J_{r_{n,i}}^{A_i} u_n\right\| \\
&\le (1-a_0)\|u_{n+1}-u_n\| + \frac{2M_2}{\varepsilon}\sum_{i=1}^{\infty}|r_{n,i}-r_{n+1,i}|.
\end{aligned} \tag{3.6}$$

On the other hand,

$$\|u_{n+1}-u_n\| \le \|x_{n+1}-x_n\| + \alpha_{n+1}\|x_{n+1}\| + \alpha_n\|x_n\| + \|\alpha_{n+1}e_{n+1}-\alpha_n e_n\| + \|e_{n+1}-e_n\|. \tag{3.7}$$

In view of (3.6) and (3.7), we have

$$\begin{aligned}
\|v_{n+1}-v_n\| &\le (1-\beta_{n+1})\|x_{n+1}-x_n\| + a_0\beta_{n+1}\|u_{n+1}-u_n\| \\
&\quad + \beta_{n+1}\left\|\sum_{i=1}^{\infty} a_i J^{A_i}_{r_{n+1,i}} u_{n+1} - \sum_{i=1}^{\infty} a_i J^{A_i}_{r_{n,i}} u_n\right\| \\
&\quad + |\beta_{n+1}-\beta_n|\|x_n\| + a_0|\beta_{n+1}-\beta_n|\|u_n\| + |\beta_{n+1}-\beta_n|\left\|\sum_{i=1}^{\infty} a_i J^{A_i}_{r_{n,i}} u_n\right\| \\
&\le (1-\beta_{n+1})\|x_{n+1}-x_n\| + a_0\beta_{n+1}\big[\|x_{n+1}-x_n\| + \alpha_{n+1}\|x_{n+1}\| \\
&\quad + \alpha_n\|x_n\| + \|e_{n+1}-e_n\| + \|\alpha_{n+1}e_{n+1}-\alpha_n e_n\|\big] \\
&\quad + \beta_{n+1}\left[(1-a_0)\|u_{n+1}-u_n\| + \frac{2M_2}{\varepsilon}\sum_{i=1}^{\infty}|r_{n,i}-r_{n+1,i}|\right] \\
&\quad + |\beta_{n+1}-\beta_n|\|x_n\| + a_0|\beta_{n+1}-\beta_n|\|u_n\| + |\beta_{n+1}-\beta_n|\left\|\sum_{i=1}^{\infty} a_i J^{A_i}_{r_{n,i}} u_n\right\| \\
&\le \|x_{n+1}-x_n\| + \|x_n\| + \|x_{n+1}\| + \|\alpha_{n+1}e_{n+1}-\alpha_n e_n\| + \|e_{n+1}-e_n\| \\
&\quad + |\beta_{n+1}-\beta_n|\left\|\sum_{i=1}^{\infty} a_i J^{A_i}_{r_{n,i}} u_n\right\| + |\beta_{n+1}-\beta_n|\|x_n\| + a_0|\beta_{n+1}-\beta_n|\|u_n\| \\
&\quad + \frac{2M_2}{\varepsilon}\beta_{n+1}\sum_{i=1}^{\infty}|r_{n,i}-r_{n+1,i}|.
\end{aligned} \tag{3.8}$$

Thus in view of (3.8), we have

$$\begin{aligned}
\|w_{n+1}-w_n\| &\le (k\mu_{n+1}+\gamma_{n+1}+\delta_{n+1})\|x_{n+1}-x_n\| + |\mu_{n+1}-\mu_n|\|f(x_n)\| \\
&\quad + |\gamma_{n+1}-\gamma_n|\|x_n\| + \delta_{n+1}\|x_n\| + \delta_{n+1}\|x_{n+1}\| + |\delta_{n+1}-\delta_n|\|v_n\| \\
&\quad + \delta_{n+1}\|e_{n+1}-e_n\| + \delta_{n+1}\|\alpha_{n+1}e_{n+1}-\alpha_n e_n\| + \delta_{n+1}|\beta_{n+1}-\beta_n|\|x_n\| \\
&\quad + \delta_{n+1}|\beta_{n+1}-\beta_n|\left\|\sum_{i=1}^{\infty} a_i J^{A_i}_{r_{n,i}} u_n\right\| + \delta_{n+1}a_0|\beta_{n+1}-\beta_n|\|u_n\| \\
&\quad + \delta_{n+1}\beta_{n+1}\frac{2M_2}{\varepsilon}\sum_{i=1}^{\infty}|r_{n,i}-r_{n+1,i}| \\
&\le \big(1-(1-k)\mu_{n+1}\big)\|x_{n+1}-x_n\| \\
&\quad + 2M_2\delta_{n+1} + M_2\big(|\mu_{n+1}-\mu_n| + |\gamma_{n+1}-\gamma_n| + |\delta_{n+1}-\delta_n| + 3|\beta_{n+1}-\beta_n|\big) \\
&\quad + \|e_{n+1}-e_n\| + \|\alpha_{n+1}e_{n+1}-\alpha_n e_n\| \\
&\quad + \delta_{n+1}\beta_{n+1}\frac{2M_2}{\varepsilon}\sum_{i=1}^{\infty}|r_{n,i}-r_{n+1,i}|.
\end{aligned} \tag{3.9}$$

Using Lemma 2.4, we have from (3.9) $\lim_{n\to\infty}\|x_n - w_n\| = 0$ and then $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$.

Step 3. $\lim_{n\to\infty}\|x_n - u_n\| = 0$ and $\lim_{n\to\infty}\|u_n - (a_0 I + \sum_{i=1}^{\infty} a_i J_{r_{n,i}}^{A_i})u_n\| = 0$.

We compute the following:

$$\begin{aligned}\|v_{n+1} - v_n\| \le{}& (1-\beta_{n+1})\|x_{n+1} - x_n\| + |\beta_{n+1} - \beta_n|\|x_n\| \\ &+ \beta_{n+1}\left\|\left(a_0 I + \sum_{i=1}^{\infty} a_i J_{r_{n+1,i}}^{A_i}\right)u_{n+1} - \left(a_0 I + \sum_{i=1}^{\infty} a_i J_{r_{n,i}}^{A_i}\right)u_n\right\| \\ &+ |\beta_{n+1} - \beta_n|\left\|\left(a_0 I + \sum_{i=1}^{\infty} a_i J_{r_{n,i}}^{A_i}\right)u_n\right\| \\ \le{}& (1-\beta_{n+1})\|x_{n+1} - x_n\| + 2|\beta_{n+1} - \beta_n|M_2 \\ &+ \left\|\left(a_0 I + \sum_{i=1}^{\infty} a_i J_{r_{n+1,i}}^{A_i}\right)u_{n+1} - \left(a_0 I + \sum_{i=1}^{\infty} a_i J_{r_{n,i}}^{A_i}\right)u_n\right\|.\end{aligned}$$

Using the result of Step 2 and Lemma 2.4, we have

$$\lim_{n\to\infty}\left\|\left(a_0 I + \sum_{i=1}^{\infty} a_i J_{r_{n,i}}^{A_i}\right)u_n - v_n\right\| = 0. \tag{3.10}$$

Since

$$\left\|\left(a_0 I + \sum_{i=1}^{\infty} a_i J_{r_{n,i}}^{A_i}\right)u_n - v_n\right\| = (1-\beta_n)\left\|x_n - \left(a_0 I + \sum_{i=1}^{\infty} a_i J_{r_{n,i}}^{A_i}\right)u_n\right\|$$

and $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$, (3.10) implies that

$$\lim_{n\to\infty}\left\|x_n - \left(a_0 I + \sum_{i=1}^{\infty} a_i J_{r_{n,i}}^{A_i}\right)u_n\right\| = 0. \tag{3.11}$$

Moreover,

$$\|u_n - x_n\| = \left\|Q_C\left[(1-\alpha_n)(x_n + e_n)\right] - Q_C x_n\right\| \le \alpha_n\|x_n\| + (1-\alpha_n)\|e_n\|,$$

then

$$\lim_{n\to\infty}\|x_n - u_n\| = 0. \tag{3.12}$$

Noticing (3.11) and (3.12), we have

$$\lim_{n\to\infty}\left\|u_n - \left(a_0 I + \sum_{i=1}^{\infty} a_i J_{r_{n,i}}^{A_i}\right)u_n\right\| = 0. \tag{3.13}$$

Step 4. $\omega(x_n) \subset D$, where $\omega(x_n)$ is the set of all of the weak limit points of $\{x_n\}$.

Since $\{x_n\}$ is bounded, there exists a subsequence of $\{x_n\}$, which is denoted by $\{x_{n_k}\}$, such that $x_{n_k} \rightharpoonup q_0$, as $k \to \infty$. From (3.12), we have $u_{n_k} \rightharpoonup q_0$, as $k \to \infty$. Then Lemma 2.2(ii) implies that

$$\limsup_{k\to\infty} \Phi\big(\|u_{n_k} - x\|\big) = \limsup_{k\to\infty} \Phi\big(\|u_{n_k} - q_0\|\big) + \Phi\big(\|q_0 - x\|\big), \quad \forall x \in E. \tag{3.14}$$

Lemma 3.1, Lemma 2.2(i), and (3.13) imply that

$$\begin{aligned}
&\limsup_{k\to\infty} \Phi\left(\left\| u_{n_k} - \left(a_0 I + \sum_{i=1}^{\infty} a_i J_{r_{n_k,i}}^{A_i}\right) q_0 \right\|\right) \\
&\quad \le \limsup_{k\to\infty} \Phi\left(\left\| \sum_{i=1}^{\infty} a_i J_{r_{n_k,i}}^{A_i} u_{n_k} - \left(a_0 I + \sum_{i=1}^{\infty} a_i J_{r_{n_k,i}}^{A_i}\right) q_0 + a_0 u_{n_k} \right\|\right) \\
&\qquad + \limsup_{k\to\infty} \left\langle (1-a_0) u_{n_k} - \sum_{i=1}^{\infty} a_i J_{r_{n_k,i}}^{A_i} u_{n_k}, J_\varphi\left(u_{n_k} - \left(a_0 I + \sum_{i=1}^{\infty} a_i J_{r_{n_k,i}}^{A_i}\right) q_0 \right) \right\rangle \\
&\quad \le \limsup_{k\to\infty} \Phi\big(\|u_{n_k} - q_0\|\big).
\end{aligned} \tag{3.15}$$

From (3.14) and (3.15), we know that $\Phi(\|(a_0 I + \sum_{i=1}^{\infty} a_i J_{r_{n_k,i}}^{A_i}) q_0 - q_0\|) \le 0$, which implies that

$$a_0 q_0 + \sum_{i=1}^{\infty} a_i J_{r_{n_k,i}}^{A_i} q_0 = q_0.$$

Then Lemma 3.1 ensures that $q_0 \in D$.

Step 5. $x_n \rightharpoonup q_0$, as $n \to \infty$, where $q_0 \in D$ is the unique element which satisfies (3.2).

Now, we define $h(y) = \limsup_{n\to\infty} \Phi(\|x_n - y\|)$, for $y \in D$. Then $h(y) : D \to R^+$ is proper, strictly convex, lower-semi-continuous, and $h(y) \to +\infty$, as $\|y\| \to +\infty$. Therefore, there exists a unique element $q_0 \in D$ such that $h(q_0) = \min_{y\in D} h(y)$. That is, q_0 satisfies (3.2).

Next, we shall show that $x_n \rightharpoonup q_0$, as $n \to \infty$.

In fact, suppose there exists a subsequence $\{x_{n_m}\}$ of $\{x_n\}$ (for simplicity, we still denote it by $\{x_n\}$) such that $x_n \rightharpoonup v_0$, as $n \to \infty$, then $v_0 \in D$ in view of Step 4. Using Lemma 2.2(ii), $h(q_0) = h(v_0) + \Phi(\|v_0 - q_0\|)$, which implies that $\Phi(\|v_0 - q_0\|) = 0$, and then $v_0 = q_0$.

If there exists another subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup p_0$, as $k \to \infty$. Then repeating the above process, we have $p_0 = q_0$.

Since all of the weakly convergent subsequences of $\{x_n\}$ converge to the same element q_0, the whole sequence $\{x_n\}$ converges weakly to q_0. Combining the results of Steps 2 and 3, $u_n \rightharpoonup q_0$, $w_n \rightharpoonup q_0$, as $n \to \infty$.

This completes the proof. □

Remark 3.1 Compared to the work in [20], the smoothness of E is not needed. The iterative algorithm (3.1) is more general than those discussed in [5, 9, 17–21].

Remark 3.2 The properties of the function Φ defined by (1.1) are widely used in the proof of Steps 4 and 5, which can be regarded as a new proof technique compared to the existing work.

Remark 3.3 Three sequences are proved to be weakly convergent to the common zero point of an infinite family of m-accretive mappings. The characteristic of the weakly convergent point q_0 of $\{x_n\}$ is presented in Theorem 3.1.

Remark 3.4 The assumptions imposed on the real number sequences in Theorem 3.1 are reasonable if we take $\alpha_n = \frac{1}{n}$, $\mu_n = \frac{n}{n^2}$, $\gamma_n = \frac{n^2-n-1}{n^2}$, $\delta_n = \frac{1}{n^2}$, and $\beta_n = \zeta_n = \frac{n+1}{2n}$, for $n \geq 1$.

Lemma 3.2 *Let E be a real uniformly smooth and uniformly convex Banach space. Let C be a nonempty, closed, and convex sunny nonexpansive retract of E, and Q_C be the sunny nonexpansive retraction of E onto C. Let $T : C \to C$ be a nonexpansive mapping with* $\operatorname{Fix}(T) \neq \emptyset$. *Let $T_t : C \to C$ be defined by $T_t x := TQ_C[(1-t)x]$, $x \in C$. Then*:

(i) *T_t is a contraction and has a fixed point z_t, which satisfies $\|z_t - Tz_t\| \to 0$, as $t \to 0$;*

(ii) *further suppose that E has a weakly continuous duality mapping J_φ, then* $\lim_{t\to 0} z_t = z_0 \in \operatorname{Fix}(T)$.

Proof Lemma 2.7 ensures the result of (i).

To show that (ii) holds, it suffices to show that, for any sequence $\{t_n\}$ such that $t_n \to 0$, we have $\lim_{n\to\infty} z_{t_n} = z_0 \in \operatorname{Fix}(T)$.

In fact, the result of (i) implies that there exists $z_t \in \operatorname{Fix}(T)$ such that $z_t = TQ_C[(1-t)z_t]$, $t \in (0,1)$. For $\forall p \in \operatorname{Fix}(T)$, since

$$\|z_t - p\| = \bigl\|TQ_C\bigl[(1-t)z_t\bigr] - TQ_C p\bigr\| \leq \|z_t - p - tz_t\| \leq (1-t)\|z_t - p\| + t\|p\|,$$

$\{z_t\}$ is bounded. Without loss of generality, we may assume that $z_{t_n} \rightharpoonup z_0$. Using (i) and Lemma 2.5, we have $z_0 \in \operatorname{Fix}(T)$.

Using Lemma 2.2, we have, for $\forall p \in \operatorname{Fix}(T)$,

$$\begin{aligned}\Phi\bigl(\|z_t - p\|\bigr) &= \Phi\bigl(\bigl\|TQ_C\bigl[(1-t)z_t\bigr] - TQ_C p\bigr\|\bigr)\\ &\leq \Phi\bigl(\bigl\|(1-t)z_t - p\bigr\|\bigr)\\ &\leq \Phi\bigl(\|z_t - p\|\bigr) - t\bigl\langle z_t, J_\varphi(z_t - p - tz_t)\bigr\rangle\\ &= \Phi\bigl(\|z_t - p\|\bigr) - t\bigl\langle z_t - p - tz_t, J_\varphi(z_t - p - tz_t)\bigr\rangle - t\bigl\langle p + tz_t, J_\varphi(z_t - p - tz_t)\bigr\rangle,\end{aligned}$$

which implies that

$$\|z_t - p - tz_t\|\varphi\bigl(\|z_t - p - tz_t\|\bigr) \leq \bigl\langle p, J_\varphi(p + tz_t - z_t)\bigr\rangle + t\bigl\langle z_t, J_\varphi(p + tz_t - z_t)\bigr\rangle.$$

In particular,

$$\begin{aligned}&\|z_{t_n} - p - t_n z_{t_n}\|\varphi\bigl(\|z_{t_n} - p - t_n z_{t_n}\|\bigr)\\ &\quad\leq \bigl\langle p, J_\varphi(p + t_n z_{t_n} - z_{t_n})\bigr\rangle + t_n\bigl\langle z_{t_n}, J_\varphi(p + t_n z_{t_n} - z_{t_n})\bigr\rangle.\end{aligned}$$

Thus

$$\begin{aligned}&\|z_{t_n} - z_0 - t_n z_{t_n}\|\varphi\bigl(\|z_{t_n} - z_0 - t_n z_{t_n}\|\bigr)\\ &\quad\leq \bigl\langle z_0, J_\varphi(z_0 + t_n z_{t_n} - z_{t_n})\bigr\rangle + t_n\bigl\langle z_{t_n}, J_\varphi(z_0 + t_n z_{t_n} - z_{t_n})\bigr\rangle.\end{aligned}\tag{3.16}$$

Since E has a weakly continuous duality mapping J_φ, (3.16) implies that $z_{t_n} - z_0 - t_n z_{t_n} \to 0$, as $n \to \infty$.

From $\|z_{t_n} - z_0\| \leq \|z_{t_n} - z_0 - t_n z_{t_n}\| + t_n\|z_{t_n}\|$, we see that $z_{t_n} \to z_0$, as $n \to \infty$.

Suppose there exists another sequence $z_{t_m} \rightharpoonup x_0$, as $t_m \to 0$ and $m \to \infty$. Then from (i) $\|z_{t_m} - Tz_{t_m}\| \to 0$ and $I - T$ being demi-closed at zero, we have $x_0 \in \operatorname{Fix}(T)$. Moreover, repeating the above proof, we have $z_{t_m} \to x_0$, as $m \to \infty$. Next, we want to show that $z_0 = x_0$.

Similar to (3.16), we have

$$\begin{aligned}&\|z_{t_m} - z_0 - t_m z_{t_m}\|\varphi\bigl(\|z_{t_m} - z_0 - t_m z_{t_m}\|\bigr)\\&\quad \le \bigl\langle z_0, J_\varphi(z_0 + t_m z_{t_m} - z_{t_m})\bigr\rangle + t_m\bigl\langle z_{t_m}, J_\varphi(z_0 + t_m z_{t_m} - z_{t_m})\bigr\rangle.\end{aligned}$$

Letting $m \to \infty$,

$$\|x_0 - z_0\|\varphi\bigl(\|x_0 - z_0\|\bigr) \le \bigl\langle z_0, J_\varphi(z_0 - x_0)\bigr\rangle. \tag{3.17}$$

Interchanging x_0 and z_0 in (3.17), we obtain

$$\|z_0 - x_0\|\varphi\bigl(\|z_0 - x_0\|\bigr) \le \bigl\langle x_0, J_\varphi(x_0 - z_0)\bigr\rangle. \tag{3.18}$$

Then (3.17) and (3.18) ensure

$$2\|x_0 - z_0\|\varphi\bigl(\|x_0 - z_0\|\bigr) \le \bigl\langle x_0 - z_0, J_\varphi(x_0 - z_0)\bigr\rangle = \|x_0 - z_0\|\varphi\bigl(\|x_0 - z_0\|\bigr),$$

which implies that $x_0 = z_0$.

Therefore, $\lim_{t\to 0} z_t = z_0 \in \operatorname{Fix}(T)$.

This completes the proof. □

Remark 3.5 Compared to the proof of Lemma 2.7 in [20], a different method is used in the proof of the result (ii) in Lemma 3.2.

Theorem 3.2 *Further suppose that E is real uniformly convex and uniformly smooth and* (vii) $\sum_{n=1}^{\infty}(1-\zeta_n)\alpha_n\beta_n\delta_n = +\infty$; (viii) $\mu_n = o(\alpha_n\beta_n\delta_n)$. *The other restrictions are the same as those in Theorem* 3.1, *then the iterative sequence* $\{x_n\}$ *generated by* (3.1) *converges strongly to* $p_0 \in D$.

Proof We shall split the proof into six steps. The proofs of Steps 1, 2, and 3 are the same as those in Theorem 3.1.

Step 4. $\limsup_{n\to+\infty}\langle p_0, J_\varphi(p_0 - x_n)\rangle \le 0$, where p_0 is an element in D.

From Lemma 3.2, we know that there exists $y_t \in C$ such that

$$y_t = a_0 Q_C\bigl[(1-t)y_t\bigr] + \sum_{i=1}^{\infty} a_i J_{r_{n,i}}^{A_i} Q_C\bigl[(1-t)y_t\bigr]$$

for $t \in (0,1)$. Moreover, $y_t \to p_0 \in D$, as $t \to 0$.

Since $\|y_t\| \le \|y_t - p_0\| + \|p_0\|$, $\{y_t\}$ is bounded. Using Lemma 2.2, we have

$$\begin{aligned}\Phi\bigl(\|y_t - x_n\|\bigr) &\le \Phi\bigl(\bigl\|(1-t)y_t - x_n\bigr\|\bigr)\\&\quad + \biggl\langle\biggl(a_0 I + \sum_{i=1}^{\infty} a_i J_{r_{n,i}}^{A_i}\biggr)x_n - x_n, J_\varphi(y_t - x_n)\biggr\rangle\end{aligned}$$

$$\leq \Phi\big(\|y_t - x_n\|\big) - t\big\langle y_t, J_\varphi\big[(1-t)y_t - x_n\big]\big\rangle + K_1\left\|\left(a_0 I + \sum_{i=1}^{\infty} a_i J_{r_{n,i}}^{A_i}\right)x_n - x_n\right\|,$$

where $K_1 = \sup\{\varphi(\|y_t - x_n\|) : n \geq 1, t > 0\}$ and from the result of Step 1 we know that K_1 is a positive constant.

Thus $\langle y_t, J_\varphi[(1-t)y_t - x_n]\rangle \leq \frac{K_1}{t}\|(a_0 I + \sum_{i=1}^{\infty} a_i J_{r_{n,i}}^{A_i})x_n - x_n\|$. Noticing (3.12) and (3.13), we have

$$\begin{aligned}
&\left\|\left(a_0 I + \sum_{i=1}^{\infty} a_i J_{r_{n,i}}^{A_i}\right)x_n - x_n\right\| \\
&\quad\leq \left\|\left(a_0 I + \sum_{i=1}^{\infty} a_i J_{r_{n,i}}^{A_i}\right)x_n - \left(a_0 I + \sum_{i=1}^{\infty} a_i J_{r_{n,i}}^{A_i}\right)u_n\right\| \\
&\qquad + \left\|\left(a_0 I + \sum_{i=1}^{\infty} a_i J_{r_{n,i}}^{A_i}\right)u_n - u_n\right\| + \|u_n - x_n\| \\
&\quad\leq 2\|x_n - u_n\| + \left\|\left(a_0 I + \sum_{i=1}^{\infty} a_i J_{r_{n,i}}^{A_i}\right)u_n - u_n\right\| \to 0,
\end{aligned}$$

as $n \to \infty$. Therefore,

$$\lim_{t\to 0}\limsup_{n\to+\infty}\big\langle y_t, J_\varphi\big[(1-t)y_t - x_n\big]\big\rangle \leq 0.$$

From the assumption on J_φ and the following fact:

$$\begin{aligned}
\big\langle p_0, J_\varphi(p_0 - x_n)\big\rangle &= \big\langle p_0, J_\varphi(p_0 - x_n) - J_\varphi\big[(1-t)y_t - x_n\big]\big\rangle \\
&\quad + \big\langle p_0 - y_t, J_\varphi\big[(1-t)y_t - x_n\big]\big\rangle + \big\langle y_t, J_\varphi\big[(1-t)y_t - x_n\big]\big\rangle,
\end{aligned}$$

we have $\limsup_{n\to+\infty}\langle p_0, J_\varphi(p_0 - x_n)\rangle \leq 0$.

Then

$$\begin{aligned}
&\limsup_{n\to\infty}\big\langle p_0, J_\varphi\big[p_0 - x_n - (1-\alpha_n)e_n + \alpha_n x_n\big]\big\rangle \\
&\quad\leq \limsup_{n\to\infty}\big\langle p_0, J_\varphi\big[p_0 - x_n - (1-\alpha_n)e_n + \alpha_n x_n\big] - J_\varphi(p_0 - x_n)\big\rangle + \limsup_{n\to\infty}\big\langle p_0, J_\varphi(p_0 - x_n)\big\rangle \\
&\quad= \limsup_{n\to\infty}\big\langle p_0, J_\varphi(p_0 - x_n)\big\rangle \leq 0.
\end{aligned}\tag{3.19}$$

Step 5. Φ defined by (1.1) satisfies $\Phi(kt) \leq k\Phi(t)$, for $t \in [0,+\infty)$, where $k \in [0,1]$.

In fact, let $F(t) = \int_0^{kt}\varphi(s)\,ds - k\int_0^t \varphi(s)\,ds$, for $t \in [0,+\infty)$. Then $F'(t) = k\varphi(kt) - k\varphi(t) \leq 0$, since $k \in (0,1)$ and the gauge function φ is increasing. Thus F is decreasing on $[0,+\infty)$. That is, $F(t) \leq F(0)$, for $t \in [0,+\infty)$. Then $\int_0^{kt}\varphi(s)\,ds \leq k\int_0^t \varphi(s)\,ds$, for $t \in [0,+\infty)$, which implies that $\Phi(kt) \leq k\Phi(t)$, for $t \in [0,+\infty)$.

Step 6. $x_n \to p_0$, as $n \to +\infty$, where $p_0 \in D$ is the same as that in Step 4.

Since Φ is convex, we have

$$\Phi\big(\|x_{n+1} - p_0\|\big) \leq (1-\zeta_n)\Phi\big(\|w_n - p_0\|\big) + \zeta_n\Phi\big(\|x_n - p_0\|\big).\tag{3.20}$$

Using Lemma 2.2 and the result of Step 5, we have

$$\begin{aligned}\Phi\big(\|w_n - p_0\|\big) &\le \mu_n \Phi\big(\big\|f(x_n) - p_0\big\|\big) + \gamma_n \Phi\big(\|x_n - p_0\|\big) + \delta_n \Phi\big(\|v_n - p_0\|\big) \\ &\le (\mu_n k + \gamma_n)\Phi\big(\|x_n - p_0\|\big) + \mu_n \big\langle f(p_0) - p_0, J_\varphi\big(f(x_n) - p_0\big)\big\rangle \\ &\quad + \delta_n \Phi\big(\|v_n - p_0\|\big). \end{aligned} \tag{3.21}$$

Similarly, we have

$$\begin{aligned}\Phi\big(\|v_n - p_0\|\big) &\le (1-\beta_n)\Phi\big(\|x_n - p_0\|\big) + \beta_n \Phi\big(\|u_n - p_0\|\big) \\ &\le (1-\beta_n)\Phi\big(\|x_n - p_0\|\big) + \beta_n(1-\alpha_n)\Phi\big(\|x_n - p_0\|\big) \\ &\quad + \beta_n\big\langle (1-\alpha_n)e_n - \alpha_n p_0, J_\varphi\big((1-\alpha_n)(x_n + e_n) - p_0\big)\big\rangle \\ &= (1-\beta_n\alpha_n)\Phi\big(\|x_n - p_0\|\big) \\ &\quad + \beta_n(1-\alpha_n)\big\langle e_n, J_\varphi\big((1-\alpha_n)(x_n + e_n) - p_0\big)\big\rangle \\ &\quad + \beta_n\alpha_n\big\langle p_0, J_\varphi\big(p_0 - x_n - (1-\alpha_n)e_n + \alpha_n x_n\big)\big\rangle. \end{aligned} \tag{3.22}$$

Let $K_2 = \sup\{\varphi(\|(1-\alpha_n)(x_n + e_n) - p_0\|), \varphi(\|f(x_n) - p_0\|) : n \ge 1\}$. Then K_2 is a positive constant. Using (3.20)-(3.22), we have

$$\begin{aligned}\Phi\big(\|x_{n+1} - p_0\|\big) &\le \big\{1 - (1-\zeta_n)\big[\mu_n(1-k) + \alpha_n\beta_n\delta_n\big]\big\}\Phi\big(\|x_n - p_0\|\big) \\ &\quad + (1-\zeta_n)\beta_n(1-\alpha_n)\delta_n\big\langle e_n, J_\varphi\big[(1-\alpha_n)(x_n + e_n) - p_0\big]\big\rangle \\ &\quad + \alpha_n\beta_n\delta_n(1-\zeta_n)\big\langle p_0, J_\varphi\big[p_0 - x_n - (1-\alpha_n)e_n + \alpha_n x_n\big]\big\rangle \\ &\quad + (1-\zeta_n)\mu_n\big\langle f(p_0) - p_0, J_\varphi\big(f(x_n) - p_0\big)\big\rangle \\ &\le \big[1 - \alpha_n\beta_n\delta_n(1-\zeta_n)\big]\Phi\big(\|x_n - p_0\|\big) \\ &\quad + (1-\zeta_n)(1-\alpha_n)\beta_n\delta_n K_2\|e_n\| \\ &\quad + \alpha_n\beta_n\delta_n(1-\zeta_n)\big\langle p_0, J_\varphi\big[p_0 - x_n - (1-\alpha_n)e_n + \alpha_n x_n\big]\big\rangle \\ &\quad + (1-\zeta_n)\mu_n\big\langle f(p_0) - p_0, J_\varphi\big(f(x_n) - p_0\big)\big\rangle. \end{aligned} \tag{3.23}$$

Let $c_n = \alpha_n\beta_n\delta_n(1-\zeta_n)$, then (3.23) reduces to $\Phi(\|x_{n+1} - p_0\|) \le (1-c_n)\Phi(\|x_n - p_0\|) + c_n\{\langle p_0, J_\varphi[p_0 - x_n - (1-\alpha_n)e_n + \alpha_n x_n]\rangle + \frac{\mu_n}{\alpha_n\beta_n\delta_n}K_2\|f(p_0) - p_0\|\} + K_2\|e_n\|$.

From Lemma 2.3, the assumptions (vii) and (viii), (3.19), and (3.23), we know that $\Phi(\|x_n - p_0\|) \to 0$, which implies that $x_n \to p_0$, as $n \to +\infty$. Combining the results in Steps 2 and 3, we can also know that $w_n \to p_0$, $u_n \to p_0$, as $n \to +\infty$.

This completes the proof. □

Remark 3.6 Actually, the three sequences $\{x_n\}$, $\{w_n\}$, and $\{u_n\}$ are proved to be strongly convergent to the common zero point p_0 of an infinite family of m-accretive mappings. The p_0 in Theorem 3.2 also satisfies (3.2).

Remark 3.7 The assumptions imposed on the real sequences in Theorem 3.2 are reasonable if we take $\alpha_n = \delta_n = \frac{1}{n^{\frac{1}{3}}}$, $\mu_n = \frac{1}{n^2}$, $\gamma_n = 1 - \frac{1}{n^2} - \frac{1}{n^{\frac{1}{3}}}$, $\zeta_n = \frac{n+1}{2n}$, and $\beta_n = \frac{1+n^{\frac{1}{3}}}{2n^{\frac{1}{3}}}$, for $n \ge 1$.

4 Example: infinite p-Laplacian-like differential systems

Remark 4.1 In the next of this paper, we shall present an example of infinite p-Laplacian-like differential systems. Based on the example, we shall construct an infinite family of m-accretive mappings, present characteristic of the common zero point of theirs, and demonstrate the applications of Theorems 3.1 and 3.2.

Now, we investigate the following p-Laplacian-like differential systems:

$$\begin{cases} -\operatorname{div}[(C(x)+|\nabla u|^2)^{\frac{s_i}{2}}|\nabla u|^{m_i-1}\nabla u] = f(x), & \text{a.e. in } \Omega, \\ -\langle \vartheta, (C(x)+|\nabla u|^2)^{\frac{s_i}{2}}|\nabla u|^{m_i-1}\nabla u\rangle = 0, & \text{a.e. on } \Gamma, \\ \quad i = 1, 2, \ldots, \end{cases} \tag{4.1}$$

where Ω is a bounded conical domain of the Euclidean space R^N ($N \geq 1$) with its boundary $\Gamma \in C^1$ [26].

$m_i + s_i + 1 = p_i$, $m_i \geq 0$ and $\frac{2N}{N+1} < p_i < +\infty$, for $i = 1, 2, \ldots$. $|\cdot|$ is the Euclidean norm in R^N and $\langle \cdot, \cdot \rangle$ is the Euclidean inner-product. ϑ is the exterior normal derivative of Γ. $C(x) \geq 0$ and $C(x) \in L^{p_i}(\Omega)$, $i \in N^+$.

We use $\|\cdot\|_{p_i}$ and $\|\cdot\|_{1,p_i,\Omega}$ to denote the norms in $L^{p_i}(\Omega)$ and $W^{1,p_i}(\Omega)$, respectively. Let $\frac{1}{p_i} + \frac{1}{p_i'} = 1$.

Remark 4.2 If $s_i = 0$ and $m_i = p_i - 1$, $i \in N^+$, then (4.1) is reduced to the case of infinite p-Laplacian differential systems.

Remark 4.3 ([27]) The mapping $J_{p_i} : L^{p_i}(\Omega) \to L^{p_i'}(\Omega)$ defined by $J_{p_i}u = |u|^{p_i-1}\operatorname{sgn} u$, for $u \in L^{p_i}(\Omega)$, is the duality mapping with the gauge function $\varphi(r) = r^{p_i-1}$ for $i \in N^+$. This presents a vivid example of a duality mapping in $L^{p_i}(\Omega)$.

Lemma 4.1 ([28]) *Let E be a real Banach space and E^* be its duality space. If $B : E \to E^*$ is maximal monotone and coercive, then B is a surjection.*

Lemma 4.2 ([29]) *For each $i \in N^+$, define the mapping $B_i : W^{1,p_i}(\Omega) \to (W^{1,p_i}(\Omega))^*$ by*

$$\langle v, B_i u\rangle = \int_\Omega \bigl\langle \bigl(C(x) + |\nabla u|^2\bigr)^{\frac{s_i}{2}} |\nabla u|^{m_i-1}\nabla u, \nabla v\bigr\rangle \, dx$$

for any $u, v \in W^{1,p_i}(\Omega)$. Then B_i is everywhere defined, monotone, and hemi-continuous.

Remark 4.4 Based on B_i, we shall construct two groups of mappings $\widetilde{A}_i : L^2(\Omega) \to L^2(\Omega)$ and $A_i : L^{p_i}(\Omega) \to L^{p_i}(\Omega)$ in the following. Since B_i may not be coercive, different proof methods are employed while showing that both $\widetilde{A}_i$ and A_i are m-accretive mappings, compared to the work done in [29].

Definition 4.1 For each $i \in N^+$, define the mapping $\widetilde{A}_i : L^2(\Omega) \to L^2(\Omega)$ in the following way:

$D(\widetilde{A}_i) = \{u(x) \in L^2(\Omega) :$ there exists $h_i(x) \in L^2(\Omega)$ such that $h_i(x) = B_i u\}$, for any $u \in D(\widetilde{A}_i)$, $\widetilde{A}_i u = h_i(x)$.

Proposition 4.1 *The mapping $\widetilde{A}_i$ is m-accretive.*

Proof First, for every $\lambda > 0$, the mapping $T_{i,\lambda} : H^1(\Omega) \to 2^{(H^1(\Omega))^*}$ defined by $T_{i,\lambda} = u + \lambda\widetilde{A}_i u$ is maximal monotone and coercive. It follows from the fact that $L^2(\Omega) \subset (H^1(\Omega))^*$ and Lemma 4.1 that $R(I + \lambda\widetilde{A}_i) = L^2(\Omega)$ for every $\lambda > 0$.

Secondly, for any $u_i \in D(\widetilde{A}_i)$, $\langle u_1 - u_2, \widetilde{A}_i u_1 - \widetilde{A}_i u_2\rangle = \langle u_1 - u_2, B_i u_1 - B_i u_2\rangle \geq 0$ since B_i is monotone.

This complete the proof. □

Lemma 4.3 *If $f, g \in L^2(\Omega)$ and there exist $u, v \in L^2(\Omega)$ such that $f = u + \lambda\widetilde{A}_i u$, $g = v + \lambda\widetilde{A}_i v$. Then*

$$\int_\Omega |u - v|^{p_i}\,dx \leq \int_\Omega |f - g|^{p_i}\,dx,$$

where $p_i \geq 2$. (Functions $u(x)$ and $v(x)$ exist from Proposition 4.1.)

Proof Define $\psi : R \to R$ by $\psi(t) = \frac{1}{p_i}|t|^{p_i}$, let $\partial\psi : R \to R$ denote its subdifferential and for $\mu > 0$, let $(\partial\psi)_\mu : R \to R$ denote the Yosida approximation of $\partial\psi$. Let ψ_μ be an indefinite integral of $(\partial\psi)_\mu$ so that $(\partial\psi)_\mu = \partial\psi_\mu$. We write $\partial\psi_\mu = \chi_\mu : R \to R$ and observe that χ_μ is monotone, Lipschitz with constant $\frac{1}{\mu}$ and differential everywhere except at $t = 0$.

Since

$$\psi_\mu\bigl(f(x) - g(x)\bigr) - \psi_\mu\bigl(u(x) - v(x)\bigr) \geq \bigl[\chi_\mu\bigl(u(x) - v(x)\bigr)\bigr]\bigl[\bigl(f(x) - g(x)\bigr) - \bigl(u(x) - v(x)\bigr)\bigr]$$

for $x \in \Omega$, we have

$$\int_\Omega \psi_\mu(f - g)\,dx \geq \int_\Omega \psi_\mu(u - v)\,dx + \int_\Omega \bigl[\chi_\mu(u - v)\bigr]\bigl[(f - g) - (u - v)\bigr]\,dx.$$

So, to prove the lemma, it suffices to prove that

$$\int_\Omega \chi_\mu(u - v)(\widetilde{A}_i u - \widetilde{A}_i v)\,dx \geq 0$$

for every $\mu > 0$ in view of the fact that $\psi_\mu(t) \uparrow \psi(t)$ for every $t \in R$ and the monotone convergence theorem.

Now, since

$$\begin{aligned}
&\int_\Omega \chi_\mu(u - v)(\widetilde{A}_i u - \widetilde{A}_i v)\,dx \\
&\quad = \bigl\langle \chi_\mu(u - v), B_i u - B_i v\bigr\rangle \\
&\quad \geq \int_\Omega \Bigl[\bigl(C(x) + |\nabla u|^2\bigr)^{\frac{s_i}{2}}|\nabla u|^{m_i+1} - \bigl(C(x) + |\nabla u|^2\bigr)^{\frac{s_i}{2}}|\nabla u|^{m_i}|\nabla v| \\
&\qquad - \bigl(C(x) + |\nabla v|^2\bigr)^{\frac{s_i}{2}}|\nabla v|^{m_i}|\nabla u| + \bigl(C(x) + |\nabla v|^2\bigr)^{\frac{s_i}{2}}|\nabla v|^{m_i+1}\Bigr]\chi_\mu'(u - v)\,dx \\
&\quad = \int_\Omega \Bigl[\bigl(C(x) + |\nabla u|^2\bigr)^{\frac{s_i}{2}}|\nabla u|^{m_i} - \bigl(C(x) + |\nabla v|^2\bigr)^{\frac{s_i}{2}}|\nabla v|^{m_i}\Bigr]\bigl(|\nabla u| - |\nabla v|\bigr)\chi_\mu'(u - v)\,dx \\
&\quad \geq 0,
\end{aligned}$$

the last inequality is available since χ_μ is monotone and $\chi_\mu(0) = 0$.

This completes the proof. □

Definition 4.2 For $i \in N^+$, define the mapping $A_i : L^{p_i}(\Omega) \to L^{p_i}(\Omega)$ in the following way:

(i) if $p_i \geq 2$, $D(A_i) = \{u(x) \in L^{p_i}(\Omega) : \text{there exists } h_i(x) \in L^{p_i}(\Omega) \text{ such that } h_i(x) = B_i u\}$, for any $u \in D(A_i)$, $A_i u = h_i(x)$;

(ii) if $\frac{2N}{N+1} < p_i < 2$, we define $A_i : L^{p_i}(\Omega) \to L^{p_i}(\Omega)$ as the L^{p_i}-closure of $\widetilde{A}_i : L^2(\Omega) \to L^2(\Omega)$ defined in Definition 4.1.

Proposition 4.2 *For $2 \leq p_i < +\infty$, the mapping $A_i : L^{p_i}(\Omega) \to L^{p_i}(\Omega)$ is m-accretive, where $i \in N^+$.*

Proof First, we show that

$$R(I + \lambda A_i) = L^{p_i}(\Omega)$$

for every $\lambda > 0$.

In fact, since $p_i \geq 2$, then Proposition 4.1 implies that

$$R(I + \lambda \widetilde{A}_i) = L^2(\Omega) \supseteq L^{p_i}(\Omega).$$

Then for any $f(x) \in L^{p_i}(\Omega)$, there is a $u \in D(\widetilde{A}_i)$ such that $f = u + \lambda \widetilde{A}_i u$. Since $0 = 0 + \lambda \widetilde{A}_i 0$, by Lemma 4.3, we have

$$\int_\Omega |u|^{p_i}\,dx \leq \int_\Omega |f|^{p_i}\,dx < +\infty.$$

That is, $u \in L^{p_i}(\Omega)$ and so $R(I + \lambda A_i) = L^{p_i}(\Omega)$ in view of the definition of A_i.

Secondly, we shall show that A_i is accretive.

For any $u_j \in D(A_i)$, $j = 1, 2$, we are left to show that

$$\bigl\langle |u_1 - u_2|^{p_i - 1} \operatorname{sgn}(u_1 - u_2), A_i u_1 - A_i u_2 \bigr\rangle \geq 0.$$

It suffices for us to show that

$$\bigl\langle |u_1 - u_2|^{p_i - 1} \operatorname{sgn}(u_1 - u_2), B_i u_1 - B_i u_2 \bigr\rangle \geq 0.$$

To this aim, take for a constant $k > 0$, define $\chi_k : R \to R$ by

$$\chi_k(t) = \bigl|(t \wedge k) \vee (-k)\bigr|^{p_i - 1} \operatorname{sgn} t.$$

Then χ_k is monotone, Lipschitz with $\chi_k(0) = 0$, and χ_k' is continuous except at finitely many points on R. This shows that

$$\begin{aligned}
&\bigl\langle |u_1 - u_2|^{p_i - 1} \operatorname{sgn}(u_1 - u_2), B_i u_1 - B_i u_2 \bigr\rangle \\
&\quad = \lim_{k \to +\infty} \int_\Omega \Bigl\langle \bigl(C(x) + |\nabla u|^2\bigr)^{\frac{s_i}{2}} |\nabla u|^{m_i - 1} \nabla u - \bigl(C(x) + |\nabla v|^2\bigr)^{\frac{s_i}{2}} |\nabla v|^{m_i - 1} \nabla v, \nabla u - \nabla v \Bigr\rangle \\
&\qquad \times \chi_k'(u_1 - u_2)\,dx \\
&\quad \geq \int_\Omega \Bigl[\bigl(C(x) + |\nabla u|^2\bigr)^{\frac{s_i}{2}} |\nabla u|^{m_i} - \bigl(C(x) + |\nabla v|^2\bigr)^{\frac{s_i}{2}} |\nabla v|^{m_i}\Bigr]
\end{aligned}$$

$$\times \left(|\nabla u| - |\nabla v|\right)\chi_k'(u_1 - u_2)\,dx$$
$$\geq 0.$$

This completes the proof. □

Proposition 4.3 *For $\frac{2N}{N+1} < p_i < 2$, the mapping $A_i : L^{p_i}(\Omega) \to L^{p_i}(\Omega)$ is m-accretive, where $i \in N^+$.*

Proof For $f(x) \in L^{p_i}(\Omega)$, we may choose a sequence $\{f_n\} \subset L^2(\Omega)$ such that $f_n(x) \to f(x)$ in $L^{p_i}(\Omega)$, as $n \to \infty$.

Proposition 4.1 implies that there is a $u_n \in L^2(\Omega)$ such that $u_n + \lambda\widetilde{A}_i u_n = f_n$, for $n \geq 1$.

Using Lemma 4.3, we have

$$\int_\Omega |u_n - u_m|^{p_i}\,dx \leq \int_\Omega |f_n - f_m|^{p_i}\,dx.$$

This implies that there is a $u(x) \in L^{p_i}(\Omega)$ such that $u_n \to u$ in $L^{p_i}(\Omega)$ and then Definition 4.2 ensures that $R(I + \lambda A_i) = L^{p_i}(\Omega)$.

The nonexpansive property of $(I + \lambda A_i)^{-1} : L^{p_i}(\Omega) \to L^{p_i}(\Omega)$ follows from Lemma 4.3, which implies that A_i is accretive.

This completes the proof. □

Lemma 4.4 ([30]) *Let Ω be a bounded conical domain in R^N. If $mp > N$, then $W^{m,p}(\Omega) \hookrightarrow\hookrightarrow C_B(\Omega)$; if $0 < mp \leq N$ and $q_0 = \frac{Np}{N-mp}$, then $W^{m,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$, where $1 \leq q < q_0$ and '$\hookrightarrow\hookrightarrow$' means compact embedding.*

Lemma 4.5 ([31]) *Let X_0 denote the closed subspace of all constant functions in $W^{1,p}(\Omega)$. Let X be the quotient space $W^{1,p}(\Omega)/X_0$. For $u \in W^{1,p}(\Omega)$, define the mapping $P : W^{1,p}(\Omega) \to X_0$ by $Pu = \frac{1}{\operatorname{meas}(\Omega)}\int_\Omega u\,dx$. Then there is a constant $C > 0$, such that $\forall u \in W^{1,p}(\Omega)$,*

$$\|u - Pu\|_p \leq C\|\nabla u\|_{(L^p(\Omega))^N}.$$

Theorem 4.1 *For $i \in R^+$, $N(A_i) = \{u \in L^{p_i}(\Omega) : u(x) \equiv \text{constant on } \Omega\}$.*

Proof (i) $p_i \geq 2$.

Let $u(x) \in N(A_i)$, then $0 = \langle u, B_i u\rangle = \int_\Omega (C(x) + |\nabla u|^2)^{\frac{s_i}{2}}|\nabla u|^{m_i+1}\,dx \geq \int_\Omega |\nabla u|^{p_i}\,dx \geq 0$, which implies that $u(x) \equiv$ constant. That is, $N(A_i) \subset \{u \in L^{p_i}(\Omega) : u(x) \equiv \text{constant on } \Omega\}$.

On the other hand, suppose $u(x) \equiv$ constant. Then $0 = \langle v, B_i u\rangle$, for $\forall v \in W^{1,p_i}(\Omega)$. Then $u \in N(A_i)$. The result follows.

(ii) $\frac{2N}{N+1} < p_i < 2$.

Suppose $u \in L^{p_i}(\Omega)$ and $u(x) \equiv$ constant. Let $u_n \equiv u$, then $\widetilde{A}_i u_n = 0$ in view of (i). Thus $\{u \in L^{p_i}(\Omega) : u(x) \equiv \text{constant on } \Omega\} \subset N(A_i)$ in view of the definition of A_i.

On the other hand, let $u \in N(A_i)$. Then there exist $\{u_n\}$ and $\{f_n\}$ in $L^2(\Omega)$ such that $u_n \to u$, $f_n \to 0$ in $L^{p_i}(\Omega)$ and $\widetilde{A}_i u_n = f_n$. Now, define the following functions:

$$\eta(t) = \begin{cases} |t|^{p_i - 1}\operatorname{sgn} t, & \text{if } |t| \geq 1, \\ t, & \text{if } |t| < 1 \end{cases}$$

and

$$\xi(t) = \begin{cases} |t|^{2-\frac{2}{p_i}} \operatorname{sgn} t, & \text{if } |t| \geq 1, \\ t, & \text{if } |t| < 1. \end{cases}$$

Then for $u \in L^2(\Omega)$, the function $t \in R \to \int_\Omega \xi(u+t)\,dx \in R$ is continuous and $\lim_{t\to\pm\infty} \int_\Omega \xi(u+t)\,dx = \pm\infty$. Therefore, there exists $t_u \in R$ such that $\int_\Omega \xi(u+t_u)\,dx = 0$. So, for $u_n \in L^2(\Omega)$, we may assume that there exists $t_n \in R$ such that $\int_\Omega \xi(u_n+t_n)\,dx = 0$ and $\widetilde{A}_i u_n = f_n$, for $n \geq 1$. Let $v_n = u_n + t_n$, then $\widetilde{A}_i v_n = \widetilde{A}_i u_n = f_n$, for $n \geq 1$.

Now, we compute the following:

$$\begin{aligned}
&\|f_n\|_{p_i} \left(\int_{|v_n|\leq 1} |v_n|^{p_i'}\,dx + \int_{|v_n|\geq 1} |v_n|^2\,dx \right)^{\frac{p_i-1}{p_i}} \\
&\geq \|f_n\|_{p_i} \left(\int_{|v_n|\leq 1} |v_n|^{p_i'}\,dx + \int_{|v_n|\geq 1} |v_n|^{p_i}\,dx \right)^{\frac{p_i-1}{p_i}} \\
&= \|f_n\|_{p_i} \big\|\eta(v_n)\big\|_{p_i'} \geq \langle \eta(v_n), f_n \rangle \\
&= \langle \eta(v_n), \widetilde{A}_i v_n \rangle \geq \int_\Omega |\nabla v_n|^{p_i} \eta'(v_n)\,dx \\
&\geq \text{const.} \int_\Omega \big|\nabla\big(\xi(v_n)\big)\big|^{p_i}\,dx.
\end{aligned} \tag{4.2}$$

Using Lemma 4.5,

$$\int_\Omega \big|\nabla\big(\xi(v_n)\big)\big|^{p_i}\,dx \geq \text{const.} \big\|\xi(v_n)\big\|_{1,p_i,\Omega}^{p_i}. \tag{4.3}$$

Then Lemma 4.4 implies that

$$\begin{aligned}
\big\|\xi(v_n)\big\|_{1,p_i,\Omega}^{p_i} &\geq \text{const.} \big\|\xi(v_n)\big\|_{p_i'}^{p_i} \\
&= \text{const.} \left(\int_{|v_n|\leq 1} |v_n|^{p_i'}\,dx + \int_{|v_n|\geq 1} |v_n|^{(2-\frac{2}{p_i})p_i'}\,dx \right)^{\frac{p_i}{p_i'}} \\
&= \text{const.} \left(\int_{|v_n|\leq 1} |v_n|^{p_i'}\,dx + \int_{|v_n|\geq 1} |v_n|^2\,dx \right)^{\frac{p_i}{p_i'}}.
\end{aligned} \tag{4.4}$$

From (4.2)-(4.4), we know that $\|\xi(v_n)\|_{1,p_i,\Omega}^{p_i} \leq \text{const.} \|f_n\|_{p_i} \to 0$, as $n \to \infty$. Then $\xi(v_n) \to 0$ in $L^{p_i'}(\Omega)$. Since the Nemytskyi mapping $u \in L^{p_i'}(\Omega) \to \xi^{-1}(u) \in L^{p_i}(\Omega)$ is continuous, $v_n \to 0$ in $L^{p_i}(\Omega)$. Then $u(x) \equiv$ constant. Thus $N(A_i) \subset \{u \in L^{p_i}(\Omega) : u(x) \equiv \text{constant on } \Omega\}$.

This completes the proof. □

Remark 4.5 Two infinite families of m-accretive mappings related to nonlinear p-Laplacian-like differential systems are constructed, which emphasizes the importance of the study on approximating common zero points of infinite nonlinear m-accretive mappings.

Remark 4.6 Theorem 4.1 helps us to see the assumption that $\bigcap_{i=1}^\infty N(A_i) \neq \emptyset$ in Theorems 3.1 and 3.2 are valid.

Definition 4.3 If $f(x) \equiv 0$ in (4.1), then the solution $u(x)$ of (4.1) is called the equilibrium solution to the p-Laplacian-like differential systems (4.1).

Theorem 4.2 *For $i \in N^+$, $u(x) \in N(A_i)$ if and only if $u(x)$ is the equilibrium solution of* (4.1).

Proof It is easy to see that if $u(x) \in N(A_i)$, then $u(x)$ is the equilibrium solution of (4.1). On the other hand, if $u(x)$ is the equilibrium solution of (4.1), then

$$-\operatorname{div}\left[\left(C(x) + |\nabla u|^2\right)^{\frac{s_i}{2}} |\nabla u|^{m_i - 1} \nabla u\right] = 0, \quad \text{a.e. in } \Omega.$$

Thus for $\forall \varphi \in C_0^\infty(\Omega)$, by using the property of generalized function, we have

$$\begin{aligned} 0 &= \left\langle \varphi, -\operatorname{div}\left[\left(C(x) + |\nabla u|^2\right)^{\frac{s_i}{2}} |\nabla u|^{m_i - 1} \nabla u\right]\right\rangle \\ &= \int_\Omega -\operatorname{div}\left[\left(C(x) + |\nabla u|^2\right)^{\frac{s_i}{2}} |\nabla u|^{m_i - 1} \nabla u\right] \varphi\, dx \\ &= \int_\Omega \left\langle \left(C(x) + |\nabla u|^2\right)^{\frac{s_i}{2}} |\nabla u|^{m_i - 1} \nabla u, \nabla \varphi \right\rangle dx = \langle \varphi, B_i u \rangle, \end{aligned}$$

which implies that $u(x) \in N(A_i)$.

This completes the proof. □

Remark 4.7 Based on Theorem 4.2, Theorems 3.1 and 3.2 can be applied to approximate the equilibrium solution of (4.1).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally. All authors read and approved the final manuscript.

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References

1. Barbu, V: Nonlinear Semigroups and Differential Equations in Banach Space. Noordhoff, Leyden (1976)
2. Agarwal, RP, O'Regan, D, Sahu, DR: Fixed Point Theory for Lipschitz-Type Mappings with Applications. Springer, Berlin (2008)
3. Browder, FE: Convergence theorems for sequences of nonlinear operators in Banach spaces. Math. Z. **100**(3), 201-225 (1967)
4. Takahashi, W: Proximal point algorithms and four resolvents of nonlinear operators of monotone type in Banach spaces. Taiwan. J. Math. **12**(8), 1883-1910 (2008)
5. Shehu, Y, Ezeora, JN: Path convergence and approximation of common zeroes of a finite family of *m*-accretive mappings in Banach spaces. Abstr. Appl. Anal. **2010**, Article ID 285376 (2010)
6. Takahashi, W: Nonlinear Functional Analysis: Fixed Point Theory and Its Application. Yokohama Publishers, Yokohama (2000)
7. Rockafellar, RT: Monotone operators and the proximal point algorithm. SIAM J. Control Optim. **14**(5), 877-898 (1976)
8. Petrusel, A, Yao, JC: Viscosity approximation by generalized contractions for resolvents of accretive operators in Banach spaces. Acta Math. Sin. **25**(4), 553-564 (2009)

9. Qin, XL, Su, YF: Approximation of a zero point of accretive operator in Banach spaces. J. Math. Anal. Appl. **329**, 415-424 (2007)
10. Ceng, LC, Ansari, QH, Schaible, S, Yao, JC: Hybrid viscosity approximation method for zeros of *m*-accretive operators in Banach spaces. Numer. Funct. Anal. Optim. **32**(11), 1127-1150 (2011)
11. Ceng, LC, Schaible, S, Yao, JC: Hybrid steepest decent methods for zeros of nonlinear operators with applications to variational inequalities. J. Optim. Theory Appl. **141**, 75-91 (2009)
12. Ceng, LC, Wu, SY, Yao, JC: New accuracy criteria for modified approximate proximal point algorithms in Hilbert spaces. Taiwan. J. Math. **12**(7), 1691-1705 (2008)
13. Xu, HK: Strong convergence of an iterative method for nonexpansive and accretive operators. J. Math. Anal. Appl. **314**, 631-643 (2006)
14. Cho, JC, Kang, SM, Zhou, HY: Approximate proximal point algorithms for finding zeroes of maximal monotone operator in Hilbert spaces. J. Inequal. Appl. **2008**, Article ID 598191 (2008)
15. Ceng, LC, Khan, AR, Ansari, QH, Yao, JC: Strong convergence of composite iterative schemes for zeros of *m*-accretive operators in Banach spaces. Nonlinear Anal. **70**, 1830-1840 (2009)
16. Halpern, B: Fixed points of nonexpansive maps. Bull. Am. Math. Soc. **73**, 957-961 (1967)
17. Zegeye, H, Shahzad, N: Strong convergence theorems for a common zero of a finite family of *m*-accretive mappings. Nonlinear Anal. **66**, 1161-1169 (2007)
18. Hu, LY, Liu, LW: A new iterative algorithm for common solutions of a finite family of accretive operators. Nonlinear Anal. **70**, 2344-2351 (2009)
19. Yao, Y, Liou, YC, Marino, G: Strong convergence of two iterative algorithms for nonexpansive mappings in Hilbert spaces. Fixed Point Theory Appl. **2009**, Article ID 279058 (2009)
20. Wei, L, Tan, RL: Iterative scheme with errors for common zeros of finite accretive mappings and nonlinear elliptic systems. Abstr. Appl. Anal. **2014**, Article ID 646843 (2014)
21. Wang, SH, Zhang, P: Some results on an infinite family of accretive operators in a reflexive Banach space. Fixed Point Theory Appl. **2015**, Article ID 8 (2015)
22. Bruck, RE: Properties of fixed-point sets of nonexpansive mappings in Banach spaces. Trans. Am. Math. Soc. **179**, 251-262 (1973)
23. Lim, TC: Fixed point theorems for asymptotically nonexpansive mapping. Nonlinear Anal. **22**, 1345-1355 (1994)
24. Liu, LS: Ishikawa and Mann iterative process with errors for nonlinear strongly accretive mappings in Banach spaces. J. Math. Anal. Appl. **194**, 114-125 (1995)
25. Browder, FE: Nonexpansive nonlinear operators in a Banach space. Proc. Natl. Acad. Sci. USA **54**, 1041-1044 (1965)
26. Wei, L, Agarwal, RP: Existence of solutions to nonlinear Neumann boundary value problems with generalized *p*-Laplacian operator. Comput. Math. Appl. **56**, 530-541 (2008)
27. Calvert, BD, Gupta, CP: Nonlinear elliptic boundary value problems in L^p-spaces and sums of ranges of accretive operators. Nonlinear Anal. **2**, 1-26 (1978)
28. Pascali, R, Sburlan, S: Nonlinear Mappings of Monotone Type. Editura Academiei, Bucharest (1978)
29. Wei, L, Agarwal, RP, Wong, PJY: Results on the existence of solution of *p*-Laplacian-like equation. Adv. Math. Sci. Appl. **23**(1), 153-167 (2013)
30. Li, LK, Guo, YT: The Theory of Sobolev Space. Shanghai Science and Technology Press, Shanghai (1981) (in Chinese)
31. Wei, L, He, Z: The applications of sums of ranges of accretive operators to nonlinear equations involving the *p*-Laplacian operator. Nonlinear Anal. **24**, 185-193 (1995)