AlNemer et al. *Fixed Point Theory and Applications* (2015) 2015:237
DOI 10.1186/s13663-015-0489-9

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH

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On best proximity points of upper semicontinuous multivalued mappings

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Abstract

In this paper we study the existence of best proximity points of a nonself upper semicontinuous multivalued mapping $T : A \to 2^B$ in a strictly convex Banach space. This multivalued mapping commutes with affine, noncyclic, and relatively u-continuous single-valued mapping $f : A \cup B \to A \cup B$. Also, we study the case when T commutes with a family of commuting, affine, noncyclic, and relatively u-continuous single-valued mappings on $A \cup B$. Moreover, we present some examples to illustrate our results.

MSC: 47H10; 54H25

Keywords: best proximity point; multivalued mapping; fixed point; upper semicontinuous mapping; relatively u-continuous mapping

1 Introduction

Let A, B be nonempty subsets of a metric space (X, d) and $T : A \to 2^B$, where 2^B is the family of all nonempty subsets of B. If $A \cap B = \emptyset$, the operator inclusion $x \in T(x)$ has no solution. In this case, it is logical to look for a point $x \in A$ such that $\operatorname{dist}(x, T(x))$ is minimum. Because $\operatorname{dist}(x, T(x))$ is at least $\operatorname{dist}(A, B)$, the point x is the solution of the equation $\operatorname{dist}(x, T(x)) = \operatorname{dist}(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$. This point is called the best proximity point of T. Indeed, best proximity point theorems examine the existence of such optimal approximate solutions of the operator inclusion $x \in T(x)$ when there is no exact solution. If $A \cap B \neq \emptyset$, the best proximity point is the fixed point of T.

For multivalued mappings, the existence of best proximity points was established by many authors, *e.g.*, Abkar and Gabeleh in [1] and [2], Al-Thagafi and Shahzad in [3], Amini-Harandi in [4], De la Sen in [5], Kirk *et al.* in [6] and Włodarczyk *et al.* in [7]. Best proximity point theorems for relatively nonexpansive single-valued mapping were studied in [8] in 2005. Since then there has been a lot of activity in this area and a number of results appeared by various authors. Best proximity point theorems for relatively u-continuous mapping were proved in [9] and [10]. For other related results, we refer the reader to [11–16] and [17]. In this paper, we study the existence of best proximity points for an upper semicontinuous multivalued mapping with nonempty, compact, and convex values $T : A \to 2^B$ which commutes with an affine and relatively u-continuous single-valued mapping $f : A \cup B :\to A \cup B$ such that $f(A) \subseteq A$ and $f(B) \subseteq B$ (noncyclic). In addition, we present some support examples for our results and we also give an example showing that the condition '$T(x) \cap B_0 \neq \emptyset$ for each $x \in A_0$' is necessary. Moreover, we add a similar theorem for a multivalued mapping which commutes with a family of commuting, affine, noncyclic, and relatively u-continuous single-valued mappings on $A \cup B$.

2 Preliminaries

Definition 2.1 [9] Let A, B be nonempty subsets of a metric space X. A mapping $f : A \cup B \to A \cup B$ is said to be relatively u-continuous if for each $\epsilon > 0$ there exists $\delta > 0$ such that $d(f(x), f(y)) < \epsilon + \operatorname{dist}(A, B)$ whenever $d(x, y) < \delta + \operatorname{dist}(A, B)$ for each $x \in A$, $y \in B$.

Definition 2.2 [8] Let A, B be nonempty subsets of a metric space X. A mapping $f : A \cup B \to A \cup B$ is called relatively nonexpansive if $d(f(x), f(y)) \leq d(x, y)$ for each $x \in A$, $y \in B$.

Every relatively nonexpansive mapping is relatively u-continuous. However, the converse is not true (see [9]).

Definition 2.3 [3] Let A, B be nonempty subsets of a metric space X and $T : A \to 2^B$ a multivalued mapping. A point $x \in A$ is called a (i) fixed point of T if $x \in T(x)$ and (ii) best proximity point of T if $\operatorname{dist}(x, T(x)) = \operatorname{dist}(A, B)$. Note that if $\operatorname{dist}(A, B) = 0$, then we get a fixed point of T.

Definition 2.4 Let A, B be nonempty subsets of a metric space X. A multivalued mapping $T : A \to 2^B$ is called upper semicontinuous if $T^{-1}(C) = \{x \in A : T(x) \cap C \neq \emptyset\}$ is closed in A whenever C is closed in B.

Proposition 2.5 [18] *Let X be a strictly convex Banach space, A a nonempty, compact, and convex subset of X, and B a nonempty closed subset of X. Let $\{x_n\}$ be a sequence in A and $y \in B$. If $\|x_n - y\| \to \operatorname{dist}(A, B)$, then $x_n \to P_A(y)$.*

Definition 2.6 [9] Let A, B be nonempty convex subsets of a Banach space X. A mapping $f : A \cup B \to A \cup B$ is called affine if $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$ for all $x, y \in A$ or $x, y \in B$ and $\alpha, \beta \in [0, 1]$ with $\alpha + \beta = 1$.

Lemma 2.7 [19] *If A is a nonempty, compact, and convex subset of a Banach space, and $T : A \to 2^A$ can be expressed as a composition of finitely many upper semicontinuous multivalued mappings with nonempty, compact, and convex values, then T has a fixed point.*

Let A, B be nonempty subsets of a Banach space X. $f : A \cup B \to A \cup B$ a relatively nonexpansive mapping such that $f(A) \subseteq A$, $f(B) \subseteq B$, $T : A \to \mathrm{KC}(B)$, where $\mathrm{KC}(B)$ is the set of all nonempty, compact, and convex subsets of B. The mapping f and T are said to commute if for each $x \in A$, $f(T(x)) \subseteq T(f(x))$. Define

$$A_0 = \{x \in A : \|x - y\| = \operatorname{dist}(A, B) \text{ for some } y \in B\},$$
$$B_0 = \{y \in B : \|x - y\| = \operatorname{dist}(A, B) \text{ for some } x \in A\}.$$

Remark 2.8 Note that if A and B are nonempty, compact, and convex sets, then A_0 and B_0 are nonempty, compact, and convex sets with $\operatorname{dist}(A_0, B_0) = \operatorname{dist}(A, B)$. For details see [6] and [8]. Also, $f(A_0) \subseteq A_0$ and $f(B_0) \subseteq B_0$ [10].

Remark 2.9 [3, 9] Let A be a nonempty subset of a normed space X. The metric projection operator is defined by $P_A(x) = \{y \in A : \|x - y\| = \operatorname{dist}(x, A)\}$ for each $x \in X$. If A is a nonempty, compact, and convex subset of a Banach space X, then P_A is upper semicontinuous with nonempty, compact, and convex values. Observe that when A is a nonempty, compact, and convex subset of a strictly convex Banach space X, P_A is a single-valued mapping from X to A.

Theorem 2.10 [10] *Let A, B be nonempty, compact, and convex subsets of a strictly convex Banach space X. If $f : A \cup B :\rightarrow A \cup B$ is relatively u-continuous such that $f(A) \subseteq A$ and $f(B) \subseteq B$. Then there exists $(x_0, y_0) \in A \times B$ such that $f(x_0) = x_0, f(y_0) = y_0$, and $\|x_0 - y_0\| = \operatorname{dist}(A, B)$.*

3 Main results

The following proposition is a noncyclic version of Proposition 3.2 in [9].

Proposition 3.1 *Let A, B be nonempty, compact, and convex subsets of a strictly convex Banach space X. Let $f : A \cup B \rightarrow A \cup B$ be a relatively u-continuous mapping such that $f(A) \subseteq A$ and $f(B) \subseteq B$. $P : A \cup B \rightarrow A \cup B$ is a mapping defined by*

$$P(x) = \begin{cases} P_B(x) & \text{if } x \in A, \\ P_A(x) & \text{if } x \in B. \end{cases}$$

Then $f(P(x)) = P(f(x))$ for each $x \in A_0 \cup B_0$, i.e., $P_A(f(y)) = f(P_A(y))$ for each $y \in B_0$ and $P_B(f(x)) = f(P_B(x))$ for each $x \in A_0$.

Proof Let $x \in A_0$. Then there exists $y \in B$ such that $\|x - y\| = \operatorname{dist}(A, B)$. So, $y = P_B(x)$ and $x = P_A(y)$. Then for each $\delta > 0$, $\|x - y\| < \delta + \operatorname{dist}(A, B)$. Since f is relatively u-continuous, for each $\epsilon > 0$ we have $\operatorname{dist}(A, B) \leq \|f(x) - f(y)\| < \epsilon + \operatorname{dist}(A, B)$. Thus, $\|f(x) - f(y)\| = \operatorname{dist}(A, B)$. So, $f(x) = P_A(f(y))$ and $f(y) = P_B(f(x))$. Since A, B are nonempty, compact, and convex subsets of a strictly convex Banach space, the metric projection is unique. Now, $x = P_A(y) \Longrightarrow f(x) = f(P_A(y)) \Longrightarrow P_A(f(y)) = f(P_A(y))$ for each $y \in B_0$. Also, $y = P_B(x) \Longrightarrow f(y) = f(P_B(x)) \Longrightarrow P_B(f(x)) = f(P_B(x))$ for each $x \in A_0$. Hence, $f(P(x)) = P(f(x))$ for each $x \in A_0 \cup B_0$. □

A cyclic version of the following proposition can be found in [9] (see the proof of Theorem 3.1 in [9]).

Proposition 3.2 *Let A, B be nonempty, compact, and convex subsets of a strictly convex Banach space X. Let $f : A \cup B \rightarrow A \cup B$ be a relatively u-continuous mapping such that $f(A) \subseteq A$ and $f(B) \subseteq B$. Then f is continuous on A_0 and B_0.*

Proof Let $x_0 \in A_0$ and $\{x_n\} \subseteq A_0$ such that $x_n \rightarrow x_0$. We want to show that $f(x_n) \rightarrow f(x_0)$. Using the triangle inequality, we obtain

$$\begin{aligned} \left\|x_n - P_B(x_0)\right\| &\leq \|x_n - x_0\| + \left\|x_0 - P_B(x_0)\right\| \\ &= \|x_n - x_0\| + \operatorname{dist}(A, B) \\ &\rightarrow \operatorname{dist}(A, B). \end{aligned}$$

Then for each $\delta > 0$ there exists $N_0 \in \mathbb{N}$ such that for each $n \geq N_0$, we have $|\|x_n - P_B(x_0)\| - \operatorname{dist}(A,B)| < \delta$. So, $n \geq N_0 \Longrightarrow \|x_n - P_B(x_0)\| < \delta + \operatorname{dist}(A,B)$. By relative u-continuity of f, $\|f(x_n) - f(P_B(x_0))\| < \epsilon + \operatorname{dist}(A,B)$ for each $n \geq N_0$. Since $\{f(x_n)\} \subseteq A$ and $P_B(f(x_0)) \in B$, Proposition 2.5 gives

$$f(x_n) \to P_A\bigl(f\bigl(P_B(x_0)\bigr)\bigr) = f\bigl(P_A\bigl(P_B(x_0)\bigr)\bigr) = f(x_0).$$

Hence, $f(x_n) \to f(x_0)$. Since $x_0 \in A_0$ was arbitrary, f is continuous on A_0. Similarly, f is continuous on B_0. Therefore, f is continuous on $A_0 \cup B_0$. □

Theorem 3.3 *Let A, B be nonempty, compact, and convex subsets in a strictly convex Banach space X. Suppose $f : A \cup B \to A \cup B$ is an affine relatively u-continuous mapping with $f(A) \subseteq A$, $f(B) \subseteq B$. Then there exists $(x_0, y_0) \in A \times B$ such that $f(x_0) = x_0$, $f(y_0) = y_0$ and $\|x_0 - y_0\| = \operatorname{dist}(A,B)$.*

In addition, if $T : A \to \mathrm{KC}(B)$ is an upper semicontinuous multivalued mapping, f and T commute, and $T(x) \cap B_0 \neq \emptyset$ for each $x \in A_0$, then there exists $a \in A$ such that $f(a) = a$ and $\operatorname{dist}(a, T(a)) = \operatorname{dist}(A,B)$.

Proof For $u \in A_0$, there is a $v \in B$ such that $\|u - v\| = \operatorname{dist}(A,B)$. Then by the relative u-continuity of f, $\|f(u) - f(v)\| = \operatorname{dist}(A,B)$, implying that $f(u) \in A_0$. Therefore, the compact convex set A_0 is invariant under the continuous mapping f, and the Schauder fixed point theorem implies the existence of a fixed point $x_0 = f(x_0) \in A_0$. Let y_0 be the unique closest point to x_0 in B. Then by the relative u-continuity of f and the uniqueness of the closest point projection onto B, $y_0 = f(y_0)$ and $\|x_0 - y_0\| = \operatorname{dist}(A,B)$.

Now, we will prove that there exists $a \in A$ such that $\operatorname{dist}(a, T(a)) = \operatorname{dist}(A,B)$. Define $\operatorname{Fix}(f) = \{x \in A \cup B : f(x) = x\}$, $\operatorname{Fix}_A(f) = \operatorname{Fix}(f) \cap A_0$ and $\operatorname{Fix}_B(f) = \operatorname{Fix}(f) \cap B_0$. Clearly, $\operatorname{Fix}_A(f)$ and $\operatorname{Fix}_B(f)$ are nonempty, because $x_0 \in \operatorname{Fix}_A(f)$ and $y_0 \in \operatorname{Fix}_B(f)$. The set $\operatorname{Fix}_A(f)$ is closed. Indeed, let $\{x_n\} \subseteq \operatorname{Fix}_A(f)$ such that $x_n \to x_0$. Since $\{x_n\} \subseteq A_0$ and A_0 is closed by Remark 2.8, we have $x_0 \in A_0 \subseteq A$. Using Proposition 3.2, $f(x_n) \to f(x_0)$. But $f(x_n) = x_n$ for each n. So $x_n \to f(x_0)$. Consequently $x_0 = f(x_0)$. Thus $x_0 \in \operatorname{Fix}_A(f)$. Therefore, $\operatorname{Fix}_A(f)$ is closed. Similarly, $\operatorname{Fix}_B(f)$ is closed. So, $\operatorname{Fix}_A(f)$ and $\operatorname{Fix}_B(f)$ are compact sets as they are closed subsets of the compact sets A_0, B_0. In addition, $\operatorname{Fix}_A(f)$ is a convex set. Indeed, let $x, y \in \operatorname{Fix}_A(f)$ and $\alpha, \beta \in [0,1]$ with $\alpha + \beta = 1$. Since f is affine, $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y) = \alpha x + \beta y$, *i.e.*, $\alpha x + \beta y \in \operatorname{Fix}(f)$. Also, $\alpha x + \beta y \in A_0$ as A_0 is convex and $x, y \in A_0$. Consequently, $\alpha x + \beta y \in \operatorname{Fix}(f) \cap A_0 = \operatorname{Fix}_A(f)$. Similarly, $\operatorname{Fix}_B(f)$ is a convex set.

Assume $x \in \operatorname{Fix}_A(f)$ and choose $v \in T(x)$. Since f and T commute, $f(v) \in T(f(x)) = T(x)$, which implies that $T(x)$ is invariant under f. Then the invariance of B_0 under f shows that the compact convex set $T(x) \cap B_0$ is invariant under f. Since f is continuous on B_0, by the Schauder fixed point theorem f has a fixed point in $T(x) \cap B_0$, implying that $T(x) \cap \operatorname{Fix}_B(f) \neq \emptyset$ for each $x \in \operatorname{Fix}_A(f)$.

Now, define $F : \operatorname{Fix}_A(f) \to 2^{\operatorname{Fix}_B(f)}$ by $F(x) = T(x) \cap \operatorname{Fix}_B(f)$ for each $x \in \operatorname{Fix}_A(f)$. Then F is an upper semicontinuous multivalued mapping with nonempty, compact, and convex values. Note that $P_A : \operatorname{Fix}_B(f) \to \operatorname{Fix}_A(f)$. To see this, let $x \in \operatorname{Fix}_B(f) \subseteq B_0$. Then there exists $y \in A$ such that $\|x - y\| = \operatorname{dist}(A,B)$. So, $y = P_A(x)$ and $x = P_B(y)$. For each $\delta > 0$, we have $\|x - y\| < \delta + \operatorname{dist}(A,B)$. Using the relative u-continuity for any f, $\operatorname{dist}(A,B) \leq \|f(x) - f(y)\| < \epsilon + \operatorname{dist}(A,B)$ for each $\epsilon > 0$. Thus, $\|f(x) - f(y)\| = \operatorname{dist}(A,B)$. This implies that $f(y) = P_A(f(x))$

and $f(x) = P_B(f(y))$. Since $x \in \mathrm{Fix}_B(f)$ and $y = P_A(x)$, we have $f(y) = f(P_A(x)) = P_A(f(x)) = P_A(x)$ and so $P_A(x) \in \mathrm{Fix}_A(f) \subseteq A$. Note that $P_A \circ F : \mathrm{Fix}_A(f) \to 2^{\mathrm{Fix}_A(f)}$. By Lemma 2.7, there exists $a \in \mathrm{Fix}_A(f) \subseteq A_0$ such that $a \in (P_A \circ F)(a)$, *i.e.*, $a = f(a)$ and $a \in P_A(F(a))$. So, there exists $b \in F(a) = T(a) \cap \mathrm{Fix}_B(f) \subseteq B_0$ such that $a = P_A(b) \subseteq \mathrm{Fix}_A(f)$. As $a = P_A(b)$, $\|a - b\| = \mathrm{dist}(b, A)$. Since $b \in F(a) = T(a) \cap \mathrm{Fix}_B(f) \subseteq B_0$, then $b \in T(a)$ and $b \in B_0$. Since $b \in B_0$, there exists $a' \in A$ such that $\|a' - b\| = \mathrm{dist}(A, B)$. Since $a \in A$ and $T(a) \subseteq B$, we have

$$\begin{aligned}
\mathrm{dist}(A,B) &\le \mathrm{dist}\bigl(a, T(a)\bigr) \\
&\le \|a - b\| \\
&= \mathrm{dist}(b, A) \\
&\le \bigl\|b - a'\bigr\| \\
&= \mathrm{dist}(A, B).
\end{aligned}$$

Thus, $\mathrm{dist}(a, T(a)) = \mathrm{dist}(A, B)$. □

Remark 3.4 The condition $T(x) \cap B_0 \neq \emptyset$ for each $x \in A_0$ is necessary in Theorem 3.3. For example, in the real space if $A = [1,5] \times [-5,5]$, $B = [-1, \frac{-1}{25}] \times [-5,5]$. Define

$$f : A \cup B \to A \cup B \quad \text{by} \quad f(x,y) = \left(x, \frac{y+1}{2}\right)$$

and

$$T : A \to \mathrm{KC}(B) \quad \text{by} \quad T(x,y) = \left[-1, \frac{-1}{x^2}\right] \times \{y\}.$$

Clearly, T is upper semicontinuous and f is affine and relatively u-continuous. Also, $f(A) \subseteq A$ and $f(B) \subseteq B$. There are fixed points of f, $x_0 = (1,1) \in A$, $y_0 = (\frac{-1}{25}, 1) \in B$ such that $\|x_0 - y_0\| = \mathrm{dist}(A, B) = 1.04$. In addition, f and T commute. Suppose that there exists $a \in \mathrm{Fix}(f) \cap A$ such that $\mathrm{dist}(a, T(a)) = 1.04$. Then $a = (z, 1)$, for some $1 \le z \le 5$. So,

$$\mathrm{dist}\bigl(a, T(a)\bigr) = \mathrm{dist}\left((z,1), \left[-1, \frac{-1}{z^2}\right] \times \{1\}\right) = \left\|(z,1) - \left(\frac{-1}{z^2}, 1\right)\right\| = 1.04.$$

Consequently, $z^3 - 1.04z^2 + 1 = 0$. So, $z_1 = 0.893939214944 + 0.7334769205376i$, $z_2 = 0.893939214944 - 0.7334769205376i$, which are not real numbers, and $z_3 = -0.747878429888$, which does not belong to $[1,5]$. Note that $A_0 = \{1\} \times [-5,5]$, $B_0 = \{\frac{-1}{25}\} \times [-5,5]$. For $x = (1,y) \in A_0$, we have $T(x) = T(1,y) = \{(-1,y)\}$. So, $T(x) \cap B_0 = \{(-1,y)\} \cap \{(\frac{-1}{25}, y) : -5 \le y \le 5\} = \emptyset$.

Corollary 3.5 *Let A, B be nonempty, compact, and convex sets in a strictly convex Banach space X. If $T : A \to \mathrm{KC}(B)$ is an upper semicontinuous multivalued mapping and $T(x) \cap B_0 \neq \emptyset$ for each $x \in A_0$, then there exists $a \in A$ such that* $\mathrm{dist}(a, T(a)) = \mathrm{dist}(A, B)$.

Proof Taking $f = I$ (the identity mapping on $A \cup B$) in Theorem 3.3, we obtain the desired result. □

Corollary 3.6 *Let A be a nonempty, compact, and convex set in a strictly convex Banach space. Suppose $f : A \to A$ is an affine continuous mapping. If $T : A \to \mathrm{KC}(A)$ is an upper semicontinuous multivalued mapping and f, T commute, then there exists $a \in A$ such that $a \in \mathrm{Fix}(f) \cap \mathrm{Fix}(T)$.*

Proof Since any continuous mapping on a compact set is relatively u-continuous on that set, taking $A = B$ in Theorem 3.3, we see that there exists $a \in$A such that $f(a) = a$ and $\mathrm{dist}(a, T(a)) = \mathrm{dist}(A,A) = 0$, *i.e.*, $a \in T(a)$. So, $f(a) = a \in T(a)$. Therefore, $a \in \mathrm{Fix}(f) \cap \mathrm{Fix}(T)$. □

Theorem 3.7 *Let X be a strictly convex Banach space. Let A, B be nonempty, compact, and convex subsets of X and let $f,g : A \cup B \to A \cup B$ be commuting, affine, and relatively u-continuous mappings such that $f(A) \subseteq A$, $f(B) \subseteq B$ and $g(A) \subseteq A$, $g(B) \subseteq B$. Then there exist points $x_0 \in A$ and $y_0 \in B$ such that $x_0 = f(x_0) = g(x_0)$, $y_0 = f(y_0) = g(y_0)$ and $\|x_0 - y_0\| = \mathrm{dist}(A,B)$.*

Proof For $u \in A_0$, there is a $v \in B$ such that $\|u - v\| = \mathrm{dist}(A,B)$. Then by the relative u-continuity of f, $\|f(u) - f(v)\| = \mathrm{dist}(A,B)$, implying that $f(u) \in A_0$. Therefore, the compact convex set A_0 is invariant under the continuous mapping f, and the Schauder fixed point theorem implies the existence of a fixed point $x = f(x) \in A_0$. The set of fixed points of f in A_0 (denoted by $\mathrm{Fix}_A(f)$) is closed and convex since f is continuous and affine. If $x \in \mathrm{Fix}_A(f)$, commutativity of f and g implies $f(g(x)) = g(f(x)) = g(x)$. Therefore, $\mathrm{Fix}_A(f)$ is invariant under g, and since g is continuous it has a fixed point in $\mathrm{Fix}_A(f)$. Let x_0 be a common fixed point of f and g in A_0, that is, $x_0 = f(x_0) = g(x_0)$, and let y_0 be the unique closest point to x_0 in B. Then by the relative u-continuity of f and g and the uniqueness of the closest point projection onto B, $y_0 = f(y_0) = g(y_0)$ and $\|x_0 - y_0\| = \mathrm{dist}(A,B)$. □

The previous theorem can be extended to an arbitrary family of commuting affine and noncyclic mappings. The proof depends on the following common fixed point result for commuting affine u-continuous mappings in strictly convex Banach spaces. The proof of this result is adapted from Przebieracz ([20], Theorem 1.1) and is included for convenience of the reader.

Lemma 3.8 (Markov-Kakutani theorem) *Let X be a strictly convex Banach space. Let A, B be nonempty, compact, and convex subsets of X and let $\mathfrak{F}$ be a family of commuting affine and relatively u-continuous mappings on $A \cup B$ such that $f(A) \subseteq A$ and $f(B) \subseteq B$. Then there is an $x_0 \in A_0$ such that $f(x_0) = x_0$ for every $f \in \mathfrak{F}$. There is a $y_0 \in B_0$ such that $f(y_0) = y_0$ for every $f \in \mathfrak{F}$.*

Proof Notice that the mappings in the family $\mathfrak{F}$ are continuous on $A_0 \cup B_0$. Let $\mathrm{Fix}(f) = \{x \in A \cup B : f(x) = x\}$, $\mathrm{Fix}_A(f) = \mathrm{Fix}(f) \cap A_0, f \in \mathfrak{F}$. As shown in the proof of Theorem 3.7, $\mathrm{Fix}_A(f) \neq \emptyset$ and $\mathrm{Fix}_A(f)$ is convex and compact. To prove that $\bigcap_{f \in \mathfrak{F}} \mathrm{Fix}_A(f) \neq \emptyset$, consider any finite collection from $\mathfrak{F}$, say $f_1, \ldots, f_n$. Assume that

$$C = \bigcap_{1 \le i \le n} \mathrm{Fix}_A(f_i) \neq \emptyset.$$

For each $x \in C$ and $k \in \{1, \ldots, n\}$, $f_k f_{n+1}(x) = f_{n+1} f_k(x) = f_{n+1}(x)$, which implies that $f_{n+1}(x) \in C$. Therefore, the compact convex set C is invariant under f_{n+1}, implying that $\mathrm{Fix}_A(f_{n+1}) \cap C \neq \emptyset$ since f_{n+1} is continuous on A_0. Since every finite collection of the sets $\mathrm{Fix}_A(f)$, $f \in \mathfrak{F}$, has a nonempty intersection, we have $\bigcap_{f\in\mathfrak{F}} \mathrm{Fix}_A(f) \neq \emptyset$. Similarly, $\bigcap_{f\in\mathfrak{F}} \mathrm{Fix}_B(f) \neq \emptyset$. □

Theorem 3.9 *Let X be a strictly convex Banach space. Let A, B be nonempty, compact, and convex subsets of X and let $\mathfrak{F}$ be a family of commuting affine and relatively u-continuous mappings on $A \cup B$ such that $f(A) \subseteq A$ and $f(B) \subseteq B$. Then there exist points $x_0 \in A$ and $y_0 \in B$ such that $x_0 = f(x_0)$ and $y_0 = f(y_0)$, for all $f \in \mathfrak{F}$ where $\|x_0 - y_0\| = \mathrm{dist}(A, B)$.*

Proof By Lemma 3.8 the mappings in the family $\mathfrak{F}$ have a common fixed point $x_0 \in A$, that is, $f(x_0) = x_0$ for $f \in \mathfrak{F}$. Let $y_0 \in B$ be the unique closest point to x_0 in B. Then, for any $f \in \mathfrak{F}$, $\|f(x_0) - y_0\| = \mathrm{dist}(A, B)$, but by the relative u-continuity of f, $\|f(x_0) - f(y_0)\| = \mathrm{dist}(A, B)$. By the uniqueness of the closest point, $y_0 = f(y_0)$ for $f \in \mathfrak{F}$. □

Theorem 3.10 *Let A, B be nonempty, compact, and convex subsets of a strictly convex Banach space X and let $\mathfrak{F}$ be a family of commuting, affine and relatively u-continuous mappings on $A \cup B$ with $f(A) \subseteq A$, $f(B) \subseteq B$ for each $f \in \mathfrak{F}$. Let $T : A \to \mathrm{KC}(B)$ be an upper semicontinuous mapping such that $T(x) \cap B_0 \neq \emptyset$ for each $x \in A_0$. If $\mathfrak{F}$ and T commute, then there exists a point $a \in A$ such that $f(a) = a$ for each $f \in \mathfrak{F}$ and $\mathrm{dist}(a, T(a)) = \mathrm{dist}(A, B)$.*

Proof By Lemma 3.8, $\bigcap_{f\in\mathfrak{F}} \mathrm{Fix}_A(f)$ and $\bigcap_{f\in\mathfrak{F}} \mathrm{Fix}_B(f)$ are nonempty.

As in the proof of Theorem 3.3, $T(x)$ is invariant under each $f \in \mathfrak{F}$, for $x \in \mathrm{Fix}_A(f)$. Since $\bigcap_{f\in\mathfrak{F}} \mathrm{Fix}_A(f) \neq \emptyset$, for $x \in \bigcap_{f\in\mathfrak{F}} \mathrm{Fix}_A(f)$, $T(x)$ is invariant under $\mathfrak{F}$. Also, B_0 is invariant under $\mathfrak{F}$. Therefore as in the proof of Theorem 3.3, since $T(x) \cap B_0$ is a compact convex set, $T(x) \cap (\bigcap_{f\in\mathfrak{F}} \mathrm{Fix}_B(f)) \neq \emptyset$. By the proof of Theorem 3.3, $\mathrm{Fix}_A(f)$ and $\mathrm{Fix}_B(f)$ are compact and convex sets for $f \in \mathfrak{F}$. Therefore, $\bigcap_{f\in\mathfrak{F}} \mathrm{Fix}_A(f)$ and $\bigcap_{f\in\mathfrak{F}} \mathrm{Fix}_B(f)$ are compact and convex.

Now define $F : \bigcap_{f\in\mathfrak{F}} \mathrm{Fix}_A(f) \to 2^{\bigcap_{f\in\mathfrak{F}} \mathrm{Fix}_B(f)}$ by $F(x) = T(x) \cap (\bigcap_{f\in\mathfrak{F}} \mathrm{Fix}_B(f))$ for each $x \in \bigcap_{f\in\mathfrak{F}} \mathrm{Fix}_A(f)$. Clearly, F is an upper semicontinuous multivalued mapping with compact convex values. Now, $P_A : \bigcap_{f\in\mathfrak{F}} \mathrm{Fix}_B(f) \to \bigcap_{f\in\mathfrak{F}} \mathrm{Fix}_A(f)$. To see this, let $x \in \bigcap_{f\in\mathfrak{F}} \mathrm{Fix}_B(f)$. Then $x \in B_0$ and $f(x) = x$ for each $f \in \mathfrak{F}$. So, there exists $y \in A$ such that $\|x - y\| = \mathrm{dist}(A, B)$. This implies $x = P_B(y)$ and $y = P_A(x)$. For each $\delta > 0$, we have $\|x - y\| < \delta + \mathrm{dist}(A, B)$. Using the relative u-continuity for any $f \in \mathfrak{F}$, $\mathrm{dist}(A, B) \leq \|f(x) - f(y)\| < \epsilon + \mathrm{dist}(A, B)$ for each $\epsilon > 0$. Thus, $\|f(x) - f(y)\| = \mathrm{dist}(A, B)$. Therefore, $f(y) = P_A(f(x))$ and $f(x) = P_B(f(y))$ for each $f \in \mathfrak{F}$. Now, $y = P_A(x) \Longrightarrow f(y) = f(P_A(x)) \Longrightarrow P_A(x) = f(P_A(x))$ for each $f \in \mathfrak{F}$. Hence, $P_A(x) \in \bigcap_{f\in\mathfrak{F}} \mathrm{Fix}_A(f)$ for each $x \in \bigcap_{f\in\mathfrak{F}} \mathrm{Fix}_B(f)$. Note that $P_A \circ F : \bigcap_{f\in\mathfrak{F}} \mathrm{Fix}_A(f) \to 2^{\bigcap_{f\in\mathfrak{F}} \mathrm{Fix}_A(f)}$. By Lemma 2.7, $P_A \circ F$ has a fixed point. So, there exists $a \in \bigcap_{f\in\mathfrak{F}} \mathrm{Fix}_A(f)$ such that $a \in (P_A \circ F)(a)$. So, $f(a) = a$ for each $f \in \mathfrak{F}$ and $a \in P_A(F(a))$, *i.e.*, there exists $b \in F(a)$ such that $a = P_A(b)$. Since $b \in F(a)$, $b \in T(a) \cap (\bigcap_{f\in\mathfrak{F}} \mathrm{Fix}_B(f))$. So, $b \in T(a)$, $b \in B_0$, and $f(b) = b$ for each $f \in \mathfrak{F}$. $a = P_A(b)$ implies $\|a - b\| = \mathrm{dist}(b, A)$. Since $b \in B_0$, there exists $a' \in A$ such that $\|a' - b\| = \mathrm{dist}(A, B)$. Since $a \in A$ and $T(a) \subseteq B$, we have

$$
\begin{aligned}
\mathrm{dist}(A, B) &\leq \mathrm{dist}\bigl(a, T(a)\bigr) \\
&\leq \|a - b\| \\
&= \mathrm{dist}(b, A)
\end{aligned}
$$

$$\begin{aligned} &\leq \|b - a'\| \\ &= \operatorname{dist}(A, B). \end{aligned}$$

Thus, $\operatorname{dist}(a, T(a)) = \operatorname{dist}(A, B)$. □

Corollary 3.11 *Let A be a nonempty, compact, and convex subset of a strictly convex Banach space X and let $\mathfrak{F}$ be a family of commuting, affine and continuous self-mappings of A. Let $T : A \to \mathrm{KC}(A)$ be an upper semicontinuous mapping. If $\mathfrak{F}$ and T commute, then there exists a point $a \in A$ such that $a = f(a) \in T(a)$ for each $f \in \mathfrak{F}$.*

4 Examples

Examples 4.1 to 4.4 are related to Theorem 3.3. On other hand, the last two examples are related to Theorem 3.7 (and Theorem 3.10).

Example 4.1 Let $X = \mathbb{R}^2$ with the usual metric. The sets $A = \{(x, y) : 0 \leq x \leq 4, 1 \leq y \leq 5\}$, $B = \{(x, 0) : 0 \leq x \leq 4\}$ are nonempty, compact, and convex with $\operatorname{dist}(A, B) = 1$. Define $f : A \cup B \to A \cup B$ by $f(x, y) = (\frac{2x+1}{3}, y)$ and $T : A \to \mathrm{KC}(B)$ by $T(x, y) = [x, 4] \times \{0\}$. Then T is upper semicontinuous and f is relatively u-continuous and affine with $f(A) \subseteq A$ and $f(B) \subseteq B$. As $\operatorname{Fix}(f) = \{(1, y) : 1 \leq y \leq 5 \text{ or } y = 0\}$, we get $x_0 = (1, 1) \in \operatorname{Fix}(f) \cap A$, $y_0 = (1, 0) \in \operatorname{Fix}(f) \cap B$ with $\|x_0 - y_0\| = 1$. In addition, f and T commute. Indeed, $f(T(x, y)) = f([x, 4] \times \{0\}) = \{\frac{2z+1}{3} : z \in [x, 4]\} \times \{0\}$ and $T(f(x, y)) = T(\frac{2x+1}{3}, y) = [\frac{2x+1}{3}, 4] \times \{0\}$. For $z \in [x, 4]$, $\frac{2z+1}{3} \in [\frac{2x+1}{3}, 3] \subseteq [\frac{2x+1}{3}, 4]$. Thus, $f(T(x, y)) \subseteq T(f(x, y))$ for each $(x, y) \in A$. Also, $T(x) \cap B_0 \neq \emptyset$ for each $x \in A_0$ since $A_0 = \{(x, 1) : 0 \leq x \leq 4\}$ and $B_0 = B$. For $(1, 1) \in A$, we have $f(a) = a$ and $\operatorname{dist}(a, T(a)) = \operatorname{dist}(A, B) = 1$.

Example 4.2 Let $X = \mathbb{R}^2$ with the usual metric. The sets $A = \{(0, a) : 1 \leq a \leq 3\}$, $B = \{(x, y) : 1 \leq x \leq 5, 1 \leq y \leq 5\}$ are nonempty, compact, and convex with $\operatorname{dist}(A, B) = 1$. Define $f : A \cup B \to A \cup B$ by $f(x, y) = (x, \frac{y+3}{2})$ and $T : A \to \mathrm{KC}(B)$ by $T(0, a) = [1, a] \times \{3\}$. Then T is upper semicontinuous and f is relatively u-continuous and affine with $f(A) \subseteq A$ and $f(B) \subseteq B$. As $\operatorname{Fix}(f) = \{(x, 3) : x = 0 \text{ or } 1 \leq x \leq 5\}$, we get $x_0 = (0, 3) \in \operatorname{Fix}(f) \cap A$, $y_0 = (1, 3) \in \operatorname{Fix}(f) \cap B$ with $\|x_0 - y_0\| = 1$. In addition, f and T commute. Indeed, $f(T(0, a)) = f([1, a] \times \{3\}) = [1, a] \times \{3\}$ and $T(f(0, a)) = T(0, \frac{a+3}{2}) = [1, \frac{a+3}{2}] \times \{3\}$. For $a \in [1, 3]$, $\frac{a+3}{2} \geq a$, *i.e.*, $[1, a] \subseteq [1, \frac{a+3}{2}]$. Thus, $f(T(0, a)) \subseteq T(f(0, a))$ for each $(0, a) \in A$. Also, $T(x) \cap B_0 \neq \emptyset$ for each $x \in A_0$ since $A_0 = A$ and $B_0 = \{(1, y) : 1 \leq y \leq 3\}$. For $a = (0, 3) \in A$, we have $f(a) = a$ and $\operatorname{dist}(a, T(a)) = \operatorname{dist}(A, B) = 1$.

Example 4.3 Let $X = \mathbb{R}^2$ with the usual metric. The sets $A = \{(x, y) : -1 \leq x \leq -0.04, -5 \leq y \leq 5\}$, $B = \{(x, y) : 0 \leq x \leq 5, -5 \leq y \leq 5\}$ are nonempty, compact, and convex with $\operatorname{dist}(A, B) = 0.04$. Define $f : A \cup B \to A \cup B$ by $f(x, y) = (x, \frac{y+1}{2})$ and $T : A \to \mathrm{KC}(B)$ by $T(x, y) = [0, x^2] \times \{y\}$. Then T is upper semicontinuous and f is relatively u-continuous and affine with $f(A) \subseteq A$ and $f(B) \subseteq B$. As $\operatorname{Fix}(f) = \{(x, 1) : -1 \leq x \leq -0.04 \text{ or } 0 \leq x \leq 5\}$, we get $x_0 = (-0.04, 1) \in \operatorname{Fix}(f) \cap A$, $y_0 = (0, 1) \in \operatorname{Fix}(f) \cap B$ with $\|x_0 - y_0\| = 0.04$. In addition, f and T commute. Also, $T(x) \cap B_0 \neq \emptyset$ for each $x \in A_0$ since $A_0 = \{(-0.04, y) : -5 \leq y \leq 5\}$ and $B_0 = \{(0, y) : -5 \leq y \leq 5\}$. For $a = (-0.04, 1) \in A$, we have $f(a) = a$ and $\operatorname{dist}(a, T(a)) = \operatorname{dist}(A, B) = 0.04$.

Example 4.4 Let $X = \mathbb{R}^2$ with the usual metric. The sets $A = \{(x, y) : -3 \leq x \leq 3, -1 \leq y \leq -0.25\}$, $B = \{(x, y) : -3 \leq x \leq 3, 0 \leq y \leq 4\}$ are nonempty, compact, and convex with $\operatorname{dist}(A, B) = 0.25$. Define $f : A \cup B \to A \cup B$ by $f(x, y) = (\frac{x}{2}, y)$ and $T : A \to \mathrm{KC}(B)$ by $T(x, y) = \{x\} \times [0, y^2]$. Then T is upper semicontinuous and f is relatively u-continuous and affine with $f(A) \subseteq A$ and $f(B) \subseteq B$. As $\operatorname{Fix}(f) = \{(0, y) : 0 \leq y \leq 4 \text{ or } -1 \leq y \leq -0.25\}$, we get $x_0 = (0, -0.25) \in \operatorname{Fix}(f) \cap A$, $y_0 = (0, 0) \in \operatorname{Fix}(f) \cap B$ with $\|x_0 - y_0\| = 0.25$. In addition, f and T commute. Also, $T(x) \cap B_0 \neq \emptyset$ for each $x \in A_0$ since $A_0 = \{(x, -0.25) : -3 \leq x \leq 3\}$ and $B_0 = \{(x, 0) : -3 \leq x \leq 3\}$. For $a = (0, -0.25)$, we have $f(a) = a$ and $\operatorname{dist}(a, T(a)) = \operatorname{dist}(A, B) = 0.25$.

Example 4.5 Let $X = \mathbb{R}^2$ with the usual metric. The sets $A = \{(x, y) : 0 \leq x \leq 5, y = -1\}$, $B = \{(x, y) : -5 \leq x \leq 0, y = 1\}$ are nonempty, compact, and convex subsets of a strictly convex Banach space with $\operatorname{dist}(A, B) = 2$. Define $f, g : A \cup B \to A \cup B$ by $f(x, y) = (\frac{2x}{5}, y)$ and $g(x, y) = (\frac{x}{2}, y)$. Then f, g are relatively u-continuous and affine with $f(A) \subseteq A$, $f(B) \subseteq B$, $g(A) \subseteq A$, and $g(B) \subseteq B$. Also f, g commute. Now, define $T : A \to \mathrm{KC}(B)$ by $T(x, y) = [-5, -x] \times \{y^2\}$. Then T is upper semicontinuous with nonempty, compact, and convex values. In addition, T commutes with f and g. Clearly, $A_0 = \{(0, -1)\}$, $B_0 = \{(0, 1)\}$, and $(0, 1) \in T(0, -1) = [-5, 0] \times \{1\}$. For $a = (0, -1) \in A$ and $b = (0, 1) \in B$, we have $f(a) = g(a) = a$, $f(b) = g(b) = b$, and $\|a - b\| = \operatorname{dist}(A, B) = 2$. Moreover, $\operatorname{dist}(a, T(a)) = \operatorname{dist}(A, B)$.

Example 4.6 Let $X = \mathbb{R}^2$ with the usual metric. The sets $A = \{(x, y) : -4 \leq x \leq -1, -6 \leq y \leq 6\}$, $B = \{(x, y) : 0 \leq x \leq 4, -6 \leq y \leq 6\}$ are nonempty, compact, and convex subsets of a strictly convex Banach space with $\operatorname{dist}(A, B) = 1$. Define $f, g : A \cup B \to A \cup B$ by $f(x, y) = (x, \frac{y}{3})$ and $g(x, y) = (x, \frac{y}{2})$. Then f, g are relatively u-continuous and affine with $f(A) \subseteq A$, $f(B) \subseteq B$, $g(A) \subseteq A$, and $g(B) \subseteq B$. Also f, g commute. Now, define $T : A \to \mathrm{KC}(B)$ by $T(x, y) = [0, -x] \times \{y\}$. Then T is upper semicontinuous with nonempty, compact, and convex values. In addition, T commutes with f and g. Clearly, $A_0 = \{(-1, y) : -6 \leq y \leq 6\}$, $B_0 = \{(0, y) : -6 \leq y \leq 6\}$. So, $(0, y) \in T(-1, y) \cap B_0 = ([0, 1] \times \{y\}) \cap B_0$ for each $(-1, y) \in A_0$. For $a = (-1, 0) \in A$ and $b = (0, 0) \in B$, we have $f(a) = g(a) = a$, $f(b) = g(b) = b$, and $\|a - b\| = \operatorname{dist}(A, B) = 1$. Moreover, $\operatorname{dist}(a, T(a)) = \operatorname{dist}(A, B)$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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Acknowledgements
This article was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah. The authors, therefore, acknowledge with thanks DSR for technical and financial support. The authors are grateful to three anonymous referees for their useful suggestions and comments.

Received: 29 June 2015 Accepted: 11 December 2015 Published online: 12 January 2016

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