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Some results on best proximity point on star-shaped sets in probabilistic Banach (Menger) spaces

Hamid Shayanpour*, Maryam Shams and Asiyeh Nematizadeh

*Correspondence:
h.shayanpour@sci.sku.ac.ir
Department of Pure Mathematics,
University of Shahrekord,
Shahrekord, 88186-34141, Iran

Abstract

We first present the concepts of proximal contraction and proximal nonexpansive mappings on star-shaped sets in probabilistic Banach (Menger) spaces. We derive some results about the best proximity points for these mappings in probabilistic Banach (Menger) spaces. Next, we bring some examples that defend our main results.

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1 Introduction and preliminaries

The equation $Tx = x$ for a mapping $T : A \to B$ may have no solution whenever $A \cap B = \emptyset$, where A, B are two nonempty subsets in a metric space (X, d). Under this condition, it is beneficial to determine a point $a_0 \in A$ such that $d(a_0, Ta_0)$ is minimal. If $d(a_0, Ta_0)$ is the global minimum value of $\operatorname{dist}(A, B)$, *i.e.*, $d(a_0, Ta_0) = \operatorname{dist}(A, B) = \min\{d(a, b) : a \in A, b \in B\}$, then a_0 is called best proximity point of T.

In 1969, Fan [1] proved one of the most classical theorems in best approximation theory. He showed that if (V, ρ) is a topological vector space with seminorm p, $W \subseteq V$, and $T : W \to V$ is a mapping, then under certain conditions, there exists an element $w_0 \in W$ such that

$$\rho(w_0 - Tw_0) = d(Tw_0, W).$$

Thereafter, this theorem has been generalized for continuous multivalued mappings by Reich [2, 3] and Sehgal and Singh [4].

Eldred *et al.* [5] showed that every relatively nonexpansive mapping has a proximal point under certain conditions. For further existence results of a best proximity point for several types of contractions, we refer to [6–25].

In 1942, a probabilistic metric (PM) space was introduced by Menger [26]. Schweizer and Sklar [27, 28] were two pioneers in the study of PM spaces.

PM spaces are very useful in probabilistic functional analysis, quantum particle physics, ϵ^{∞} theory, nonlinear analysis, and applications; see [29–33].

Indeed, the study of fixed point results in PM spaces is one of the most active research areas in fixed point theory. Sehgal and Bharucha-Reid [34] were two pioneers in this study.

For further existence results of a fixed point and common fixed point in PM spaces, we refer, for example, to [35–37]. In 2014, Su and Zhang [38], proved some best proximity point theorems in PM spaces.

Let Δ^+ be the set of all distribution functions F (*i.e.*, a nondecreasing and left-continuous function $F:\mathbb{R}\to[0,1]$ such that $\inf_{t\in\mathbb{R}} F(t)=0$ and $\sup_{t\in\mathbb{R}} F(t)=1$) such that $F(0)=0$. Let X be a nonempty set, $\epsilon_0=\chi_{(0,\infty)}\in\Delta^+$, and $F:X\times X\to\Delta^+$ ($F(p,q)=F_{p,q}$) be a mapping such that

(PM1) $F_{p,q}=\epsilon_0$ iff $p=q$,

(PM2) $F_{p,q}=F_{q,p}$, and

(PM3) if $F_{p,q}(t)=1$ and $F_{q,r}(s)=1$, then $F_{p,r}(t+s)=1$

for all $p,q,r\in X$ and $t,s\geq 0$. Then (X,F) is called a probabilistic metric space.

For well-known definitions (such as t-norm, t-norm of H-type, probabilistic Menger space, complete probabilistic Menger space, probabilistic normed (PN) space, *etc.*) and known results, we refer to [27, 39].

First, we state some notation, definitions, and known results; afterward, we introduce concepts of proximal contraction, proximal nonexpansive, P-property, weak P-property, and semisharp proximinal pair in PM spaces. Throughout this paper, the minimum t-norm will be denoted by $\Delta_m(a,b)=\min\{a,b\}$.

Lemma 1.1 ([39]) *Let (x_n) be a sequence in a probabilistic Menger space (X,F,Δ) such that Δ is a t-norm of H-type. If*

$$F_{x_n,x_{n+1}}(kt)\geq F_{x_{n-1},x_n}(t)\quad (n\geq 1, t>0)$$

for some $k\in(0,1)$, then (x_n) is a Cauchy sequence.

Definition 1.2 Suppose that A is a nonempty subset of a probabilistic Menger space (X,F,Δ). Then the probabilistic diameter of A is the mapping D_A defined on $[0,\infty]$ by $D_A(\infty)=1$ and $D_A(x)=\lim_{t\to x^-}\varphi_A(t)$, where $\varphi_A(t)=\inf\{F_{a,b}(t):a,b\in A\}$.

A nonempty set A in a probabilistic Menger space is bounded if $\lim_{x\to\infty} D_A(x)=1$. It is easy to see that $F_{a,b}(t)\geq D_A(t)$ for all $a,b\in A$ and $t\geq 0$.

Definition 1.3 Let (X,F,Δ) be a probabilistic Menger space, $A\subseteq X$, and $T:A\to A$ be a mapping. The mapping T is said to be an isometry if

$$F_{Tx,Ty}(t)=F_{x,y}(t)\quad \forall x,y\in X, \forall t\geq 0.$$

Definition 1.4 Let (X,F,Δ) be a probabilistic Menger space, and $A,B\subseteq X$. A mapping $T:A\to B$ is said to be continuous at $x\in A$ if for every sequence (x_n) in A that converges to x, the sequence (Tx_n) in B converges to Tx.

Remark 1.5 If T is an isometry mapping on subset A of a probabilistic Menger space (X,F,Δ), then T is a continuous mapping because

$$F_{Tx_n,Tx}(t)=F_{x_n,x}(t)\to 1\quad \forall t>0.$$

Also, it is easy to see that T is an injective mapping.

An immediate consequence of the definition of a PN space ([27], Section 15.1) is the following lemma.

Lemma 1.6 ([27]) *Let (X,ν,Δ) be a PN space, and F^ν be the function from $X \times X$ into Δ^+ defined by*

$$F^\nu(p,q) = \nu_{p-q}.$$

Then (X,F^ν,Δ) is a probabilistic Menger space.

We call this probabilistic metric F^ν on X the probabilistic metric induced by the probabilistic norm ν.

Definition 1.7 A PN space (X,ν,Δ) is said to be a probabilistic Banach space if (X,F^ν,Δ) is a complete probabilistic Menger space.

Remark 1.8 Let A, B, C be a nonempty subsets of a PN space (X,ν,Δ) such that Δ is continuous t-norm and $x \in A$. If two mappings $T : A \to B$ and $S : A \to C$ are continuous at x, then $T + S$ is continuous at x because

$$\nu_{(T+S)(x)-(T+S)(x_n)}(t) \geq \Delta\left(\nu_{T(x)-T(x_n)}\left(\frac{t}{2}\right), \nu_{S(x)-S(x_n)}\left(\frac{t}{2}\right)\right) \to 1 \quad \forall t > 0.$$

Definition 1.9 Let A be a nonempty subset of a PM space (X,F). A mapping $T : A \to X$ is called a contraction (nonexpansive) if $F_{Tx,Ty}(t) \geq F_{x,y}(\frac{t}{\alpha})$ ($F_{Tx,Ty}(t) \geq F_{x,y}(t)$) for some $0 < \alpha < 1$ and for all $x,y \in A$ and $t > 0$.

Definition 1.10 Suppose that A and B are nonempty subsets of a PM space (X,F). Then the probabilistic distance of A, B is the mapping $F_{A,B}$ defined on $[0,\infty]$ by

$$F_{A,B}(t) = \sup_{x\in A, y\in B} F_{x,y}(t) \quad \forall t \geq 0.$$

Also, if A and B are nonempty subsets of a PN space (X,ν,Δ), then ${F^\nu}_{A,B}(t) = \nu_{A-B}(t) = \sup_{x\in A,y\in B} \nu_{x-y}(t)$, where F^ν is the probabilistic metric induced by the probabilistic norm ν.

Definition 1.11 Let (X,F) be a PM space. For subsets A and B of X, define:

$$A_0 = \{x \in A : \exists y \in B \text{ s.t. } \forall t \geq 0, F_{x,y}(t) = F_{A,B}(t)\},$$
$$B_0 = \{y \in B : \exists x \in A \text{ s.t. } \forall t \geq 0, F_{x,y}(t) = F_{A,B}(t)\}.$$

Clearly, if A_0 (or B_0) is a nonempty subset, then A and B are nonempty subsets.

Definition 1.12 Let (X,F) be a PM space, and (A,B) be a pair of nonempty subsets of X. A mapping $T : A \to B$ is called the proximal contraction (proximal nonexpansive) if there exists a real number $0 < \alpha < 1$ such that

$$F_{u,Tx}(t) = F_{A,B}(t) = F_{v,Ty}(t) \quad \Longrightarrow \quad F_{u,v}(t) \geq F_{x,y}\left(\frac{t}{\alpha}\right)$$

$$\big(F_{u,Tx}(t) = F_{A,B}(t) = F_{v,Ty}(t) \quad \Longrightarrow \quad F_{u,v}(t) \geq F_{x,y}(t)\big)$$

for all $u, v, x, y \in A$ and $t > 0$.

Example 1.13 Let $X = [0, 2]$, and $T : X \to X$ be the mapping defined by $Tx = \frac{1}{8}x$. If $F_{x,y}(t) = \frac{t}{t+|x-y|}$, then it is easy to check that $F_{X,X}(t) = 1$. If $F_{u,Tx}(t) = 1 = F_{v,Ty}(t)$, then for $\alpha = \frac{1}{8}$, we have $F_{u,v}(t) = F_{x,y}(\frac{t}{\alpha})$, where $u, v, x, y \in X$. Therefore, T is a proximal contraction.

Definition 1.14 Let X be a vector space, and A be a nonempty subset of X. Then the subset A is called a p-star-shaped set if there exists a point $p \in A$ such that $\alpha p + (1 - \alpha)x \in A$ for all $x \in A$, $\alpha \in [0, 1]$, and p is called the center of A.

Clearly, each convex set C is a p-star-shaped set for each $p \in C$. Let (X, ν, Δ_m) be a PN space, A be a p-star-shaped set, B be a q-star-shaped set, and $\nu_{p-q} = \nu_{A-B}$. If $x \in A_0$, then there exists a point $y \in B$ such that $\nu_{x-y}(t) = \nu_{A-B}(t)$ for all $t > 0$. So we have

$$\begin{aligned}
\nu_{A-B}(t) &\geq \nu_{(\alpha p+(1-\alpha)x)-(\alpha q+(1-\alpha)y)}(t) \\
&\geq \Delta_m\big(\nu_{\alpha(p-q)}(\alpha t), \nu_{(1-\alpha)(x-y)}\big((1-\alpha)t\big)\big) \\
&= \Delta_m\big(\nu_{p-q}(t), \nu_{x-y}(t)\big) \\
&= \Delta_m\big(\nu_{A-B}(t), \nu_{A-B}(t)\big) \\
&= \nu_{A-B}(t)
\end{aligned}$$

for all $t > 0$. Therefore, $\nu_{(\alpha p+(1-\alpha)x)-(\alpha q+(1-\alpha)y)}(t) = \nu_{A-B}(t)$, which means that A_0 is a p-star-shaped set and, similarly, that B_0 is a q-star-shaped set.

Definition 1.15 Let (X, F) be a PM space. A pair (A, B) of nonempty subsets of X is said to have the P-property (weak P-property) if $A_0 \neq \emptyset$ and

$$F_{u,x}(t) = F_{A,B}(t) = F_{v,y}(t) \quad \Longrightarrow \quad F_{u,v}(t) = F_{x,y}(t)$$
$$\big(F_{u,x}(t) = F_{A,B}(t) = F_{v,y}(t) \quad \Longrightarrow \quad F_{u,v}(t) \geq F_{x,y}(t)\big)$$

for all $u, v \in A_0$, $x, y \in B_0$, and $t > 0$.

Example 1.16 Let $X = \mathbb{R}^2$ and define

$$F_{(x,y),(u,v)}(t) = \frac{t}{t + \sqrt{(x-u)^2 + (y-v)^2}}.$$

Clearly, (X, F, Δ_m) is a complete probabilistic Menger space. Let

$$A = \left\{\left(0, \frac{1}{n}\right) : n \in \mathbb{N}\right\} \cup \big\{(0, 0)\big\},$$
$$B = \left\{\left(1, \frac{1}{n}\right) : n \in \mathbb{N}\right\} \cup \big\{(1, 0)\big\}.$$

Then it is easy to check that $A_0 = A$, $B_0 = B$, and $F_{A,B}(t) = \frac{t}{t+1}$. If

$$F_{(0,x),(1,y)}(t) = F_{A,B}(t) = \frac{t}{t+1} = F_{(0,u),(1,v)}(t),$$

then $x = y$ and $u = v$, so that

$$F_{(0,x),(0,u)}(t) = \frac{t}{t+|x-u|} = \frac{t}{t+|y-v|} = F_{(1,y),(1,v)}(t).$$

Therefore, the pair (A,B) has the P-property.

Example 1.17 Let $X = \mathbb{R}^2$ and define

$$F_{(x,y),(u,v)}(t) = \frac{t}{t+\sqrt{(x-u)^2+(y-v)^2}}.$$

Let $A = \{(0,0)\}$ and $B = \{(x,y) \in X : y = 1 + \sqrt{1-x^2}\}$. Clearly, $A_0 = \{(0,0)\}$ and $B_0 = \{(-1,1),(1,1)\}$. If

$$F_{(0,0),(x,y)}(t) = F_{A,B}(t) = \frac{t}{t+\sqrt{2}} = F_{(0,0),(u,v)}(t),$$

then

$$1 = F_{(0,0),(0,0)}(t) \geq F_{(x,y),(u,v)}(t),$$

where $(x,y),(u,v) \in B_0$. Therefore, the pair (A,B) has the weak P-property.

Definition 1.18 Let (X,F) be a PM space. A pair (A,B) of nonempty subsets of X is called a semisharp proximinal pair if there exists at most one $(x_0,y_0) \in A \times B$ such that $F_{x,y_0}(t) = F_{A,B}(t) = F_{x_0,y}(t)$ for all $(x,y) \in A \times B$.

It is easy to check that if a pair (A,B) has the P-property, then the pair (A,B) is a semisharp proximinal pair. Clearly, a semisharp proximinal pair (A,B) does not necessarily have the P-property.

Example 1.19 Suppose that $X = \mathbb{R}$, $A = \{-10,10\}$, $B = \{-2,2\}$, and $F_{x,y}(t) = \frac{t}{t+|x-y|}$. It is easy to verify that $F_{A,B}(t) = \frac{t}{t+8}$, $A_0 = A$, $B_0 = B$, and (A,B) is a semisharp proximinal pair but does not have the P-property.

Remark 1.20 It is easy to check that the P-property is stronger than the weak P-property. If a pair (A,B) has the weak P-property and $T : A \to B$ is a nonexpansive mapping, then for all $u,v,x,y \in A$, we have

$$F_{u,Tx}(t) = F_{A,B}(t) = F_{v,Ty}(t) \quad \Longrightarrow \quad F_{u,v}(t) \geq F_{Tx,Ty}(t) \geq F_{x,y}(t).$$

That is, T is a proximal nonexpansive mapping. Similarly, if a pair (A,B) has the weak P-property and $T : A \to B$ is a contraction mapping, then T is a proximal contraction mapping. Also, a pair (A,B) has the P-property if and only if both pairs (A,B) and (B,A) have the weak P-property.

Definition 1.21 Let X and Y be vector spaces. A mapping $T : X \to Y$ is affine if

$$T\left(\sum_{i=1}^{n} \lambda_i x_i\right) = \sum_{i=1}^{n} \lambda_i T(x_i)$$

for all $n \in \mathbb{N}$, $x_1, \dots, x_n \in X$, and $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ such that $\sum_{i=1}^{n} \lambda_i = 1$.

In Section 2, we show some results on the best proximity points in probabilistic Banach (Menger) spaces. For example, if (A, B) is a semisharp proximinal pair of a probabilistic Banach space (X, ν, Δ_m) such that A is a p-star-shaped set, A_0 is a nonempty compact set, B is a q-star-shaped set and $\nu_{p-q}(t) = \nu_{A-B}(t)$ for all $t > 0$, then every proximal nonexpansive mapping $T : A \to B$ with $T(A_0) \subseteq B_0$ has a best proximity point. We also prove that if A is a nonempty, compact, and convex subset of a probabilistic Banach space (X, ν, Δ_m) and $T : A \to A$ is a nonexpansive mapping, then T has a fixed point. Finally, we give some examples which defend our main results.

2 Proximity point for proximal contraction and proximal nonexpansive mappings

We first give the following lemma and then we state the main results of this paper. We recall that if A_0 (or B_0) is a nonempty subset, then A and B are nonempty subsets.

Lemma 2.1 *Let (X, F, Δ) be a complete probabilistic Menger space such that Δ is a t-norm of H-type, and $A, B \subseteq X$ be such that A_0 is a nonempty closed set. If $T : A \to B$ is a proximal contraction mapping such that $T(A_0) \subseteq B_0$, then there exists a unique $x \in A_0$ such that $F_{x,Tx}(t) = F_{A,B}(t)$ for all $t > 0$.*

Proof Since A_0 is nonempty and $T(A_0) \subseteq B_0$, there exist $x_1, x_0 \in A_0$ such that $F_{x_1,Tx_0}(t) = F_{A,B}(t)$. Since $Tx_1 \in B_0$, there exists $x_2 \in A_0$ such that $F_{x_2,Tx_1}(t) = F_{A,B}(t)$. Continuing this process, we obtain a sequence $(x_n) \subseteq A_0$ such that $F_{x_{n+1},Tx_n}(t) = F_{A,B}(t)$ for all $n \in \mathbb{N}$ and $t > 0$. Since for all $n \in \mathbb{N}$,

$$F_{x_n,Tx_{n-1}}(t) = F_{A,B}(t) = F_{x_{n+1},Tx_n}(t) \quad (t > 0)$$

and T is a proximal contraction, we have

$$F_{x_{n+1},x_n}(t) \geq F_{x_n,x_{n-1}}\left(\frac{t}{\alpha}\right) \quad (0 < \alpha < 1, t > 0).$$

Therefore, by Lemma 1.1, (x_n) is a Cauchy sequence and so converges to some $x \in A_0$. Again by the assumption $T(A_0) \subseteq B_0$, $Tx \in B_0$. Then there exists an element $u \in A_0$ such that $F_{u,Tx}(t) = F_{A,B}(t)$ for all $t > 0$. Since for all $n \in \mathbb{N}$,

$$F_{u,Tx}(t) = F_{A,B}(t) = F_{x_{n+1},Tx_n}(t) \quad (t > 0),$$

by the hypothesis we have

$$F_{u,x_{n+1}}(t) \geq F_{x,x_n}\left(\frac{t}{\alpha}\right) \geq F_{x,x_n}(t) \quad (t > 0).$$

Letting $n \to \infty$ shows that $x_n \to u$ and thus $x = u$, so $F_{x,Tx}(t) = F_{A,B}(t)$. If there exists another element y such that $F_{y,Ty}(t) = F_{A,B}(t)$, then by the hypothesis we have $F_{x,y}(t) \geq F_{x,y}(\frac{t}{\alpha})$, which means that $x = y$. □

Proposition 2.2 *Let (X, F, Δ) be a probabilistic Menger space, and $A, B \subseteq X$ be such that A_0 is a nonempty set. Suppose that $T : A \to B$ is a proximal contraction mapping such that $T(A_0) \subseteq B_0$ and $g : A \to A$ is an isometry mapping such that $A_0 \subseteq g(A_0)$. Denote $G = g(A)$ and*

$$G_0 = \{z \in G : \exists y \in B \text{ s.t. } \forall t \geq 0, F_{z,y}(t) = F_{G,B}(t)\}.$$

Then Tg^{-1} is a proximal contraction, and $G_0 = A_0$.

Proof Since $G \subseteq A$, $F_{G,B}(t) \leq F_{A,B}(t)$ for all $t > 0$. Assume that $x \in A_0 \subseteq g(A_0)$. Then $x = g(x')$ for some $x' \in A_0$, and so there exists $y \in B$ such that $F_{A,B}(t) = F_{g(x'),y}(t) \leq F_{G,B}(t)$ for all $t > 0$. Thus, $F_{A,B}(t) = F_{G,B}(t)$ for all $t > 0$. Now we show that Tg^{-1} is a proximal contraction. To this end, suppose that $u, v, x, y \in G$ are such that

$$F_{u,Tg^{-1}x}(t) = F_{G,B}(t) = F_{A,B}(t) = F_{v,Tg^{-1}y}(t) \quad (t > 0).$$

By the hypothesis we have

$$F_{u,v}(t) \geq F_{g^{-1}x,g^{-1}y}\left(\frac{t}{\alpha}\right) = F_{gg^{-1}x,gg^{-1}y}\left(\frac{t}{\alpha}\right) = F_{x,y}\left(\frac{t}{\alpha}\right) \quad (t > 0)$$

for some $\alpha \in (0, 1)$. Therefore, Tg^{-1} is a proximal contraction. If $x \in G_0$, then $x \in G \subseteq A$, and there exists $y \in B$ such that $F_{x,y}(t) = F_{G,B}(t) = F_{A,B}(t)$ for all $t > 0$, so that $x \in A_0$. If $x \in A_0 \subseteq A$, then there exists $y \in B$ such that $F_{x,y}(t) = F_{A,B}(t) = F_{G,B}(t)$ for all $t > 0$. On the other hand, by the hypothesis $x \in G$, and therefore $G_0 = A_0$. □

Corollary 2.3 *Let the hypotheses of Lemma* 2.1 *be satisfied. Suppose that $T : A \to B$ is a proximal contraction mapping such that $T(A_0) \subseteq B_0$ and $g : A \to A$ is an isometry mapping such that $A_0 \subseteq g(A_0)$. Then there exists a unique $x \in A_0$ such that $F_{gx,Tx}(t) = F_{A,B}(t)$.*

Proof By Proposition 2.2, $Tg^{-1} : G = g(A) \to B$ is proximal contraction, and $Tg^{-1}(G_0) = Tg^{-1}(A_0) \subseteq T(A_0) \subseteq B_0$. Now by Lemma 2.1 there exists a unique $x' \in A_0$ such that $F_{x',Tg^{-1}x'}(t) = F_{A,B}(t)$. Since $A_0 \subseteq g(A_0)$, there exists $x \in A_0$ such that $x' = g(x)$, so that $F_{g(x),Tx}(t) = F_{A,B}(t)$. Note that g is an injective mapping, therefore, by Lemma 2.1, x is unique, and hence the result follows. □

Theorem 2.4 *Let (X, ν, Δ_m) be a probabilistic Banach space, $A, B \subseteq X$ be such that A is a convex set, A_0 be a nonempty compact set, and B be a bounded convex set. Suppose that $T : A \to B$ is a continuous affine and proximal nonexpansive mapping such that $T(A_0) \subseteq B_0$ and $g : A \to A$ is an isometry mapping such that $A_0 \subseteq g(A_0)$. Then there exists an element $x \in A_0$ such that $\nu_{gx-Tx}(t) = \nu_{A-B}(t)$ for all $t > 0$.*

Proof Fix $z \in A_0$ and $i \in (0, 1)$. We define the mapping $T_i : A \to B$ by

$$T_i x = (1 - i)Tz + iTx.$$

We show that T_i is a proximal contraction. Let $u, v, x, y \in A$ be such that

$$\nu_{u-T_i x}(t) = \nu_{A-B}(t) = \nu_{v-T_i y}(t) \quad (t > 0).$$

Since T is an affine mapping, we have

$$\nu_{u-T((1-i)z+ix)}(t) = \nu_{A-B}(t) = \nu_{v-T((1-i)z+iy)}(t) \quad (t > 0).$$

So by the hypothesis we have

$$\begin{aligned} \nu_{u-v}(t) &\geq \nu_{(1-i)z+ix-(1-i)z-iy}(t) \\ &= \nu_{i(x-y)}(t) = \nu_{x-y}\left(\frac{t}{i}\right) \quad (t > 0). \end{aligned}$$

Hence, T_i is a proximal contraction. Let $x \in A_0$, so that $Tx \in B_0$ and $Tz \in B_0$. Therefore, there exist $u, v \in A_0$ such that

$$\nu_{u-Tx}(t) = \nu_{A-B}(t) = \nu_{v-Tz}(t) \quad (t > 0).$$

Put $y = iu + (1-i)v \in A$. Then

$$\begin{aligned} \nu_{y-T_i x}(t) &= \nu_{iu+(1-i)v-(1-i)Tz-iTx}(t) \\ &= \nu_{i(u-Tx)+(1-i)(v-Tz)}(t) \\ &\geq \Delta_m\big(\nu_{i(u-Tx)}(it), \nu_{(1-i)(v-Tz)}\big((1-i)t\big)\big) \\ &= \Delta_m\big(\nu_{u-Tx}(t), \nu_{v-Tz}(t)\big) \\ &= \Delta_m\big(\nu_{A-B}(t), \nu_{A-B}(t)\big) = \nu_{A-B}(t) \quad (t > 0), \end{aligned}$$

and thus $T_i(A_0) \subseteq B_0$. By Corollary 2.3 there exists a unique $x_i \in A_0$ such that $\nu_{gx_i - T_i x_i}(t) = \nu_{A-B}(t)$ for all $t > 0$. Fix $j \in (0,1)$. Then

$$\begin{aligned} \nu_{gx_i - Tx_i}(t) &\geq \Delta_m\big(\nu_{gx_i - T_i x_i}(jt), \nu_{T_i x_i - Tx_i}\big((1-j)t\big)\big) \\ &= \Delta_m\big(\nu_{A-B}(jt), \nu_{(1-i)(Tz-Tx_i)}\big((1-j)t\big)\big) \\ &= \Delta_m\left(\nu_{A-B}(jt), \nu_{Tz-Tx_i}\left(\frac{(1-j)t}{1-i}\right)\right) \\ &\geq \Delta_m\left(\nu_{A-B}(jt), D_B\left(\frac{(1-j)t}{1-i}\right)\right) \quad (t > 0). \end{aligned}$$

Now letting $i \to 1$, we obtain

$$\lim_{i\to 1} \nu_{gx_i - Tx_i}(t) \geq \Delta_m\big(\nu_{A-B}(jt), 1\big) = \nu_{A-B}(jt) \quad \big(\forall j \in (0,1), t > 0\big).$$

Then letting $j \to 1$, we have

$$\lim_{i\to 1} \nu_{gx_i - Tx_i}(t) = \nu_{A-B}(t) \quad (t > 0).$$

So we can create a sequence (x_n) in A_0 such that

$$\nu_{gx_n - Tx_n}(t) \to \nu_{A-B}(t) \quad (t > 0).$$

Since A_0 is compact, the sequence (x_n) has a subsequence (x_{n_k}) such that $x_{n_k} \to x \in A_0$. By Remark 1.5, g is continuous mapping, and so $g - T$ is a continuous mapping by Remark 1.8. Indeed, since Δ_m is a continuous t-norm, $p \to \nu_P$ is continuous ([27], Chapter 12), and we get

$$\nu_{gx-Tx}(t) = \lim_{k\to\infty} \nu_{gx_{n_k} - Tx_{n_k}}(t) = \nu_{A-B}(t),$$

as required. □

Theorem 2.5 *Let (X, F, Δ) be a complete probabilistic Menger space such that Δ is a t-norm of H-type, and (A, B) be a pair of subsets of X with the weak P-property such that A_0 is a nonempty closed set. If $T : A \to B$ is a contraction mapping such that $T(A_0) \subseteq B_0$, then there exists a unique x in A such that $F_{x,Tx}(t) = F_{A,B}(t)$ for all $t > 0$.*

Proof It is a direct consequence of Remark 1.20 and Lemma 2.1. □

Clearly, the pair (A, A) has the P-property, so we have the following result.

Corollary 2.6 *Let (X, F, Δ) be a complete probabilistic Menger space such that Δ is a t-norm of H-type. Then every contraction self-mapping from each nonempty closed subset of X has a unique fixed point.*

Theorem 2.7 *Let (X, ν, Δ_m) be a probabilistic Banach space, and (A, B) be a semisharp proximinal pair of X such that A is a p-star-shaped set, A_0 be a nonempty compact set, B be a q-star-shaped set, and let $\nu_{p-q}(t) = \nu_{A-B}(t)$ for all $t > 0$. If $T : A \to B$ is a proximal nonexpansive mapping such that $T(A_0) \subseteq B_0$, then there exists an element $x \in A_0$ such that $\nu_{x-Tx}(t) = \nu_{A-B}(t)$ for all $t > 0$.*

Proof For each integer $i \geq 1$, define $T_i : A_0 \to B_0$ by

$$T_i(x) = \left(1 - \frac{1}{i}\right) Tx + \frac{1}{i} q \quad (x \in A_0).$$

Then by the hypothesis we have $T_i(A_0) \subseteq B_0$. Next, we show that for each i, T_i is a proximal contraction with $\alpha = 1 - \frac{1}{i} < 1$. To do this, suppose that $x, y, u, v, s, r \in A_0$ and $t > 0$ are such that

$$\nu_{u-T_i x}(t) = \nu_{v-T_i y}(t) = \nu_{A_0-B_0}(t) = \nu_{A-B}(t) = \nu_{s-Tx}(t) = \nu_{r-Ty}(t).$$

Now we define

$$u' = \left(1 - \frac{1}{i}\right) s + \frac{1}{i} p \in A_0, \qquad v' = \left(1 - \frac{1}{i}\right) r + \frac{1}{i} p \in A_0,$$

so we have

$$\begin{aligned}
\nu_{A-B}(t) &\geq \nu_{u'-T_i x}(t) = \nu_{(1-\frac{1}{i})s+\frac{1}{i}p-(1-\frac{1}{i})Tx-\frac{1}{i}q}(t) \\
&= \nu_{(1-\frac{1}{i})(s-Tx)+\frac{1}{i}(p-q)}(t) \\
&\geq \Delta_m\left(\nu_{(1-\frac{1}{i})(s-Tx)}\left(t\left(1-\frac{1}{i}\right)\right), \nu_{\frac{1}{i}(p-q)}\left(t\left(\frac{1}{i}\right)\right)\right) \\
&= \Delta_m\bigl(\nu_{s-Tx}(t), \nu_{p-q}(t)\bigr) \\
&= \Delta_m\bigl(\nu_{A-B}(t), \nu_{A-B}(t)\bigr) = \nu_{A-B}(t).
\end{aligned}$$

Hence, $\nu_{u'-T_i x}(t) = \nu_{A-B}(t)$. Since $\nu_{u-T_i x}(t) = \nu_{A-B}(t)$ and (A,B) is a semisharp proximinal pair, we have $u' = u$. By the same method we also have $v' = v$. Since T is a proximal nonexpansive mapping, we have

$$\begin{aligned}
\nu_{u-v}(t) &= \nu_{u'-v'}(t) = \nu_{(1-\frac{1}{i})(s-r)}(t) \\
&= \nu_{s-r}\left(\frac{t}{1-\frac{1}{i}}\right) \geq \nu_{x-y}\left(\frac{t}{1-\frac{1}{i}}\right).
\end{aligned}$$

Therefore, T_i is a proximal contraction with $\alpha = 1 - \frac{1}{i} < 1$. By Lemma 2.1, for each $i \geq 1$, there exists a unique $u_i \in A_0$ such that $\nu_{u_i - T_i u_i}(t) = \nu_{A_0-B_0}(t) = \nu_{A-B}(t)$. Since A_0 is compact and $(u_i) \subseteq A_0$, without loss of generality, we can assume that u_i is a convergent sequence and $u_i \to x \in A_0$.

For each $i \geq 1$, since $T(u_i) \in T(A_0) \subseteq B_0$, there exists $v_i \in A_0$ such that $\nu_{v_i - Tu_i}(t) = \nu_{A-B}(t)$. So we have

$$\begin{aligned}
\nu_{A-B}(t) &\geq \nu_{(1-\frac{1}{i})v_i+\frac{1}{i}p-T_i u_i}(t) \\
&= \nu_{(1-\frac{1}{i})v_i+\frac{1}{i}p-(1-\frac{1}{i})Tu_i-\frac{1}{i}q}(t) \\
&\geq \Delta_m\left(\nu_{(1-\frac{1}{i})(v_i-Tu_i)}\left(t\left(1-\frac{1}{i}\right)\right), \nu_{\frac{1}{i}(p-q)}\left(t\left(\frac{1}{i}\right)\right)\right) \\
&= \Delta_m\bigl(\nu_{v_i-Tu_i}(t), \nu_{p-q}(t)\bigr) \\
&= \Delta_m\bigl(\nu_{A-B}(t), \nu_{A-B}(t)\bigr) = \nu_{A-B}(t).
\end{aligned}$$

Thus, $\nu_{A-B}(t) = \nu_{(1-\frac{1}{i})v_i+\frac{1}{i}p-T_i u_i}(t)$. Since (A,B) is a semisharp proximinal pair and $\nu_{A-B}(t) = \nu_{u_i-T_i u_i}(t)$, we have $u_i = (1-\frac{1}{i})v_i + \frac{1}{i}p$, and so

$$\nu_{u_i-v_i}(t) = \nu_{\frac{1}{i}(v_i-p)}(t) = \nu_{v_i-p}(it).$$

Since A_0 is compact and $(v_i) \subseteq A_0$, without loss of generality, we can assume that v_i is a convergent sequence and $v_i \to z \in A_0$. For every $j \leq i$, we have

$$\nu_{u_i-v_i}(t) = \nu_{v_i-p}(it) \geq \nu_{v_i-p}(jt) \geq \Delta_m\left(\nu_{v_i-z}\left(\frac{j}{2}t\right), \nu_{z-p}\left(\frac{j}{2}t\right)\right).$$

Letting $i \to \infty$, we have

$$\lim_{i\to\infty} \nu_{u_i - v_i}(t) \geq \nu_{z-p}\left(\frac{j}{2}t\right) \quad (\forall j \geq 1).$$

Now letting $j \to \infty$, we have

$$\lim_{i\to\infty} \nu_{u_i - v_i}(t) \geq \lim_{j\to\infty} \nu_{z-p}\left(\frac{j}{2}t\right) = 1.$$

Therefore, $\nu_{u_i - v_i}(t) \to 1$, so that $z = \lim_{i\to\infty} v_i = \lim_{i\to\infty} u_i = x$. Since $Tx \in B_0$, there must exist $u \in A_0$ such that $\nu_{A-B}(t) = \nu_{u-Tx}(t)$. Since we know that $\nu_{A-B}(t) = \nu_{v_i - Tu_i}(t)$ and T is a proximal nonexpansive mapping, it follows that $\nu_{v_i - u}(t) \geq \nu_{u_i - x}(t) \to 1$. This implies that $u = \lim_{i\to\infty} v_i = x$ and then $\nu_{A-B}(t) = \nu_{x-Tx}(t)$, as required. □

Theorem 2.8 *Let (X, ν, Δ_m) be a probabilistic Banach space, (A, B) be a semisharp proximinal pair of X with the weak P-property such that A is a p-star-shaped set, A_0 be a nonempty compact set, B be a q-star-shaped set, and let $\nu_{p-q}(t) = \nu_{A-B}(t)$ for all $t > 0$. If $T : A \to B$ is a nonexpansive mapping such that $T(A_0) \subseteq B_0$, then T has a best proximity point in A_0.*

Proof It is a direct consequence of Remark 1.20 and Theorem 2.7. □

Proposition 2.9 *Let (X, F, Δ) be a probabilistic Menger space, and $A, B \subseteq X$ be such that A_0 is a nonempty set. Suppose that $T : A \to B$ is a proximal nonexpansive mapping such that $T(A_0) \subseteq B_0$ and $g : A \to A$ is an isometry mapping such that $A_0 \subseteq g(A_0)$. Denote $G = g(A)$ and*

$$G_0 = \{z \in G : \exists y \in B \text{ s.t. } \forall t \geq 0, F_{z,y}(t) = F_{G,B}(t)\}.$$

Then Tg^{-1} is a proximal nonexpansive, and $G_0 = A_0$.

Proof The result follows by using a similar argument as in the proof of Proposition 2.2. □

The following theorem is an immediate consequence of Theorem 2.7 and Proposition 2.9.

Theorem 2.10 *Let (X, ν, Δ_m) be a probabilistic Banach space, (A, B) be a semisharp proximinal pair of X such that A is a p-star-shaped set, A_0 be a nonempty compact set, B be a q-star-shaped set, and let $\nu_{p-q}(t) = \nu_{A-B}(t)$ for all $t > 0$. If $T : A \to B$ is a proximal nonexpansive mapping such that $T(A_0) \subseteq B_0$ and $g : A \to A$ is an isometry mapping such that $A_0 \subseteq g(A_0)$, then there exists an element $x \in A_0$ such that $\nu_{gx-Tx}(t) = \nu_{A-B}(t)$ for all $t > 0$.*

Corollary 2.11 *Let (X, ν, Δ_m) be a probabilistic Banach space, and let (A, B) be a pair of convex subsets of X with the P-property such that A_0 is a nonempty compact set. If $T : A \to B$ is a nonexpansive mapping such that $T(A_0) \subseteq B_0$ and $g : A \to A$ is an isometry mapping such that $A_0 \subseteq g(A_0)$, then there exists an element $x \in A_0$ such that $\nu_{gx-Tx}(t) = \nu_{A-B}(t)$ for all $t > 0$.*

In Corollary 2.11, if $g(x) = x$, then we have the following corollary.

Corollary 2.12 *With the hypotheses of the previous corollary, if $T : A \to B$ is a nonexpansive mapping such that $T(A_0) \subseteq B_0$, then T has a best proximity point.*

In Corollary 2.12, if $A = B$, then we have the following corollary.

Corollary 2.13 *If A is a nonempty, compact, and convex subset of a probabilistic Banach space (X, ν, Δ_m) and $T : A \to A$ is a nonexpansive mapping, then T has a fixed point.*

In the following, we give some examples that defend our main results.

Example 2.14 Let $X = \mathbb{R}^2$, $A = \{(0,y) : y \in \mathbb{R}\}$ and $B = \{(1,y) : y \in \mathbb{R}\}$. Suppose that $T : A \to B$ is defined by $T(0,y) = (1, \frac{y}{4})$, $g : A \to A$ is defined by $g(0,y) = (0,-y)$, and $F_{(x,x'),(y,y')}(t) = \frac{t}{t+|x-y|+|x'-y'|}$. It is easy to see that (X, F, Δ_m) is a complete probabilistic Menger space, $F_{A,B}(t) = \frac{t}{t+1}$, $A_0 = A$, $B_0 = B$, $T(A_0) \subseteq B_0$, and

$$F_{g(0,x),g(0,y)}(t) = F_{(0,-x),(0,-y)}(t) = \frac{t}{t + |x - y|} = F_{(0,x),(0,y)}(t).$$

If $(0,u), (0,x), (0,v), (0,y) \in A$ are such that

$$\frac{t}{t + 1 + |u - \frac{x}{4}|} = F_{(0,u),T(0,x)}(t) = F_{A,B}(t) = F_{(0,v),T(0,y)}(t) = \frac{t}{t + 1 + |v - \frac{y}{4}|},$$

then $u = \frac{x}{4}$ and $v = \frac{y}{4}$, so that

$$F_{(0,u),(0,v)}(t) = F_{(0,\frac{x}{4}),(0,\frac{y}{4})}(t) = \frac{t}{t + \frac{1}{4}|x - y|} = F_{(0,x),(0,y)}\left(\frac{t}{\frac{1}{4}}\right).$$

Therefore, all the hypothesis of Corollary 2.3 are satisfied, and we also have

$$F_{(0,0),T(0,0)}(t) = F_{(0,0),(1,0)}(t) = \frac{t}{t + 1} = F_{A,B}(t).$$

Example 2.15 Let $X = \mathbb{R}$, $A = [0,2]$ and $B = [3,5]$. For every $x \in X$, define $\nu_x(t) = \frac{t}{t+|x|}$. It is easy to see that (X, ν, Δ_m) is a probabilistic Banach space, $\nu_{A-B}(t) = \frac{t}{t+1}$, $A_0 = \{2\}$, and $B_0 = \{3\}$. For every $x \in A$, define $T : A \to B$ by $Tx = 5 - x$ and let g be the identity mapping. Clearly, T is a continuous affine and proximal nonexpansive mapping, and $T(A_0) = \{T(2)\} = \{3\} = B_0$. Therefore, all the hypotheses of Theorem 2.4 are satisfied, and also we have

$$\nu_{2-T2}(t) = \nu_{2-3}(t) = \frac{t}{t + 1} = \nu_{A-B}(t).$$

The following example shows that the weak P-property of the pair (A, B) cannot be removed from Theorem 2.5.

Example 2.16 Let $X = \mathbb{R}$, $A = \{-10, 10\}$, $B = \{-2, 2\}$, and $F_{p,q}(t) = \frac{t}{t+|p-q|}$. Clearly, (X, F, Δ_m) is a complete probabilistic Menger space. Then $A_0 = A$, $B_0 = B$, and $F_{A,B}(t) = \frac{t}{t+8}$. Let T :

$A \to B$ be a mapping given by $T(-10) = 2$ and $T(10) = -2$. It is easy to see that for $\alpha = \frac{1}{5}$, T is a contraction mapping with $T(A_0) \subseteq B_0$. The mapping T does not have any best proximity point because $F_{x,Tx}(t) = \frac{t}{t+12} < \frac{t}{t+8} = F_{A,B}(t)$ for all $x \in A$. It should be noted that the pair (A,B) does not have the weak P-property.

Example 2.17 Let $X = \mathbb{R}$, $A = [0,1]$, and $B = [2,3]$. For every $x \in X$, define $\nu_x(t) = \frac{t}{t+|x|}$. It is easy to see that (X, ν, Δ_m) is a probabilistic Banach space, A is 1-star-shaped set, B is 2-star-shaped set,

$$\nu_{A-B}(t) = \sup_{x\in A, y\in B} \nu_{x-y}(t) = \frac{t}{t+1}, \qquad A_0 = \{1\}, \qquad B_0 = \{2\},$$

and

$$\nu_{1-2}(t) = \frac{t}{t+|1-2|} = \frac{t}{t+1} = \nu_{A-B}(t).$$

Also, (A,B) is a semisharp proximinal pair. Now for each $x \in A$, define $T : A \to B$ by $Tx = 3 - x$. If $u, v, x, y \in A$, then

$$\frac{t}{t+|u-3+x|} = \nu_{u-Tx}(t) = \nu_{A-B}(t) = \nu_{v-Ty}(t) = \frac{t}{t+|v-3+y|},$$

so that $u = x = 1$ and $v = y = 1$. Thus,

$$\nu_{u-v}(t) = 1 = \nu_{x-y}(t).$$

So T is a proximal nonexpansive, and $T(A_0) = B_0$. Therefore, all the hypotheses of Theorem 2.7 are satisfied, and we also have

$$\nu_{1-T1}(t) = \nu_{1-2}(t) = \frac{t}{t+1} = \nu_{A-B}(t).$$

Example 2.18 Let $X = \mathbb{R}^2$, $A = \{(x,0) : 0 \le x \le 1\}$, $B_1 = \{(x,y) : x+y = 1, -1 \le x \le 0\}$, $B_2 = \{(x,1) : 0 \le x \le 1\}$, $B = B_1 \cup B_2$, and $\nu_{(x,x')}(t) = \frac{t}{t+|x|+|x'|}$. It is easy to see that (X, ν, Δ_m) is a probabilistic Banach space, $\nu_{A-B}(t) = \frac{t}{t+1}$, B is not convex but is a $(0,1)$-star-shaped set, and A is $(0,0)$-star-shaped set. Clearly, $A_0 = A$ and $B_0 = B_2$. So

$$\nu_{(0,0)-(0,1)}(t) = \frac{t}{t+|0|+|1|} = \frac{t}{t+1} = \nu_{A-B}(t),$$

and (A,B) is a semisharp proximinal pair. Suppose that $T : A \to B$ is defined by

$$T(x,0) = \begin{cases} (0,1), & x = 0, \\ (\sin x, 1), & x \neq 0, \end{cases}$$

and $(u,0), (v,0), (x,0), (y,0) \in A$ are such that

$$\nu_{(u,0)-T(x,0)}(t) = \nu_{A-B}(t) = \frac{t}{t+1} = \nu_{(v,0)-T(y,0)}(t).$$

If $x = y = 0$, then $u = v = 0$, and therefore

$$\nu_{(u,0)-(v,0)}(t) = \nu_{(0,0)-(0,0)}(t) = 1 = \nu_{(x,0)-(y,0)}(t).$$

If $x, y \neq 0$, then $u = \sin x$, $v = \sin y$, and therefore

$$\begin{aligned}\nu_{(u,0)-(v,0)}(t) = \nu_{(\sin x,0)-(\sin y,0)}(t) &= \frac{t}{t + |\sin x - \sin y|}\\ &\geq \frac{t}{t + |x - y|}\\ &= \nu_{(x,0)-(y,0)}(t).\end{aligned}$$

If $x = 0$ and $y \neq 0$, then $u = 0$ and $v = \sin y$, and therefore

$$\nu_{(u,0)-(v,0)}(t) = \nu_{(0,0)-(\sin y,0)}(t) = \frac{t}{t + |\sin y|} \geq \frac{t}{t + |y|} \geq \nu_{(0,0)-(y,0)}(t).$$

If $x \neq 0$ and $y = 0$, then $u = \sin x$ and $v = 0$, and therefore

$$\nu_{(u,0)-(v,0)}(t) = \nu_{(\sin x,0)-(0,0)}(t) = \frac{t}{t + |\sin x|} \geq \frac{t}{t + |x|} \geq \nu_{(x,0)-(0,0)}(t).$$

Hence, T is proximal nonexpansive, and $T(A_0) \subseteq B_2 = B_0$, so all the hypotheses of Theorem 2.7 are satisfied, and we also have

$$\nu_{(0,0)-T(0,0)}(t) = \nu_{(0,0)-(0,1)}(t) = \frac{t}{t+1} = \nu_{A-B}(t).$$

Example 2.19 Let $X = \mathbb{R}$, $A = [0,1]$, $B = [\frac{15}{8}, 2]$, and $\nu_x(t) = \frac{t}{t+|x|}$. Clearly, (X, ν, Δ_m) is a probabilistic Banach space, $\nu_{A-B}(t) = \frac{t}{t+\frac{7}{8}}$, the pair (A, B) has the P-property, $A_0 = \{1\}$, and $B_0 = \{\frac{15}{8}\}$. If $Tx = -\frac{1}{8}x + 2$, then $T(A_0) = \{T(1)\} = \{\frac{15}{8}\} = B_0$. Let $x, y \in A$. Then we have

$$\nu_{Tx-Ty}(t) = \nu_{-\frac{1}{8}(x-y)}(t) = \nu_{x-y}(8t) \geq \nu_{x-y}(t).$$

Therefore, all the hypotheses of Corollary 2.12 are satisfied, and hence T has a best proximity point, and we also have

$$\nu_{1-T1}(t) = \nu_{1-\frac{15}{8}}(t) = \frac{t}{t+\frac{7}{8}} = \nu_{A-B}(t).$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors have contributed in obtaining the new results presented in this article. All authors read and approved the final manuscript.

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