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The Banach contraction principle in C^*-algebra-valued b-metric spaces with application

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Abstract
We introduce the notion of a C^*-algebra-valued b-metric space. We generalize the Banach contraction principle in this new setting. As an application of our result, we establish an existence result for an integral equation in a C^*-algebra-valued b-metric space.

1 Introduction

The Banach contraction principle [1], also known as the Banach fixed point theorem, is one of the main pillars of the theory of metric fixed points. According to this principle, if T is a contraction on a Banach space X, then T has a unique fixed point in X. Many researchers investigated the Banach fixed point theorem in many directions and presented generalizations, extensions, and applications of their findings. Among them, Bakhtin [2] introduced a prominent generalization of the idea of a metric space, which is later used by Czerwick [3, 4]. They introduced and used the concept of real-valued b-metric space to establish certain fixed point results. The idea clearly is an extension of the metric space as follows from the following definition.

Definition 1.1 ([5]) Let X be a nonempty set, and $b \in \mathbb{R}$ be such that $b \geq 1$. A b-metric on X is a real-valued mapping $d_b\colon X \times X \to \mathbb{R}$ that satisfies the following conditions for all $x, y, z \in X$:

(1) $d_b(x,y) \geq 0$ and $d_b(x,y) = 0 \Leftrightarrow x = y$.
(2) $d_b(y,x) = d(x,y)$ (symmetry).
(3) $d_b(y,z) \leq b[d_b(y,x) + d_b(x,z)]$.

By a b-metric space with coefficient b we mean the pair (X, d_b).

For recent development on b-metric spaces, we refer to [5–10].

Recently, Ma *et al.* [11] presented their work on the extension of Banach contraction principle for C^*-algebra-valued metric spaces. Later, Batul and Kamran [12] introduced the notion of a C^*-valued contractive type mapping and established a fixed point result in this setting. Motivated by the ideas and results presented in [11, 12], in this paper, we will introduce a new notion of C^*-algebra-valued b-metric space and establish a fixed point result in such spaces.

We now recollect some basic definitions, notation, and results. The details on C^*-algebras are available in [13, 14].

An algebra $\mathbb{A}$, together with a conjugate linear involution map $a \mapsto a^*$, is called a $*$-algebra if $(ab)^* = b^*a^*$ and $(a^*)^* = a$ for all $a, b \in \mathbb{A}$. Moreover, the pair $(\mathbb{A}, *)$ is called a unital $*$-algebra if $\mathbb{A}$ contains the identity element $1_{\mathbb{A}}$. By a Banach $*$-algebra we mean a complete normed unital $*$-algebra $(\mathbb{A}, *)$ such that the norm on $\mathbb{A}$ is submultiplicative and satisfies $\|a^*\| = \|a\|$ for all $a \in \mathbb{A}$. Further, if for all $a \in \mathbb{A}$, we have $\|a^*a\| = \|a\|^2$ in a Banach $*$-algebra $(\mathbb{A}, *)$, then $\mathbb{A}$ is known as a C^*-algebra. A positive element of $\mathbb{A}$ is an element $a \in \mathbb{A}$ such that $a = a^*$ and its spectrum $\sigma(a) \subset \mathbb{R}_+$, where $\sigma(a) = \{\lambda \in \mathbb{R} : \lambda 1_{\mathbb{A}}\text{-}a \text{ is noninvertible}\}$. The set of all positive elements will be denoted by $\mathbb{A}_+$. Such elements allow us to define a partial ordering '$\succeq$' on the elements of $\mathbb{A}$. That is,

$$b \succeq a \quad \text{if and only if} \quad b - a \in \mathbb{A}_+.$$

If $a \in \mathbb{A}$ is positive, then we write $a \succeq 0_{\mathbb{A}}$, where $0_{\mathbb{A}}$ is the zero element of $\mathbb{A}$. Each positive element a of a C^*-algebra $\mathbb{A}$ has a unique positive square root. From now on, by $\mathbb{A}$ we mean a unital C^*-algebra with identity element $1_{\mathbb{A}}$. Further, $\mathbb{A}_+ = \{a \in \mathbb{A} : a \succeq 0_{\mathbb{A}}\}$ and $(a^*a)^{1/2} = |a|$. Using the concept of positive elements in $\mathbb{A}$, a C^*-algebra-valued metric d on a nonempty set X is defined in [11] as a mapping $d \colon X \times X \to \mathbb{A}_+$ that satisfies, for all $x_1, x_2, x_3 \in X$, (i) $d(x_1, x_2) = 0_{\mathbb{A}} \Leftrightarrow x_1 = x_2$, (ii) $d(x_1, x_2) = d(x_2, x_1)$, and (iii) $d(x_1, x_2) \preceq d(x_1, x_3) + d(x_3, x_2)$. The triplet $(X, \mathbb{A}, d)$ is then called a C^*-algebra-valued metric space.

2 Main results

In this section, we extend Definition 1.1 to introduce the notion b-metric space in the setting of C^*-algebras as follows.

Definition 2.1 Let $\mathbb{A}$ be a C^*-algebra, and X be a nonempty set. Let $b \in \mathbb{A}$ be such that $\|b\| \geq 1$. A mapping $d_b \colon X \times X \to \mathbb{A}_+$ is said to be a C^*-algebra-valued b-metric on X if the following conditions hold for all $x_1, x_2, x_3 \in \mathbb{A}$:

(BM1) $d_b(x_1, x_2) = 0_{\mathbb{A}} \Leftrightarrow x_1 = x_2$.

(BM2) d_b is symmetric, that is, $d_b(x_1, x_2) = d_b(x_2, x_1)$.

(BM3) $d_b(x_1, x_2) \preceq b[d_b(x_1, x_3) + d_b(x_3, x_2)]$.

The triplet $(X, \mathbb{A}, d_b)$ is called a C^*-algebra-valued b-metric space with coefficient b.

Remark 2.1 Note that:

(1) If we take $\mathbb{A} = \mathbb{R}$, then the new notion of C^*-algebra-valued b-metric space becomes equivalent to Definition 1.1 of the real b-metric space.

(2) If we take $b = 1_{\mathbb{A}}$ in Definition 2.1, then d_b becomes the usual C^*-algebra-valued metric as defined in [11].

Thus, the class of ordinary C^*-algebra-valued metric spaces is clearly smaller than the class of C^*-algebra-valued b-metric spaces. In fact, there are C^*-algebra-valued b-metric spaces that are not C^*-algebra-valued metric spaces, as illustrated by the following example.

Example 2.1 Let $X = \ell_p$ be the set of sequences $\{x_n\}$ in $\mathbb{R}$ such that $\sum_{n=1}^{\infty} |x_n|^p < \infty$ and $0 < p < 1$. Let $\mathbb{A} = M_2(\mathbb{R})$. For $x = x_n, y = y_n \in \ell_p$, define $d_b : X \times X \to \mathbb{A}$ as follows:

$$d_b(x,y) = \begin{pmatrix} (\sum_{n=1}^{\infty} |x_n - y_n|^p)^{\frac{1}{p}} & 0 \\ 0 & (\sum_{n=1}^{\infty} |x_n - y_n|^p)^{\frac{1}{p}} \end{pmatrix}.$$

Then one can show that d_b is a C^*-algebra-valued b-metric space with coefficient $b = \begin{pmatrix} 2^{\frac{1}{p}} & 0 \\ 0 & 2^{\frac{1}{p}} \end{pmatrix}$ such that $\|b\| = 2^{\frac{1}{p}}$. The claim follows from the following observation in [4]:

$$\left(\sum_{n=1}^{\infty} |x_n - z_n|^p\right)^{\frac{1}{p}} \leq 2^{\frac{1}{p}} \left[\left(\sum_{n=1}^{\infty} |x_n - y_n|^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{\infty} |y_n - z_n|^p\right)^{\frac{1}{p}}\right].$$

Note that here d_b is not a usual C^*-algebra-valued metric on X.

From now on, we call a C^*-algebra-valued b-metric space simply a C^*-valued b-metric, and the triplet $(X, \mathbb{A}, d_b)$ is then called a C^*-valued b-metric space. Given $(X, \mathbb{A}, d_b)$, the following are natural deductions from the corresponding notions in C^*-valued metric spaces.

(1) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ with respect to the algebra $\mathbb{A}$ if and only if for any $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that $\|d_b(x_n, x)\| < \epsilon$ for all $n > N$. Symbolically, we then write $\lim_{n\to\infty} x_n = x$.
(2) If for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\|d_b(x_n, x_m)\| < \epsilon$ for all $n, m > N$, then the sequence $\{x_n\}$ is called a Cauchy sequence with respect to $\mathbb{A}$.
(3) If every Cauchy sequence in X is convergent with respect to $\mathbb{A}$, then the triplet $(X, \mathbb{A}, d)$ is called a complete C^*-valued b-metric space.

Definition 2.2 Let $(X, \mathbb{A}, d_b)$ be a C^*-valued b-metric space. A contraction on X is a mapping $T\colon X \to X$ if there exists an $a \in \mathbb{A}$ with $\|a\| < 1$ such that

$$d_b(Tx, Ty) \preceq a^* d_b(x,y) a \quad \text{for all } x, y \in X. \tag{1}$$

Example 2.2 Let $\mathbb{A} = \mathbb{R}^2$ and $X = [0, \infty)$. Let $\preceq$ be the partial order on $\mathbb{A}$ given by

$$(a_1, b_1) \preceq (a_2, b_2) \quad \Leftrightarrow \quad a_1 \leq a_2 \text{ and } b_1 \leq b_2.$$

Define

$$d_b\colon X \times X \to \mathbb{A}, \qquad d_b(x,y) = \bigl((x-y)^2, 0\bigr).$$

Then d_b is C^*-valued b-metric with coefficient $(2, 0)$, and with this d_b, the triplet $(X, \mathbb{A}, d_b)$ becomes a C^*-valued b-metric. Consider $T\colon X \to X$ given by $Tx = \frac{x}{3} + 5$; then T is a contraction on X with $a = (\frac{1}{3}, 0)$:

$$d_b(Tx, Ty) = \bigl((Tx - Ty)^2, 0\bigr) = \left(\left(\frac{x}{3} - \frac{y}{3}\right)^2, 0\right) = \left(\frac{1}{3}, 0\right) d_b(x,y) \left(\frac{1}{3}, 0\right).$$

Theorem 2.1 *Consider a complete C^*-valued b-metric space $(X, \mathbb{A}, d_b)$ with coefficient b. Let $T: X \to X$ be a contraction with the contraction constant a such that $\|b\|\|a\|^2 < 1$. Then T has a unique fixed point in X.*

Proof If $\mathbb{A} = \{0_{\mathbb{A}}\}$, then there is nothing to prove. Assume that $\mathbb{A} \neq \{0_{\mathbb{A}}\}$.

Choose $x_0 \in X$ and define inductively a sequence $\{x_n\}$ by the iterative scheme as

$$x_{n+1} = Tx_n.$$

Then it follows that $x_n = T^n x_0$ for $n = 0, 1, 2, \ldots$. From the contraction condition (1) on T it follows that

$$\begin{aligned} d_b(x_n, x_{n+1}) &= d_b(Tx_{n-1}, Tx_n) \\ &\preceq a^* d_b(x_{n-1}, x_n) a \\ &= a^* d_b(Tx_{n-2}, Tx_{n-1}) a \\ &\preceq (a^*)^2 d_b(x_{n-2}, x_{n-1}) a^2 \\ &\preceq (a^*)^3 d_b(x_{n-3}, x_{n-2}) a^3 \preceq (a^*)^n d_b(x_0, x_1) a^n = (a^*)^n D a^n, \end{aligned}$$

where $D = d_b(x_0, x_1)$.

Now suppose that $m > n$; then the triangle inequality (BM3) for the b-metric d_b implies

$$\begin{aligned} d_b(x_n, x_m) &\preceq bd(x_n, x_{n+1}) + b^2 d(x_{n+1}, x_{n+2}) + \cdots + b^{m-n-1} d(x_{m-2}, x_{m-1}) \\ &\quad + b^{m-n-1} d(x_{m-1}, x_m) \\ &\preceq b(a^*)^n D a^n + b^2 (a^*)^{n+1} D a^{n+1} + \cdots + b^{m-n-1} (a^*)^{m-2} D a^{m-2} \\ &\quad + s^{m-n-1} (a^*)^{m-1} D a^{m-1} \\ &= b\bigl[(a^*)^n D a^n + b(a^*)^{n+1} D a^{n+1} + \cdots + b^{m-n-2} (a^*)^{m-2} D a^{m-2}\bigr] \\ &\quad + b^{m-n-1} (a^*)^{m-1} D a^{m-1} \\ &= b \sum_{k=n}^{m-2} b^{k-n} (a^*)^k D a^k + b^{m-n-1} (a^*)^{m-1} D a^{m-1} \\ &= b \sum_{k=n}^{m-1} b^{k-n} (a^*)^k D^{\frac{1}{2}} D^{\frac{1}{2}} a^k + b^{m-n-1} (a^*)^{m-1} D^{\frac{1}{2}} D^{\frac{1}{2}} a^{m-1} \\ &= b \sum_{k=n}^{m-1} b^{k-n} \bigl(D^{\frac{1}{2}} a^k\bigr)^* \bigl(D^{\frac{1}{2}} a^k\bigr) + b^{m-n-1} \bigl(D^{\frac{1}{2}} a^{m-1}\bigr)^* \bigl(D^{\frac{1}{2}} a^{m-1}\bigr) \\ &= b \sum_{k=n}^{m-1} b^{k-n} \bigl|D^{\frac{1}{2}} a^k\bigr|^2 + b^{m-n-1} \bigl|D^{\frac{1}{2}} a^{m-1}\bigr|^2 \\ &\preceq \left\| b \sum_{k=n}^{m-1} b^{k-n} \bigl|D^{\frac{1}{2}} a^k\bigr|^2 \right\| 1_{\mathbb{A}} + \bigl\| b^{m-n-1} \bigl|D^{\frac{1}{2}} a^{m-1}\bigr|^2 \bigr\| 1_{\mathbb{A}} \\ &\preceq \|b\| \sum_{k=n}^{m-1} \bigl\|b^{k-n}\bigr\| \bigl\|D^{\frac{1}{2}}\bigr\|^2 \bigl\|a^k\bigr\|^2 1_{\mathbb{A}} + \bigl\|b^{m-n-1}\bigr\| \bigl\|D^{\frac{1}{2}}\bigr\|^2 \bigl\|a^{m-1}\bigr\|^2 1_{\mathbb{A}} \end{aligned}$$

$$
\begin{aligned}
&\preceq \|b\| \sum_{k=n}^{m-1} \|b\|^{k-n} \big\|D^{\frac{1}{2}}\big\|^2 \big\|a^k\big\|^2 1_{\mathbb{A}} + \|b\|^{m-n-1} \big\|D^{\frac{1}{2}}\big\|^2 \big\|a^{m-1}\big\|^2 1_{\mathbb{A}} \\
&\preceq \|b\|^{1-n} \big\|D^{\frac{1}{2}}\big\|^2 \sum_{k=n}^{m-1} \|b\|^k \big\|a^2\big\|^k 1_{\mathbb{A}} + \|b\|^{-n} \|b\|^{m-1} \big\|D^{\frac{1}{2}}\big\|^2 \big\|a^{m-1}\big\|^2 1_{\mathbb{A}} \\
&\preceq \|b\|^{1-n} \big\|D^{\frac{1}{2}}\big\|^2 \sum_{k=n}^{m-1} \big(\|b\| \big\|a^2\big\|\big)^k 1_{\mathbb{A}} + \|b\|^{-n} \big\|D^{\frac{1}{2}}\big\|^2 \big(\|b\| \big\|a^2\big\|\big)^{m-1} 1_{\mathbb{A}} \\
&\longrightarrow 0_{\mathbb{A}} \quad \text{as } m, n \to \infty,
\end{aligned}
$$

which follows from the observation that the summation in the first term is a geometric series, and $\|b\| \|a^2\| < 1$ implies that both $(\|b\| \|a^2\|)^{m-1} \to 0$ and $(\|b\| \|a^2\|)^{n-1} \to 0$. This proves that $\{x_n\}$ is a Cauchy sequence in X with respect to $\mathbb{A}$, and from the completeness of $(X, \mathbb{A}, d)$ it follows that $x_n \to x \in X$, that is,

$$
\lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_{n-1} = x.
$$

We claim that x is a fixed point of T. In fact, from the triangle inequality (BM3) and the contraction condition (1) we have:

$$
\begin{aligned}
0_{\mathbb{A}} &\preceq d(Tx, x) \\
&\preceq b\big[d(Tx, Tx_n) + d(Tx_n, x)\big] \\
&\preceq ba^* d(x, x_n) a + d(x_{n-1}, x) \longrightarrow 0_{\mathbb{A}} \quad \text{as } n \to \infty.
\end{aligned}
$$

This shows that $Tx = x$.

To prove that x is the unique fixed point, we suppose that $y \in X$ is another fixed point of T. Then again from the contraction condition (1) we have

$$
0_{\mathbb{A}} \preceq d(x, y) = d(Tx, Ty) \preceq a^* d(x, y) a.
$$

Using the norm of $\mathbb{A}$, we have

$$
0 \le \big\|d(x, y)\big\| \le \big\|a^* d(x, y) a\big\| \le \big\|a^*\big\| \big\|d(x, y)\big\| \|a\| = \|a\|^2 \big\|d(x, y)\big\|.
$$

The above inequality holds only when $d(x, y) = 0_{\mathbb{A}}$. Hence, $x = y$. □

Example 2.3 The mapping T of Example 2.2 satisfies the hypothesis of Theorem 2.1, and T has unique fixed point $x = 1.5$ in X.

Remark 2.2 Theorem 2.1 generalizes the following results.

(1) By taking $\mathbb{A} = \mathbb{R}$, the C^*-valued b-metric becomes simply the b-metric, and we immediately get the Banach contraction principle in b-metric spaces from Theorem 2.1.

(2) Taking $b = 1$, [11], Theorem 2.1, becomes a special case of Theorem 2.1.

3 Application

As an application of the fixed point theorem for contractions on a C^*-valued complete b-metric space, we provide an existence result for a class of integral equations.

Example 3.1 Let E be a Lebesgue-measurable set and $X = L^\infty(E)$. Consider the Hilbert space $L^2(E)$. Let the set of all bounded linear operators on $L^2(E)$ be denoted by $BL(L^2(E))$. Note that $BL(L^2(E))$ is a C^*-algebra with usual operator norm. For $S, T \in X$, define

$$d_b\colon X \times X \to BL\bigl(L^2(E)\bigr), \qquad d_b(T,S) = \pi_{(T-S)^2},$$

where $\pi_h\colon L^2(E) \to L^2(E)$ is the product operator given by

$$\pi_h(f) = h \cdot f \quad \text{for } f \in L^2(E).$$

Working in the same lines as in [11], Example 2.1, we can show that $(X, BL(L^2(E)), d_b)$ is a complete C^*-valued b-metric space. With these settings, suppose that there exist a continuous function $f\colon E \times E \to \mathbb{R}$ and a constant $0 < \alpha < 1$ such that for all $x, y \in X$ and $u, v \in E$, we have

$$\bigl|K\bigl(u,v,x(v)\bigr) - K\bigl(u,v,y(v)\bigr)\bigr| \le \alpha\bigl|f(u,v)\bigl(x(v) - y(v)\bigr)\bigr|, \tag{2}$$

where K is a function from $E \times E \times \mathbb{R}$ to $\mathbb{R}$, and $\sup_{t\in E}\int_E |f(u,v)|\,dv \le 1$. Then the integral equation

$$x(u) = \int_E K\bigl(u,v,x(v)\bigr)\,dv, \quad u \in E$$

has a unique solution.

Proof Here $(X, BL(L^2(E)), d_b)$ is a C^*-valued complete b-metric space with respect to $BL(L^2(E))$.

Let

$$T\colon X \to X, \quad Tx(u) = \int_E K\bigl(u,v,x(v)\bigr)\,dv, \quad u \in E.$$

Then

$$\begin{aligned}
\bigl\|d(Tx,Ty)\bigr\| &= \|\pi_{(Tx-Ty)^2}\| \\
&= \sup_{\|g\|=1}\langle \pi_{(Tx-Ty)^2}g, g\rangle \quad \text{for every } g \in L^2(E) \\
&= \sup_{\|g\|=1}\int_E (Tx-Ty)^2 g(u)\overline{g(u)}\,dv \\
&= \sup_{\|g\|=1}\int_E \biggl[\int_E \bigl(K\bigl(u,v,x(v)\bigr) - K\bigl(u,v,y(v)\bigr)\bigr)\,dv\biggr]^2 g(u)\overline{g(u)}\,du \\
&\le \sup_{\|g\|=1}\int_E \biggl[\int_E \bigl(K\bigl(u,v,x(v)\bigr) - K\bigl(u,v,y(v)\bigr)\bigr)\,dv\biggr]^2 \bigl|g(u)\bigr|^2\,du
\end{aligned}$$

$$
\begin{aligned}
&\leq \sup_{\|g\|=1} \int_E \alpha^2 \left[\int_E \bigl(f(u,v)\bigl(x(v)-y(v)\bigr)\bigr)\, dv \right]^2 \bigl|g(u)\bigr|^2\, du \\
&\leq \alpha^2 \sup_{\|g\|=1} \int_E \left[\int_E \bigl|f(u,v)\bigr|\, dv \right]^2 \bigl|g(u)\bigr|^2\, du \cdot \bigl\|(x-y)^2\bigr\|_\infty \\
&\leq \alpha^2 \sup_{t\in E} \int_E \bigl|f(u,v)\bigr|^2\, dv \cdot \sup_{\|g\|=1} \int_E \bigl|g(u)\bigr|^2\, du \cdot \bigl\|(x-y)^2\bigr\|_\infty \\
&\leq \alpha^2 \bigl\|(x-y)^2\bigr\|_\infty \\
&= \|a\| \bigl\|d(x,y)\bigr\|.
\end{aligned}
$$

Setting $a = \alpha I_2$, we have $a \in BL(L^2(E))_+$ and $\|a\| = \alpha^2 < 1$. Thus, all the conditions of Theorem 2.1 hold, and hence the conclusion. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally to this work. All authors read and approved the final manuscript.

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