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Best proximity point results in partially ordered metric spaces via simulation functions

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Abstract

We obtain sufficient conditions for the existence and uniqueness of best proximity points for a new class of non-self mappings involving simulation functions in a metric space endowed with a partial order. Some interesting consequences including fixed point results via simulation functions are presented.

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1 Introduction

Recently, in [1] the authors introduced the class of simulation functions as follows.

Definition 1.1 We say that $\xi : [0,\infty) \times [0,\infty) \to \mathbb{R}$ is a simulation function if it satisfies the following conditions:

(i) $\xi(0,0) = 0$;

(ii) $\xi(t,s) < s - t$, for every $t, s > 0$;

(iii) if $\{a_n\}$ and $\{b_n\}$ are two sequences in $(0,\infty)$, then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n > 0 \quad \Longrightarrow \quad \limsup_{n\to\infty} \xi(a_n, b_n) < 0.$$

Various examples of simulation functions were presented in [1]. The class of such functions will be denoted by $\mathcal{Z}$.

Definition 1.2 ([1]) Let $T : X \to X$ be a given operator, where X is a nonempty set equipped with a metric d. We say that T is a $\mathcal{Z}$-contraction with respect to $\xi \in \mathcal{Z}$ if

$$\xi\bigl(d(Tx, Ty), d(x, y)\bigr) \geq 0, \quad \text{for all } x, y \in X.$$

In [1], the authors established the following fixed point theorem that generalizes many previous results from the literature including the Banach fixed point theorem.

Theorem 1.3 ([1]) *Let $T : X \to X$ be a given map, where X is a nonempty set equipped with a metric d such that (X, d) is complete. Suppose that T is a $\mathcal{Z}$-contraction with respect* *to $\xi \in \mathcal{Z}$. Then T has a unique fixed point. Moreover, for any $x \in X$, the sequence $\{T^n x\}$ converges to this fixed point.*

For other results via simulation functions, we refer to [2–7].

Let (X,d) be a metric space. Consider a mapping $T : A \to B$, where A and B are nonempty subsets of X. If $d(x, Tx) > 0$ for every $x \in A$, then the set of fixed points of T is empty. In this case, we are interested in finding a point $x \in A$ such that $d(x, Tx)$ is minimum in some sense.

Definition 1.4 We say that $z \in A$ is a best proximity point of T if

$$d(z, Tz) = d(A,B) := \inf\{d(x,y) : x \in A, y \in B\}.$$

Observe that if $d(A,B) = 0$, then a best proximity point of T is a fixed point of T.

The study of the existence of best proximity points is an interesting field of optimization and it attracted recently the attention of several researchers (see [1, 8–23] and the references therein).

In the sequel, we will use the following notations. Set

$$A_0 = \{x \in A : d(x,y) = d(A,B), \text{for some } y \in B\}$$

and

$$B_0 = \{y \in B : d(x,y) = d(A,B), \text{for some } x \in A\}.$$

We refer to [19] for sufficient conditions that guarantee that A_0 and B_0 are nonempty.

Now, we endow the set X with a partial order $\preceq$. Let us introduce the following class of mappings. For a given simulation function $\xi \in \mathcal{Z}$, we denote by $\mathcal{T}_\xi$ the set of mappings $T : A \to B$ satisfying the following conditions:

(C1) for every $x_1, x_2, y_1, y_2 \in A$, we have

$$y_1 \preceq y_2, \quad d(x_1, Ty_1) = d(x_2, Ty_2) = d(A,B) \quad \Longrightarrow \quad x_1 \preceq x_2;$$

(C2) for every $x, y, u_1, u_2 \in A$, we have

$$x \preceq y, x \neq y, \quad d(u_1, Tx) = d(u_2, Ty) = d(A,B) \quad \Longrightarrow \quad \xi\big(d(u_1,u_2), m(x,y)\big) \geq 0,$$

where

$$m(x,y) = \max\left\{\frac{d(x,u_1)d(y,u_2)}{d(x,y)}, d(x,y)\right\}.$$

Our aim in this paper is to study the existence and uniqueness of best proximity points for non-self mappings $T : A \to B$ that belong to the class $\mathcal{T}_\xi$, for some simulation function $\xi \in \mathcal{Z}$.

2 Main results

Our first main result is the following.

Theorem 2.1 *Let $T \in \mathcal{T}_\xi$, for some $\xi \in \mathcal{Z}$. Suppose that the following conditions hold*:

(1) *(X,d) is complete*;
(2) *A is closed with respect to the metric d*;
(3) *$T(A_0) \subseteq B_0$*;
(4) *there exist $x_0, x_1 \in A_0$ such that*

$$d(x_1, Tx_0) = d(A,B), \quad x_0 \preceq x_1;$$

(5) *T is continuous.*

Then T has a best proximity point, that is, there is some $z \in A$ such that $d(z,Tz) = d(A,B)$.

Proof By condition (4), we have

$$d(x_1, Tx_0) = d(A,B),$$

for some $x_0, x_1 \in A_0$ such that $x_0 \preceq x_1$. Condition (3) implies that $Tx_1 \in B_0$, which yields

$$d(x_2, Tx_1) = d(A,B),$$

for some $x_2 \in A_0$. Since $x_0 \preceq x_1$, condition (C1) implies that $x_1 \preceq x_2$. Continuing this process, by induction, we can construct a sequence $\{x_n\} \subset A_0$ such that

$$d(x_{n+1}, Tx_n) = d(A,B), \quad n = 0,1,2,\ldots \tag{2.1}$$

and

$$x_0 \preceq x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots.$$

Suppose that for some $p = 0,1,2,\ldots$, we have $x_{p+1} = x_p$. In this case, we get $d(x_p, Tx_p) = d(A,B)$, that is, x_p is a best proximity point of T. So, without restriction of the generality, we may suppose that

$$x_n \neq x_{n+1}, \quad n = 0,1,2,\ldots.$$

Since

$$x_n \preceq x_{n+1}, x_n \neq x_{n+1}, \quad d(x_n, Tx_{n-1}) = d(x_{n+1}, Tx_n) = d(A,B), \quad n = 1,2,3,\ldots,$$

it follows from condition (C2) that

$$\xi\bigl(d(x_n,x_{n+1}), m(x_{n-1},x_n)\bigr) \geq 0, \quad n = 1,2,3,\ldots,$$

where

$$\begin{aligned} m(x_{n-1},x_n) &= \max\left\{\frac{d(x_{n-1},x_n)d(x_n,x_{n+1})}{d(x_{n-1},x_n)}, d(x_{n-1},x_n)\right\} \\ &= \max\bigl\{d(x_n,x_{n+1}), d(x_{n-1},x_n)\bigr\}. \end{aligned}$$

Suppose that for some $n_0 = 1, 2, 3, \dots$, we have

$$\max\{d(x_{n_0}, x_{n_0+1}), d(x_{n_0-1}, x_{n_0})\} = d(x_{n_0}, x_{n_0+1}).$$

In this case, we obtain

$$0 \le \xi\bigl(d(x_{n_0}, x_{n_0+1}), d(x_{n_0}, x_{n_0+1})\bigr).$$

On the other hand, since $d(x_{n_0}, x_{n_0+1}) > 0$, using the property (ii) of a simulation function, we obtain

$$\xi\bigl(d(x_{n_0}, x_{n_0+1}), d(x_{n_0}, x_{n_0+1})\bigr) < 0,$$

which is a contradiction. As a consequence,

$$\max\{d(x_n, x_{n+1}), d(x_{n-1}, x_n)\} = d(x_{n-1}, x_n), \quad n = 1, 2, 3, \dots. \tag{2.2}$$

Thus, we obtain

$$\xi\bigl(d(x_n, x_{n+1}), d(x_{n-1}, x_n)\bigr) \ge 0, \quad n = 1, 2, 3, \dots. \tag{2.3}$$

From (2.2), we deduce that the sequence $\{r_n\}$ defined by

$$r_n = d(x_n, x_{n+1}), \quad n = 0, 1, 2, \dots$$

is decreasing, which yields

$$\lim_{n\to\infty} r_n = r,$$

where $r \in [0, \infty)$. Suppose that $r > 0$. Using (2.3) and the property (iii) of a simulation function, we deduce that

$$0 \le \limsup_{n\to\infty} \xi\bigl(d(x_n, x_{n+1}), d(x_{n-1}, x_n)\bigr) < 0,$$

which is a contradiction. As consequence, we have

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.4}$$

Let us prove now that $\{x_n\}$ is a Cauchy sequence. We argue by contradiction by supposing that $\{x_n\}$ is not a Cauchy sequence. In this case, there is some $\varepsilon > 0$ for which there are subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ such that

$$n(k) > m(k) > k, \quad d(x_{m(k)}, x_{n(k)}) \ge \varepsilon, \qquad d(x_{m(k)}, x_{n(k)-1}) < \varepsilon.$$

Using the triangle inequality, we have

$$\varepsilon \le d(x_{m(k)}, x_{n(k)}) \le d(x_{m(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{n(k)}) < \varepsilon + d(x_{n(k)-1}, x_{n(k)}).$$

Thus we have

$$\varepsilon \leq d(x_{m(k)}, x_{n(k)}) < \varepsilon + d(x_{n(k)-1}, x_{n(k)}), \quad \text{for all } k.$$

Letting $k \to \infty$ and using (2.4), we obtain

$$\lim_{n\to\infty} d(x_{m(k)}, x_{n(k)}) = \varepsilon. \tag{2.5}$$

Again, the triangle inequality yields

$$\bigl|d(x_{n(k)-1}, x_{m(k)}) - d(x_{m(k)}, x_{n(k)})\bigr| \leq d(x_{n(k)-1}, x_{n(k)}), \quad \text{for all } k.$$

Letting $k \to \infty$, using (2.4) and (2.5), we obtain

$$\lim_{n\to\infty} d(x_{n(k)-1}, x_{m(k)}) = \varepsilon. \tag{2.6}$$

Similarly, we have

$$\bigl|d(x_{n(k)-1}, x_{m(k)-1}) - d(x_{n(k)-1}, x_{m(k)})\bigr| \leq d(x_{m(k)-1}, x_{m(k)}), \quad \text{for all } k.$$

Letting $k \to \infty$, using (2.4) and (2.6), we obtain

$$\lim_{n\to\infty} d(x_{n(k)-1}, x_{m(k)-1}) = \varepsilon. \tag{2.7}$$

Observe that for k large enough, we have

$$x_{m(k)-1} \preceq x_{n(k)-1}, \qquad x_{m(k)-1} \neq x_{n(k)-1}$$

and

$$d(x_{m(k)}, Tx_{m(k)-1}) = d(x_{n(k)}, Tx_{n(k)-1}) = d(A,B).$$

Then condition (C2) yields

$$\xi\bigl(d(x_{m(k)}, x_{n(k)}), m(x_{m(k)-1}, x_{n(k)-1})\bigr) \geq 0, \quad \text{for all } k. \tag{2.8}$$

On the other hand, for all k, we have

$$m(x_{m(k)-1}, x_{n(k)-1}) = \max\left\{\frac{d(x_{m(k)-1}, x_{m(k)})d(x_{n(k)-1}, x_{n(k)})}{d(x_{m(k)-1}, x_{n(k)-1})}, d(x_{m(k)-1}, x_{n(k)-1})\right\}.$$

Passing $k \to \infty$ and using (2.4) and (2.7), we get

$$\lim_{k\to\infty} m(x_{m(k)-1}, x_{n(k)-1}) = \varepsilon. \tag{2.9}$$

Using (2.5), (2.9), (2.8) and the condition (iii) of a simulation function, we have

$$0 \leq \limsup_{k\to\infty} \xi\bigl(d(x_{m(k)}, x_{n(k)}), m(x_{m(k)-1}, x_{n(k)-1})\bigr) < 0,$$

which is a contradiction. As consequence, the sequence $\{x_n\}$ is Cauchy. Since A is a closed subset of the complete metric space (X,d) (from conditions (1) and (2)), there is some $z \in A$ such that

$$\lim_{n\to\infty} d(x_n, z) = 0.$$

The continuity of T (from condition (5)) yields

$$\lim_{n\to\infty} d(Tx_n, Tz) = 0.$$

Since $d(x_{n+1}, Tx_n) = d(A,B)$ for all $n = 0,1,2,\ldots$, we obtain

$$d(A,B) = \lim_{n\to\infty} d(x_{n+1}, Tx_n) = d(z, Tz),$$

that is, $z \in A$ is a best proximity point of T. This ends the proof. □

Next, we obtain a best proximity point result for mappings $T \in \mathcal{T}_\xi$ that are not necessarily continuous.

We say that the set A is $(d,\preceq)$-regular if it satisfies the following property:

$$\{a_n\} \subset A \text{ is nondecreasing w.r.t. } \preceq \quad \text{and} \quad \lim_{n\to\infty} d(a_n, a) = 0 \quad \Longrightarrow \quad a = \sup\{a_n\}.$$

Theorem 2.2 *Let $T \in \mathcal{T}_\xi$, for some $\xi \in \mathcal{Z}$. Suppose that the following conditions hold*:

(1) (X,d) *is complete*;

(2) A_0 *is closed*;

(3) $T(A_0) \subseteq B_0$;

(4) *there exist $x_0, x_1 \in A_0$ such that*

$$d(x_1, Tx_0) = d(A,B), \quad x_0 \preceq x_1;$$

(5) *A is $(d,\preceq)$-regular.*

Then T has a best proximity point, that is, there is some $z \in A$ such that $d(z,Tz) = d(A,B)$.

Proof Let us consider the sequence $\{x_n\} \subset A_0$ defined by (2.1). Following the proof of Theorem 2.1, we know that $\{x_n\}$ is a Cauchy sequence. Since A_0 is closed, there is some $z \in A_0$ such that

$$\lim_{n\to\infty} d(x_n, z) = 0.$$

From condition (3), we have $Tz \in B_0$, which yields

$$d(y_1, Tz) = d(A,B),$$

for some $y_1 \in A_0$. On the other hand, the regularity condition (5) implies that

$$x_n \preceq z, \quad \text{for all } n.$$

Since for all n,

$$x_n \preceq z, \quad d(x_{n+1}, Tx_n) = d(y_1, Tz) = d(A, B),$$

condition (C1) yields

$$x_{n+1} \preceq y_1, \quad \text{for all } n.$$

On the other hand, we know that $z = \sup\{x_n\}$, which implies that

$$z \preceq y_1.$$

Thus we have

$$d(y_1, Tz) = d(A, B), \quad z \preceq y_1.$$

Again, since $Ty_1 \in B_0$, there is some $y_2 \in A_0$ such that $d(y_2, Ty_1) = d(A, B)$. Condition (C1) yields $y_1 \preceq y_2$. Thus we have

$$d(y_2, Ty_1) = d(A, B), \quad y_1 \preceq y_2.$$

Set $y_0 = z$ and continuing this process, we can build a sequence $\{y_n\} \subset A_0$ such that

$$d(y_{n+1}, Ty_n) = d(A, B), \quad n = 0, 1, 2, \ldots$$

and

$$y_0 \preceq y_1 \preceq y_2 \preceq \cdots \preceq y_n \preceq y_{n+1} \preceq \cdots.$$

Following similar arguments as in the proof of Theorem 2.1, we can prove that $\{y_n\}$ is a Cauchy sequence in the closed subset A_0 of the complete metric space (X, d), which yields

$$\lim_{n\to\infty} d(y_n, y) = 0,$$

for some $y \in A_0$. The regularity assumption (5) implies that $y = \sup\{y_n\}$. So, we have

$$x_n \preceq z = y_0 \preceq y_1 \preceq \cdots \preceq y_n \preceq y, \quad \text{for all } n.$$

We claim that $z = y$. In order to prove our claim, suppose that $d(z, y) > 0$. Set

$$I = \{n : x_n = z\}.$$

We consider two cases.

Case 1. If $|I| = \infty$.

In this case, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$x_{n_k} = z, \quad \text{for all } k,$$

which implies that z is a best proximity point. So, this case is trivial.

Case 2. If $|I| < \infty$.

In this case, for n large enough, we have

$$x_n \neq z, \quad x_n \preceq z \preceq y_n, \quad \text{for all } n.$$

From condition (C2), for n large enough, we obtain

$$\xi\big(d(x_{n+1}, y_{n+1}), m(x_n, y_n)\big) \geq 0,$$

where

$$m(x_n, y_n) = \max\left\{\frac{d(x_n, x_{n+1})d(y_n, y_{n+1})}{d(x_n, y_n)}, d(x_n, y_n)\right\}.$$

Observe that

$$\lim_{n\to\infty} d(x_{n+1}, y_{n+1}) = \lim_{n\to\infty} m(x_n, y_n) = d(z, y) > 0.$$

From the property (iii) of simulation functions, we obtain

$$0 \leq \limsup_{n\to\infty} \xi\big(d(x_{n+1}, y_{n+1}), m(x_n, y_n)\big) < 0,$$

which is a contradiction. As consequence, we have $z = y$.

Since $z = y$, we obtain

$$x_n \preceq z = y_0 \preceq y_1 \preceq \cdots \preceq y_n \preceq y = z, \quad \text{for all } n,$$

which implies that

$$y_n = z, \quad \text{for all } n.$$

Since $d(y_{n+1}, Ty_n) = d(A, B)$, we have $d(z, Tz) = d(A, b)$, that is, z is a best proximity point of T. This completes the proof. □

Note that the assumptions in Theorems 2.1 and 2.2 do not guarantee the uniqueness of the best proximity point. The next example shows this fact.

Example 2.3 *Let X be the subset of $\mathbb{R}^3$ given by*

$$X = \big\{(0,0,1), (1,0,0), (0,0,-1), (-1,0,0)\big\}.$$

We endow X with the partial order $\preceq$ defined by

$$(x, y, z) \preceq \big(x', y', z'\big) \iff x \leq x', y \leq y', z \leq z'.$$

Let d be the Euclidean metric on $\mathbb{R}^3$. Then (X, d) is a complete metric space. Set

$$A = \big\{(0,0,1), (1,0,0)\big\} \quad \textit{and} \quad B = \big\{(0,0,-1), (-1,0,0)\big\}.$$

In this case, we have

$$d(A,B)=\sqrt{2}, \qquad A_0=A, \qquad B_0=B.$$

Let $T:A\to B$ *be the mapping defined by*

$$T(x,y,z)=(-z,-y,-x), \quad (x,y,z)\in A.$$

Then T *is continuous and* $T\in\mathcal{T}_\xi$ *for every* $\xi\in\mathcal{Z}$. *Moreover, it can be shown that all the other conditions of Theorems* 2.1 *and* 2.2 *are satisfied. However,* $z_1=(0,0,1)$ *and* $z_2=(1,0,0)$ *are two best proximity points of* T.

In the next theorem, we give a sufficient condition for the uniqueness of the best proximity point.

Theorem 2.4 *In addition to the assumptions of Theorem* 2.1 (*resp. Theorem* 2.2), *suppose that*

$$\textit{for every } (x,y)\in A_0\times A_0, \textit{there is some } w\in A_0 \textit{ such that } x\preceq w, y\preceq w.$$

Then T *has a unique best proximity point.*

Proof From Theorem 2.1 (resp. Theorem 2.2), the set of best proximity points of T is not empty. Suppose that $z_1,z_2\in A_0$ are two distinct best proximity points of T, that is,

$$d(z_1,Tz_1)=d(z_2,Tz_2)=d(A,B), \qquad d(z_1,z_2)>0.$$

We consider two cases.

Case 1. If z_1 and z_2 are comparable.

We may assume that $z_1\preceq z_2$. From condition (C2), we have

$$\xi\bigl(d(z_1,z_2),m(z_1,z_2)\bigr)\geq 0,$$

where

$$m(z_1,z_2)=\max\left\{\frac{d(z_1,z_1)d(z_2,z_2)}{d(z_1,z_2)},d(z_1,z_2)\right\}=d(z_1,z_2).$$

Thus we have

$$\xi\bigl(d(z_1,z_2),d(z_1,z_2)\bigr)\geq 0,$$

which is a contradiction with the property (ii) of a simulation function.

Case 2. If z_1 and z_2 are not comparable.

In this case, there is some $w\in A_0$ such that

$$z_1\preceq w, \qquad z_2\preceq w, \quad w\notin\{z_1,z_2\}.$$

Since $T(A_0) \subseteq B_0$, we can build a sequence $\{w_n\} \subset A_0$ such that

$$d(w_{n+1}, Tw_n) = d(A,B), \quad n = 0,1,2,\ldots$$

with $w_0 = w$. From condition (C1), we get

$$z_1 \preceq w_n, \quad n = 0,1,2,\ldots.$$

If for some k, we have $z_1 = w_k$, using condition (C1), we have $w_{k+1} \preceq z_1$, which yields $w_{k+1} = z_1$. Arguing similarly, we obtain $w_n = z_1$ for every $n \geq k$. Thus we have

$$\lim_{n\to\infty} d(w_n, z_1) = 0.$$

If $w_n \neq z_1$ for every n, from condition (C2), we have

$$\xi\big(d(z_1, w_{n+1}), m(z_1, w_n)\big) \geq 0, \quad n = 0,1,2,\ldots,$$

where

$$m(z_1, w_n) = \max\left\{\frac{d(z_1,z_1)d(w_n,w_{n+1})}{d(z_1,w_n)}, d(z_1,w_n)\right\} = d(z_1,w_n).$$

Thus we have

$$\xi\big(d(z_1, w_{n+1}), d(z_1, w_n)\big) \geq 0, \quad n = 0,1,2,\ldots.$$

On the other hand, from the property (ii) of a simulation function, we have

$$0 \leq \xi\big(d(z_1, w_{n+1}), d(z_1, w_n)\big) < d(z_1, w_n) - d(z_1, w_{n+1}), \quad n = 0,1,2,\ldots.$$

We deduce that the sequence $\{s_n\}$ defined by

$$s_n = d(z_1, w_n), \quad n = 0,1,2,\ldots$$

converges to some $s \geq 0$. But the property (ii) of a simulation function gives us that $s = 0$. Thus, in all cases, we have

$$\lim_{n\to\infty} d(w_n, z_1) = 0.$$

Analogously, we can prove that

$$\lim_{n\to\infty} d(w_n, z_2) = 0.$$

Finally, the uniqueness of the limit yields the desired result. □

In the following corollaries we deduce some known and some new results in best proximity point theory via various choices of simulation functions.

We denote by $\mathcal{F}$ the set of mappings $T : A \to B$ satisfying the following conditions:

(F1) for every $x_1, x_2, y_1, y_2 \in A$, we have

$$y_1 \preceq y_2, \quad d(x_1, Ty_1) = d(x_2, Ty_2) = d(A,B) \quad \Longrightarrow \quad x_1 \preceq x_2;$$

(F2) for every $x, y, u_1, u_2 \in A$, we have

$$\begin{aligned} & x \preceq y, x \neq y, \quad d(u_1, Tx) = d(u_2, Ty) = d(A,B) \\ & \quad \Longrightarrow \quad d(u_1, u_2) \leq k \max\left\{ \frac{d(x,u_1)d(y,u_2)}{d(x,y)}, d(x,y) \right\}, \end{aligned}$$

for some constant $k \in (0,1)$.

Take $\xi(t,s) = ks - t$, for $t, s \geq 0$, we deduce from Theorems 2.1, 2.2 and 2.4 the following results.

Corollary 2.5 *Let $T \in \mathcal{F}$. Suppose that the following conditions hold*:

(1) *(X,d) is complete*;

(2) *A is closed with respect to the metric d*;

(3) *$T(A_0) \subseteq B_0$*;

(4) *there exist $x_0, x_1 \in A_0$ such that*

$$d(x_1, Tx_0) = d(A,B), \quad x_0 \preceq x_1;$$

(5) *T is continuous.*

Then T has a best proximity point, that is, there is some $z \in A$ such that $d(z,Tz) = d(A,B)$.

Corollary 2.6 *Let $T \in \mathcal{F}$. Suppose that the following conditions hold*:

(1) *(X,d) is complete*;

(2) *A_0 is closed*;

(3) *$T(A_0) \subseteq B_0$*;

(4) *there exist $x_0, x_1 \in A_0$ such that*

$$d(x_1, Tx_0) = d(A,B), \quad x_0 \preceq x_1;$$

(5) *A is $(d, \preceq)$-regular.*

Then T has a best proximity point, that is, there is some $z \in A$ such that $d(z,Tz) = d(A,B)$.

Corollary 2.7 *In addition to the assumptions of Corollary* 2.5 (*resp. Corollary* 2.6), *suppose that*

$$\textit{for every } (x,y) \in A_0 \times A_0, \textit{there is some } w \in A_0 \textit{ such that } x \preceq w, y \preceq w.$$

Then T has a unique best proximity point.

We denote by $\mathcal{G}$ the set of mappings $T: A \to B$ satisfying the following conditions:

(G1) for every $x_1, x_2, y_1, y_2 \in A$, we have

$$y_1 \preceq y_2, \quad d(x_1, Ty_1) = d(x_2, Ty_2) = d(A,B) \quad \Longrightarrow \quad x_1 \preceq x_2;$$

(G2) for every $x, y, u_1, u_2 \in A$, we have

$$x \preceq y, x \neq y, \quad d(u_1, Tx) = d(u_2, Ty) = d(A, B)$$
$$\implies \quad d(u_1, u_2) \leq \max\left\{\frac{d(x, u_1)d(y, u_2)}{d(x, y)}, d(x, y)\right\} - \varphi\left(\max\left\{\frac{d(x, u_1)d(y, u_2)}{d(x, y)}, d(x, y)\right\}\right),$$

where $\varphi : [0, \infty) \to [0, \infty)$ is lower semi-continuous function and $\varphi^{-1}(\{0\}) = \{0\}$.

Take $\xi(t, s) = s - \varphi(s) - t$, for $t, s \geq 0$, we deduce from Theorems 2.1, 2.2 and 2.4 the following results obtained in [23].

Corollary 2.8 *Let $T \in \mathcal{G}$. Suppose that the following conditions hold*:

(1) *(X, d) is complete*;

(2) *A is closed with respect to the metric d*;

(3) *$T(A_0) \subseteq B_0$*;

(4) *there exist $x_0, x_1 \in A_0$ such that*

$$d(x_1, Tx_0) = d(A, B), \quad x_0 \preceq x_1;$$

(5) *T is continuous.*

Then T has a best proximity point, that is, there is some $z \in A$ such that $d(z, Tz) = d(A, B)$.

Corollary 2.9 *Let $T \in \mathcal{G}$. Suppose that the following conditions hold*:

(1) *(X, d) is complete*;

(2) *A_0 is closed*;

(3) *$T(A_0) \subseteq B_0$*;

(4) *there exist $x_0, x_1 \in A_0$ such that*

$$d(x_1, Tx_0) = d(A, B), \quad x_0 \preceq x_1;$$

(5) *A is $(d, \preceq)$-regular.*

Then T has a best proximity point, that is, there is some $z \in A$ such that $d(z, Tz) = d(A, B)$.

Corollary 2.10 *In addition to the assumptions of Corollary* 2.8 (*resp. Corollary* 2.9), *suppose that*

$$\text{for every } (x, y) \in A_0 \times A_0, \text{there is some } w \in A_0 \text{ such that } x \preceq w, y \preceq w.$$

Then T has a unique best proximity point.

We denote by $\mathcal{H}$ the set of mappings $T : A \to B$ satisfying the following conditions:

(H1) for every $x_1, x_2, y_1, y_2 \in A$, we have

$$y_1 \preceq y_2, \quad d(x_1, Ty_1) = d(x_2, Ty_2) = d(A, B) \quad \implies \quad x_1 \preceq x_2;$$

(H2) for every $x, y, u_1, u_2 \in A$, we have

$$\begin{aligned} &x \preceq y, x \neq y, \quad d(u_1, Tx) = d(u_2, Ty) = d(A, B) \\ &\quad\Longrightarrow \quad d(u_1, u_2) \leq \varphi\left(\max\left\{\frac{d(x, u_1)d(y, u_2)}{d(x, y)}, d(x, y)\right\}\right) \\ &\qquad\qquad\qquad\qquad \times \max\left\{\frac{d(x, u_1)d(y, u_2)}{d(x, y)}, d(x, y)\right\}, \end{aligned}$$

where $\varphi : [0, \infty) \to [0, 1)$ is a function such that $\limsup_{t\to r^+} \varphi(t) < 1$, for all $r > 0$.

Take $\xi(t, s) = s\varphi(s) - t$, for $t, s \geq 0$, we deduce from Theorems 2.1, 2.2 and 2.4 the following results.

Corollary 2.11 *Let $T \in \mathcal{H}$. Suppose that the following conditions hold*:

(1) (X, d) *is complete*;

(2) *A is closed with respect to the metric d*;

(3) $T(A_0) \subseteq B_0$;

(4) *there exist $x_0, x_1 \in A_0$ such that*

$$d(x_1, Tx_0) = d(A, B), \quad x_0 \preceq x_1;$$

(5) *T is continuous.*

Then T has a best proximity point, that is, there is some $z \in A$ such that $d(z, Tz) = d(A, B)$.

Corollary 2.12 *Let $T \in \mathcal{H}$. Suppose that the following conditions hold*:

(1) (X, d) *is complete*;

(2) *A_0 is closed*;

(3) $T(A_0) \subseteq B_0$;

(4) *there exist $x_0, x_1 \in A_0$ such that*

$$d(x_1, Tx_0) = d(A, B), \quad x_0 \preceq x_1;$$

(5) *A is $(d, \preceq)$-regular.*

Then T has a best proximity point, that is, there is some $z \in A$ such that $d(z, Tz) = d(A, B)$.

Corollary 2.13 *In addition to the assumptions of Corollary* 2.11 (*resp. Corollary* 2.12), *suppose that*

$$\text{for every } (x, y) \in A_0 \times A_0, \text{there is some } w \in A_0 \text{ such that } x \preceq w, y \preceq w.$$

Then T has a unique best proximity point.

Finally, take $A = B = X$ in Theorems 2.1, 2.2 and 2.4, we obtain the following fixed point theorems.

For a given simulation function $\xi \in \mathcal{Z}$, we denote by $\mathcal{C}_\xi$ the class of mappings $T : X \to X$ satisfying the following conditions:

(I) for every $x, y \in X$, we have

$$x \preceq y \quad\Longrightarrow\quad Tx \preceq Ty;$$

(II) for every $x, y \in X$, we have

$$x \preceq y, x \neq y \quad \Longrightarrow \quad \xi\left(d(Tx, Ty), \max\left\{\frac{d(x, Tx)d(y, Ty)}{d(x, y)}, d(x, y)\right\}\right) \geq 0.$$

Corollary 2.14 *Let $T \in \mathcal{C}_\xi$, for some $\xi \in \mathcal{Z}$. Suppose that*

(1) *(X, d) is complete*;

(2) *there exists some $x_0 \in X$ such that $x_0 \preceq Tx_0$*;

(3) *T is continuous.*

Then T has a fixed point, that is, there is some $z \in X$ such that $z = Tz$.

Corollary 2.15 *Let $T \in \mathcal{C}_\xi$, for some $\xi \in \mathcal{Z}$. Suppose that*

(1) *(X, d) is complete*;

(2) *there exists some $x_0 \in X$ such that $x_0 \preceq Tx_0$*;

(3) *X is $(d, \preceq)$-regular.*

Then T has a fixed point, that is, there is some $z \in X$ such that $z = Tz$.

Corollary 2.16 *In addition to the assumptions of Corollary* 2.14 (*resp. Corollary* 2.15), *suppose that*

$$\text{for every } (x, y) \in X \times X, \text{there is some } w \in X \text{ such that } x \preceq w, y \preceq w.$$

Then T has a unique fixed point.

Competing interests
The author declares that he has no competing interests.

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