Budhia et al. *Fixed Point Theory and Applications* (2016) 2016:2
DOI 10.1186/s13663-015-0480-5

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Extensions of almost-F and F-Suzuki contractions with graph and some applications to fractional calculus

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Abstract

In this paper, we introduce the two new concepts of an α-type almost-F-contraction and an α-type F Suzuki contraction and prove some fixed point theorems for such mappings in a complete metric space. Some examples and an application to a nonlinear fractional differential equation are given to illustrate the usability of the new theory.

MSC: 47H10; 54H25; 37C25

Keywords: fixed point; almost-F-contraction; F-Suzuki contraction; nonlinear fractional differential equation

1 Introduction

In recent years two interesting but different generalizations of the Banach-contraction theorem have been given by Samet *et al.* [1] and Wardowski [2]. These two results have become of recent interest of many authors (see [3–13] and references therein).

Most recently, Piri and Kumam [14] (respectively, Minak *et al.* [15]) extended the results of Wardowski [2] by introducing the concept of an F-Suzuki contraction (respectively, almost-F-contraction) and obtained some interesting fixed point results. Following this direction of research, we introduce the new concepts of an α-type almost-F-contraction and an α-type F-Suzuki contraction and prove some fixed point theorems concerning such contractions. Moreover, some examples and an application to a nonlinear fractional differential equation are given to illustrate the usability of the new theory.

2 Preliminaries

The aim of this section is to present some notions and results used in the paper. Throughout the article $\mathbb{N}$, $\mathbb{R}^+$, and $\mathbb{R}$ will denote the set of natural numbers, positive real numbers, and real numbers, respectively.

Definition 2.1 [2] Let $F:\mathbb{R}^+\to\mathbb{R}$ be a mapping satisfying:

(F1) F is strictly increasing, that is, $\alpha<\beta\Rightarrow F(\alpha)<F(\beta)$ for all $\alpha,\beta\in\mathbb{R}^+$,
(F2) for every sequence $\{\alpha_n\}$ in $\mathbb{R}^+$ we have $\lim_{n\to\infty}\alpha_n=0$ iff $\lim_{n\to\infty}F(\alpha_n)=-\infty$,
(F3) there exists a number $k\in(0,1)$ such that $\lim_{\alpha\to 0^+}\alpha^k F(\alpha)=0$.

We denote with $\mathcal{F}$ the family of all functions F that satisfy the conditions (F1)-(F3).

Example 2.1 The following function $F:\mathbb{R}^+\to\mathbb{R}$ belongs to $\mathcal{F}$:

$$F(\alpha)=\ln\alpha,\qquad F(\alpha)=\ln\alpha+\alpha,\qquad F(\alpha)=-\frac{1}{\sqrt{\alpha}}\quad\text{where }\alpha>0.$$

Definition 2.2 [2] Let (X,d) be a metric space. A mapping $T:X\to X$ is called an F-contraction on X if there exist $F\in\mathcal{F}$ and $\tau>0$ such that, for all $x,y\in X$ with $d(Tx,Ty)>0$, we have

$$\tau+F\bigl(d(Tx,Ty)\bigr)\le F\bigl(d(x,y)\bigr).$$

Definition 2.3 [15] Let (X,d) be a metric space, $T:X\to X$ be a mapping. Then the mapping T is said to be an almost-F-contraction if there exist $F\in\mathcal{F}$ and $\tau>0$, $L\ge 0$ such that

$$\begin{aligned}
&d(Tx,Ty)>0\quad\Longrightarrow\quad\tau+F\bigl(d(Tx,Ty)\bigr)\le F\bigl(d(x,y)+Ld(y,Tx)\bigr)\quad\text{and}\\
&d(Tx,Ty)>0\quad\Longrightarrow\quad\tau+F\bigl(d(Tx,Ty)\bigr)\le F\bigl(d(x,y)+Ld(x,Ty)\bigr)
\end{aligned}$$

for all $x,y\in X$.

Definition 2.4 [13, 14] Let us denote by $\mathcal{G}$ the set of all functions $F:\mathbb{R}^+\to\mathbb{R}$ satisfying the following conditions:

(G1) F is strictly increasing, that is, $\alpha<\beta\Rightarrow F(\alpha)<F(\beta)$ for all $\alpha,\beta\in\mathbb{R}^+$,

(G2) there is a sequence $\{\alpha_n\}$ of positive real numbers such that $\lim_{n\to\infty}F(\alpha_n)=-\infty$, or $\inf F=-\infty$,

(G3) F is continuous on $(0,\infty)$.

Example 2.2 The following function $F:\mathbb{R}^+\to\mathbb{R}$ belongs to $\mathcal{G}$:

$$F(\alpha)=-\frac{1}{\alpha},\qquad F(\alpha)=-\frac{1}{\alpha}+\alpha,\qquad F(\alpha)=\ln\alpha\quad\text{where }\alpha>0.$$

Lemma 2.1 [14] *Let $F:\mathbb{R}^+\to\mathbb{R}$ be an increasing function and $\{\alpha_n\}$ be a sequence of positive real numbers. Then the following holds*:

(a) *if* $\lim_{n\to\infty}F(\alpha_n)=-\infty$, *then* $\lim_{n\to\infty}\alpha_n=0$;

(b) *if* $\inf F=-\infty$, *and* $\lim_{n\to\infty}\alpha_n=0$, *then* $\lim_{n\to\infty}F(\alpha_n)=-\infty$.

Definition 2.5 [14] Let (X,d) be a metric space. A mapping $T:X\to X$ is said to be an F-Suzuki contraction if there exists $\tau>0$ such that for all $x,y\in X$ with $Tx\neq Ty$

$$\frac{1}{2}d(x,Tx)<d(x,y)\quad\Longrightarrow\quad\tau+F\bigl(d(Tx,Ty)\bigr)\le F\bigl(d(x,y)\bigr),$$

where $F\in\mathcal{G}$.

Definition 2.6 [1] Let $T:X\to X$ and $\alpha:X\times X\to[0,\infty)$ be two given mappings. Then T is called an α-admissible if

$$x,y\in X,\quad\alpha(x,y)\ge 1\quad\Longrightarrow\quad\alpha(Tx,Ty)\ge 1.$$

3 Fixed point results for α-type almost-F-contraction and α-type F-Suzuki contraction

In this section, we first introduce the concepts of an α-type almost-F-contraction and an α-type F-Suzuki contraction and then we prove some fixed point theorems for these contractions in a complete metric space.

We begin with the following definitions.

Definition 3.1 Let (X,d) be a metric space, $T : X \to X$ be a mapping, and $\alpha : X \times X \to \{-\infty\} \cup (0,\infty)$ be a symmetric function. Then the mapping T is said to be an α-type almost-F-contraction if there exist $F \in \mathcal{F}$ and $\tau > 0$ and $L \geq 0$ such that

$$\begin{aligned} d(Tx,Ty) > 0 \quad &\Longrightarrow \quad \tau + \alpha(x,y)F\bigl(d(Tx,Ty)\bigr) \leq F\bigl(d(x,y) + Ld(y,Tx)\bigr) \quad \text{and} \\ d(Tx,Ty) > 0 \quad &\Longrightarrow \quad \tau + \alpha(x,y)F\bigl(d(Tx,Ty)\bigr) \leq F\bigl(d(x,y) + Ld(x,Ty)\bigr) \end{aligned}$$

for all $x,y \in X$.

Example 3.1 Let $X = [0,3] \cup [5,6]$ with the usual metric, $T : X \to X$ be defined as

$$Tx = \begin{cases} 0 & \text{if } x \in [0,3], \\ 3 & \text{if } x \in [5,6], \end{cases}$$

and $F(\alpha) = \ln \alpha$.

Then T is not an almost-F-contraction. Since at $x = 3$ and $y = 5$, $d(Tx,Ty) > 0$ but $\tau + F(d(Tx,Ty)) = \tau + F(3)$, whereas $F(d(x,y) + Ld(x,Ty)) = F(2)$.

Define

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x,y \in [0,3] \text{ or } x,y \in [5,6], \\ 0.1 & \text{otherwise.} \end{cases}$$

Then T is an α-type almost-F-contraction with $\tau = 0.5$ and $L = 3$.

Definition 3.2 Let (X,d) be a metric space $T : X \to X$ be a mapping and $\alpha : X \times X \to \{-\infty\} \cup (0,\infty)$ be a symmetric function. A map $T : X \to X$ is said to be an α-type F-Suzuki contraction if there exists $\tau > 0$ such that for all $x,y \in X$ with $Tx \neq Ty$

$$\frac{1}{2}d(x,Tx) < d(x,y) \quad \Longrightarrow \quad \tau + \alpha(x,y)F\bigl(d(Tx,Ty)\bigr) \leq F\bigl(d(x,y)\bigr),$$

where $F \in \mathcal{G}$.

Example 3.2 Let $X = [1,3] \cup [5,9]$ with the usual metric and $F : \mathbb{R}^+ \to \mathbb{R}$ be defined as $F(\alpha) = -\frac{1}{\alpha}$. Define a mapping $T : X \to X$ as

$$Tx = \begin{cases} 5, & x \in [1,3], \\ 8, & x \in [5,9]. \end{cases}$$

Then T is not F-Suzuki contraction as the condition

$$\frac{1}{2}d(x,Tx) < d(x,y) \quad \Longrightarrow \quad \tau + F\bigl(d(Tx,Ty)\bigr) \le F\bigl(d(x,y)\bigr)$$

fails for $x = 3$ and $y = 5$.

Define $\alpha : X \times X \to \{-\infty\} \cup (0,\infty)$ as

$$\alpha(x,y) = 2, \quad \text{for all } x,y \in X.$$

Then T is α-type F-Suzuki contraction *i.e.*

$$\frac{1}{2}d(x,Tx) < d(x,y) \quad \Longrightarrow \quad \tau + \alpha(x,y)F\bigl(d(Tx,Ty)\bigr) \le F\bigl(d(x,y)\bigr)$$

holds for all $x,y \in X$ with $\tau = \frac{1}{6}$.

Now, we prove our first result.

Theorem 3.1 *Let (X,d) be a complete metric space and $T : X \to X$ be an α-type almost-F-contraction where $F \in \mathcal{F}$, satisfying the following conditions*:
(i) *T is α-admissible*,
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge 1$*,
(iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1}) \ge 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x \in X$ as $n \to \infty$ then $\alpha(x_n,x) \ge 1$ for all $n \in \mathbb{N} \cup \{0\}$.*

Then T has a fixed point $x^* \in X$.

Proof Let $x_0 \in X$ be such that $\alpha(x_0, Tx_0) \ge 1$. Define the sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$, for all $n \in \mathbb{N} \cup \{0\}$. If $x_{n+1} = x_n$ for some $n \in \mathbb{N}$, then $x^* = x_n$ is a fixed point of T. Let us assume that $x_{n+1} \ne x_n$ for all $n \in \mathbb{N} \cup \{0\}$.

Since T is α-admissible, we have $\alpha(x_0,x_1) = \alpha(x_0, Tx_0) \ge 1$, which implies $\alpha(Tx_0, Tx_1) = \alpha(x_1,x_2) \ge 1$. Continuing in this way we have in general

$$\alpha(x_n,x_{n+1}) \ge 1 \quad \text{for all } n \in \mathbb{N}. \tag{3.1}$$

Now, $F(d(x_{n+1},x_n)) = F(d(Tx_n, Tx_{n-1})) \le \alpha(x_n,x_{n-1})F(d(Tx_n, Tx_{n-1}))$.

Therefore,

$$\begin{aligned}\tau + F\bigl(d(Tx_n, Tx_{n-1})\bigr) &\le \tau + \alpha(x_n,x_{n-1})F\bigl(d(Tx_n, Tx_{n-1})\bigr) \\ &\le F\bigl(d(x_n,x_{n-1}) + Ld(x_n, Tx_{n-1})\bigr).\end{aligned}$$

So $F(d(x_{n+1},x_n)) = F(d(Tx_n, Tx_{n-1})) \le F(d(x_n,x_{n-1})) - \tau$. In general we get

$$F\bigl(d(x_{n+1},x_n)\bigr) = F\bigl(d(Tx_n, Tx_{n-1})\bigr) \le F\bigl(d(x_1,x_0)\bigr) - n\tau. \tag{3.2}$$

Thus as $n \to \infty$, we have $\lim_{n\to\infty} F(d(x_{n+1},x_n)) = -\infty$, then by (F2) we have $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$. Now, from (F3), there exists $k \in (0,1)$ such that $\lim_{n\to\infty}(d(x_{n+1},x_n))^k F(d(x_{n+1},$

$x_n)) = 0$. From (3.2) it follows that

$$\bigl(d(x_{n+1},x_n)\bigr)^k F\bigl(d(x_{n+1},x_n)\bigr) \leq \bigl(d(x_{n+1},x_n)\bigr)^k \bigl(F\bigl(d(x_1,x_0)\bigr) - n\tau\bigr).$$

Then as $n \to \infty$ we get

$$\lim_{n\to\infty} n\bigl(d(x_{n+1},x_n)\bigr)^k = 0.$$

Therefore, there exists $n_0 \in \mathbb{N}$ such that

$$n\bigl(d(x_{n+1},x_n)\bigr)^k \leq 1, \quad \forall n \geq n_0,$$

i.e.

$$d(x_{n+1},x_n) \leq \frac{1}{n^{1/k}}, \quad \forall n \geq n_0.$$

Now, for $m > n > n_0$,

$$\begin{aligned} d(x_n,x_m) &\leq d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2}) + \cdots + d(x_{m-1},x_m) \\ &\leq \sum_{n\geq n_0} \frac{1}{n^{1/k}}, \end{aligned}$$

which is convergent as $k \in (0,1)$. Therefore as $m,n \to \infty$ we get $d(x_n,x_m) \to 0$. Hence $\{x_n\}$ is a Cauchy sequence. From the completeness of X we then have $x^* \in X$ such that $x_n \to x^*$.

Now we claim that $d(x_{n+1},Tx^*) = d(Tx_n,Tx^*) \to 0$ as $n \to \infty$. If $x^* = Tx^*$, then the proof is finished. Assume that $x^* \neq Tx^*$. If $x_{n+1} = Tx_n = Tx^*$ for infinite values of $n \in \mathbb{N} \cup \{0\}$, then the sequence has a subsequence that converges to Tx^* and the uniqueness of the limit implies $x^* = Tx^*$. Then we can assume that $Tx_n \neq Tx^*$ for all $n \in \mathbb{N} \cup \{0\}$. Now using (iii), we have

$$\tau + F\bigl(d\bigl(Tx_n,Tx^*\bigr)\bigr) \leq \tau + \alpha\bigl(x_n,x^*\bigr)F\bigl(d\bigl(Tx_n,Tx^*\bigr)\bigr) \leq F\bigl(d\bigl(x_n,x^*\bigr) + Ld\bigl(Tx_n,x^*\bigr)\bigr).$$

Then, as $n \to \infty$ we get

$$\tau + \lim_{n\to\infty} F\bigl(d\bigl(Tx_n,Tx^*\bigr)\bigr) \leq -\infty,$$

which will lead to a contradiction of the assumption that $\lim_{n\to\infty} d(Tx_n,Tx^*) > 0$ (in respect of (F2)). Thus we have $x_{n+1} = Tx_n \to Tx^*$ as $n \to \infty$ and hence $Tx^* = x^*$. □

Theorem 3.2 *We further assume that $\alpha(x,y) \geq 1$ for all $x,y \in \mathrm{Fix}(T)$ and suppose T also satisfies the following condition: there exist $G \in \mathcal{F}$ and some $L \geq 0$, $\tau > 0$ such that for all $x,y \in X$*

$$\tau + \alpha(x,y)G\bigl(d(Tx,Ty)\bigr) \leq G\bigl(d(x,y) + Ld(x,Tx)\bigr)$$

holds. Then the fixed point in the above result is unique.

Proof Let $y^* \in X$, $y^* \neq x^*$ such that $Ty^* = y^*$. Then $d(Tx^*, Ty^*) > 0$, which implies

$$\tau + G\bigl(d\bigl(Tx^*, Ty^*\bigr)\bigr) \leq \tau + \alpha\bigl(x^*, y^*\bigr)G\bigl(d\bigl(Tx^*, Ty^*\bigr)\bigr) \leq G\bigl(d\bigl(x^*, y^*\bigr) + L\bigl(x^*, Tx^*\bigr)\bigr).$$

Therefore we have

$$\tau + G\bigl(d\bigl(x^*, y^*\bigr)\bigr) = \tau + G\bigl(d\bigl(Tx^*, Ty^*\bigr)\bigr) \leq G\bigl(d\bigl(x^*, y^*\bigr)\bigr),$$

which is a contradiction as $\tau > 0$. □

The following example is illustrative of Theorem 3.2.

Example 3.3 Let $X = [1,3] \cup [5,9]$ with usual metric and $F : \mathbb{R}^+ \to \mathbb{R}$ be defined as $F(\alpha) = \ln\alpha$. Define a continuous map $T : X \to X$ as

$$Tx = \begin{cases} 2x + 3, & x \in [1,3], \\ 9, & x \in [5,9]. \end{cases}$$

Then T satisfies all the conditions of Theorem 3.2 for $\tau \leq 1.9408$ and $L = 1$, and hence T has a unique fixed point $x^* = 9$.

If $\alpha(x,y) = 1$ for all $x, y \in X$ then we have following result as in [15].

Corollary 3.1 *Let (X,d) be a complete metric space and $T : X \to X$ be an almost-F-contraction. Then T has a fixed point x^* in X.*

If $\alpha(x,y) = 1$ for all $x, y \in X$ and $L = 0$ then we have following result of Wardowski's [2].

Corollary 3.2 *Let (X,d) be a complete metric space and $T : X \to X$ be an F-contraction. Then T has a unique fixed point x^* in X.*

To prove our next result, we first give the following.

Definition 3.3 [16] An α-admissible map T is said to have the K-property whenever for each sequence $\{x_n\} \subset X$ with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$, then there exists a natural number k such that $\alpha(Tx_m, Tx_n) \geq 1$ for all $m > n \geq k$.

Theorem 3.3 *Let (X,d) be a complete metric space and $T : X \to X$ be an α-type F-Suzuki contraction satisfying the following conditions*:

(i) *T is α-admissible,*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$,*
(iii) *T has the K-property,*
(iv) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x \in X$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.*

Then T has a fixed point $x^* \in X$.

Proof Let $x_0 \in X$ be such that $\alpha(x_0, Tx_0) \geq 1$. Define the sequence $\{x_n\} \subseteq X$ by $x_{n+1} = Tx_n$, for all $n \in \mathbb{N} \cup \{0\}$. Since T is α-admissible we have $\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1$, which implies $\alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1$. Continuing in this way we have in general

$$\alpha(x_n, x_{n+1}) \geq 1 \quad \text{for all } n \in \mathbb{N} \cup \{0\}. \tag{3.3}$$

If $x_{n+1} = x_n$ for some $n \in \mathbb{N} \cup \{0\}$, then $x^* = x_n$ is a fixed point of T. Let us assume that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N} \cup \{0\}$. Therefore $\frac{1}{2}d(x_n, Tx_n) < d(x_n, Tx_n)$ for all $n \in \mathbb{N} \cup \{0\}$ and hence

$$\begin{aligned} \tau + F\bigl(d\bigl(Tx_n, T^2x_n\bigr)\bigr) &\leq \tau + \alpha(x_n, Tx_n)F\bigl(d\bigl(Tx_n, T^2x_n\bigr)\bigr) \\ &\leq F\bigl(d(x_n, Tx_n)\bigr) \quad \text{for all } n \in \mathbb{N} \cup \{0\}. \end{aligned}$$

So $F(d(Tx_n, T^2x_n)) \leq F(d(x_n, Tx_n)) - \tau$ and repeating this process in general we get

$$F\bigl(d\bigl(Tx_n, T^2x_n\bigr)\bigr) \leq F\bigl(d(x_0, x_1)\bigr) - n\tau.$$

As $n \to \infty$ we obtain

$$\lim_{n\to\infty} F\bigl(d(x_{n+1}, x_{n+2})\bigr) = -\infty,$$

which together with (G2) and by Lemma 2.1, gives

$$\lim_{n\to\infty} d(x_{n+1}, x_{n+2}) = 0.$$

Suppose $\{x_n\}$ is not a Cauchy sequence. Then there exist $\varepsilon > 0$ and $p(n) > q(n) > n \geq k$ such that $d(x_{p(n)}, x_{q(n)}) \geq \varepsilon$ and $d(x_{(p(n)-1)}, x_{q(n)}) < \varepsilon$.

Now

$$\varepsilon \leq d(x_{p(n)}, x_{q(n)}) \leq d(x_{p(n)}, x_{p(n)-1}) + d(x_{p(n)-1}, x_{q(n)}) < d(x_{p(n)}, x_{p(n)-1}) + \varepsilon.$$

Therefore

$$\lim_{n\to\infty} d(x_{p(n)}, x_{q(n)}) = \varepsilon. \tag{3.4}$$

Again we have

$$d(x_{p(n)}, x_{q(n)}) \leq d(x_{p(n)}, x_{p(n)+1}) + d(x_{p(n)+1}, x_{q(n)+1}) + d(x_{q(n)+1}, x_{q(n)})$$

and

$$d(x_{p(n)+1}, x_{q(n)+1}) \leq d(x_{p(n)+1}, x_{p(n)}) + d(x_{p(n)}, x_{q(n)}) + d(x_{q(n)}, x_{q(n)+1}).$$

So as $n \to \infty$, from the above two inequalities we have

$$\lim_{n\to\infty} d(x_{p(n)+1}, x_{q(n)+1}) = \varepsilon. \tag{3.5}$$

Therefore there exists $k \in \mathbb{N}$ such that $\frac{1}{2}d(x_{p(n)}, x_{q(n)}) < d(x_{p(n)}, x_{q(n)})$ for all $n \geq k$. Using the K-property we have

$$\begin{aligned}\tau + F\bigl(d(Tx_{p(n)}, Tx_{q(n)})\bigr) &\leq \tau + \alpha(x_{p(n)}, x_{q(n)})F\bigl(d(Tx_{p(n)}, Tx_{q(n)})\bigr) \\ &\leq F\bigl(d(x_{p(n)}, x_{q(n)})\bigr).\end{aligned}$$

So as $n \to \infty$ and by (G3), we get $\tau + F(\varepsilon) \leq F(\varepsilon)$, which is a contradiction. Hence $\{x_n\}$ is a Cauchy sequence in X and so it converges to some x^* in X.

Next, we claim that

$$\begin{aligned}&\frac{1}{2}d(x_n, Tx_n) < d\bigl(x_n, x^*\bigr) \quad \text{or} \\ &\frac{1}{2}d\bigl(Tx_n, T^2x_n\bigr) < d\bigl(Tx_n, x^*\bigr) \quad \text{for all } n \in \mathbb{N}.\end{aligned}$$

Assume there exists $m \in \mathbb{N}$ such that

$$\frac{1}{2}d(x_m, Tx_m) \geq d\bigl(x_m, x^*\bigr) \quad \text{and} \quad \frac{1}{2}d\bigl(Tx_m, T^2x_m\bigr) \geq d\bigl(Tx_m, x^*\bigr).$$

Then,

$$2d\bigl(x_m, x^*\bigr) \leq d(x_m, Tx_m) \leq d\bigl(x_m, x^*\bigr) + d\bigl(x^*, Tx_m\bigr)$$

and hence

$$d\bigl(x_m, x^*\bigr) \leq d\bigl(x^*, Tx_m\bigr) \leq \frac{1}{2}d\bigl(Tx_m, T^2x_m\bigr). \tag{3.6}$$

Since $\frac{1}{2}d(x_m, Tx_m) < d(x_m, Tx_m)$, we have

$$\tau + F\bigl(d\bigl(Tx_m, T^2x_m\bigr)\bigr) \leq \tau + \alpha(x_m, Tx_m)F\bigl(d\bigl(Tx_m, T^2x_m\bigr)\bigr) \leq F\bigl(d(x_m, Tx_m)\bigr),$$

which implies $F(d(Tx_m, T^2x_m)) < F(d(x_m, Tx_m))$ and so $d(Tx_m, T^2x_m) < d(x_m, Tx_m)$. Now

$$\begin{aligned}d\bigl(Tx_m, T^2x_m\bigr) &< d(x_m, Tx_m) \\ &\leq d\bigl(x_m, x^*\bigr) + d\bigl(x^*, Tx_m\bigr) \\ &\leq \frac{1}{2}d\bigl(Tx_m, T^2x_m\bigr) + \frac{1}{2}d\bigl(Tx_m, T^2x_m\bigr) = d\bigl(Tx_m, T^2x_m\bigr).\end{aligned}$$

Thus, for every $n \in \mathbb{N}$ either

$$\begin{aligned}&\tau + F\bigl(d\bigl(Tx_n, Tx^*\bigr)\bigr) \leq \tau + \alpha\bigl(x_n, x^*\bigr)F\bigl(d\bigl(Tx_n, Tx^*\bigr)\bigr) \leq F\bigl(d\bigl(x_n, x^*\bigr)\bigr) \quad \text{or} \\ &\tau + F\bigl(d\bigl(T^2x_n, Tx^*\bigr)\bigr) \leq \tau + \alpha\bigl(x_{n+1}, x^*\bigr)F\bigl(d\bigl(T^2x_n, Tx^*\bigr)\bigr) \leq F\bigl(d\bigl(x_{n+1}, x^*\bigr)\bigr).\end{aligned}$$

As $n \to \infty$ we get from the above

$$\lim_{n\to\infty} F\bigl(d\bigl(Tx_n, Tx^*\bigr)\bigr) = -\infty \quad \text{and}$$

$$\lim_{n\to\infty} F\big(d\big(T^2x_n, Tx^*\big)\big) = -\infty,$$

respectively. This further gives

$$\lim_{n\to\infty} d\big(Tx_n, Tx^*\big) = 0 \quad \text{and} \quad \lim_{n\to\infty} d\big(T^2x_n, Tx^*\big) = 0.$$

Then

$$0 \leq d\big(x^*, Tx^*\big) \leq d\big(x^*, Tx_n\big) + d\big(Tx_n, Tx^*\big),$$

which as $n \to \infty$ gives $d(x^*, Tx^*) = 0$ and hence $Tx^* = x^*$. □

Theorem 3.4 *If we further assume that* $\alpha(x,y) \geq 1$ *for all* $x, y \in \mathrm{Fix}(T)$, *then the fixed point is unique in the above result.*

Proof Let $y^* \in X$, $y^* \neq x^*$ such that $Ty^* = y^*$. Then $\frac{1}{2}d(Tx^*, x^*) = 0 < d(x^*, y^*)$, which implies

$$\tau + F\big(d\big(Tx^*, Ty^*\big)\big) \leq \tau + \alpha\big(x^*, y^*\big)F\big(d\big(Tx^*, Ty^*\big)\big) \leq F\big(d\big(x^*, y^*\big)\big).$$

Therefore, we have

$$\tau + F\big(d\big(x^*, y^*\big)\big) = \tau + F\big(d\big(Tx^*, Ty^*\big)\big) \leq F\big(d\big(x^*, y^*\big)\big),$$

which is a contradiction as $\tau > 0$. □

The following example illustrates Theorem 3.4.

Example 3.4 Let $X = [1,3] \cup [5,9]$ with the usual metric and $F : \mathbb{R}^+ \to \mathbb{R}$ be defined as $F(\alpha) = -\frac{1}{\alpha}$. Define a continuous map $T : X \to X$ as

$$Tx = \begin{cases} 2x + 3, & x \in [1,3], \\ 9, & x \in [5,9]. \end{cases}$$

Then T is not a F-Suzuki contraction as the condition

$$\frac{1}{2}d(x, Tx) < d(x,y) \quad \Longrightarrow \quad \tau + F\big(d(Tx, Ty)\big) \leq F\big(d(x,y)\big)$$

fails for $x = 1$ and $y = 5$.

Now, we distinguish following cases:

Case 1. If $x, y \in [1,3]$ then there are no points for which $\frac{1}{2}d(x, Tx) < d(x,y)$ holds, so we are through.

Case 2. If $x, y \in [5,9]$ then $Tx = Ty$ so we are done.

Case 3. Let $x \in [1,3]$ and $y \in [5,9]$. In this case we have $d(Tx, Ty) \leq 4$ and $2 \leq d(x,y)$. Therefore $\tau + \alpha(x,y)F(d(Tx, Ty)) \leq \tau + \alpha(x,y)F(4) = \tau - \frac{\alpha(x,y)}{4}$ and $F(2) \leq F(d(x,y))$ by (F2). So for given $\alpha(x,y)$ we can choose τ for which $\tau \leq \frac{\alpha(x,y)-2}{4}$ holds. Then for that $\alpha(x,y)$ and $\tau > 0$ we are done.

In particular, if we define $\alpha : X \times X \to (0,\infty) \cup \{-\infty\}$ as

$$\alpha(x,y) = 3 \quad \text{for all } x,y \in X.$$

Then T satisfies all the conditions of the above theorem with $\tau = \frac{1}{4}$ and hence T has an unique fixed point $x^* = 9$.

If $\alpha(x,y) = 1$ for all $x,y \in X$ then we have following result

Corollary 3.3 *Let (X,d) be a complete metric space and $T : X \to X$ be an F-Suzuki contraction. Then T has a unique fixed point x^* in X.*

4 Consequences

In this section we will show that some existing results in the literature can be deduced easily from our theorems proved in Section 3.

4.1 Fixed point with graph

Following Jachymski [17], let (X,d) be a metric space and $\Delta = \{(x,x) : x \in X\}$. Consider a graph G with the set $V(G)$ of its vertices equal to X and the set $E(G)$ of its edges as a superset of Δ. Assume that G has no parallel edges, that is, $(x,y),(y,x) \in E(G)$ implies $x = y$. Also, G is directed if the edges have a direction associated with them. Now we can identify the graph G with the pair $(V(G),E(G))$.

Denote

$$\mathcal{G} := \{G : G \text{ is a directed graph with } V(G) = X \text{ and } \Delta \subseteq E(G)\}.$$

Definition 4.1 A mapping $T : X \to X$ is called G-continuous, if we have a given $x \in X$ and a sequence $\{x_n\}$ such that $x_n \to x$, as $n \to \infty$, $(x_n, x_{n+1}) \in E(G)$, $\forall n \in \mathbb{N}$ imply $Tx_n \to Tx$.

Theorem 4.1 *Let (X,d) be a metric space endowed with a graph G and let T be a self-mapping on X. Suppose that the following assertions hold*:

(i) *for all $x,y \in X$, $(x,y) \in E(G) \Rightarrow (Tx,Ty) \in E(G)$;*
(ii) *there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$;*
(iii) *there exist a number $\tau > 0$, $L \geq 0$ and $F \in \mathcal{F}$ such that*

$$d(Tx,Ty) > 0 \quad \Rightarrow \quad \tau + F\bigl(d(Tx,Ty)\bigr) \leq F\bigl[d(x,y) + Ld(y,Tx)\bigr] \quad \textit{and}$$
$$d(Tx,Ty) > 0 \quad \Rightarrow \quad \tau + F\bigl(d(Tx,Ty)\bigr) \leq F\bigl[d(x,y) + Ld(x,Ty)\bigr]$$

for all $(x,y) \in E(G)$;

(iv) *for any sequence $\{x_n\} \subset X$, $x \in X$ with $x_n \to x$ as $n \to \infty$ and $(x_n, x_{n+1}) \in E(G)$ we have $(x_n, x) \in E(G)$ or T is G-continuous.*

Then T has a fixed point.

Proof Define $\alpha : X \times X \to \{-\infty\} \cup (0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 1 & \text{if } (x,y) \in E(G), \\ -\infty & \text{otherwise.} \end{cases}$$

First we prove that T is α-admissible. If $\alpha(x,y) \geq 1$, then $(x,y) \in E(G)$. As, from (i), we have $(Tx,Ty) \in E(G)$, $\alpha(Tx,Ty) \geq 1$. So T is an α-admissible mapping. From (ii) there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$, *i.e.* $\alpha(x_0, Tx_0) \geq 1$.

Let $\alpha(x,y) \geq 1$, then $(x,y) \in E(G)$. Now, from (iii) we have

$$\begin{aligned}
&\tau + F\bigl(d(Tx,Ty)\bigr) \leq F\bigl[d(x,y) + Ld(x,Ty)\bigr] \quad \text{and} \\
&\tau + F\bigl(d(Tx,Ty)\bigr) \leq F\bigl[d(x,y) + Ld(y,Tx)\bigr] \quad \textit{i.e.} \\
&\alpha(x,y) \geq 1 \quad \Rightarrow \quad \tau + F\bigl(d(Tx,Ty)\bigr) \leq F\bigl[d(x,y) + Ld(x,Ty)\bigr] \quad \text{and} \\
&\tau + F\bigl(d(Tx,Ty)\bigr) \leq F\bigl[d(x,y) + Ld(y,Tx)\bigr].
\end{aligned}$$

Now, let $\{x_n\} \subset X$ be a sequence such that $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) \geq 1$.

Then, $(x_{n+1}, x_n) \in E(G)$ and then from (iv) $(x_n, x) \in E(G)$ *i.e.* $\alpha(x_n, x) \geq 1$. Thus, all the conditions of Theorem 3.1 are satisfied and hence T has a fixed point in X. □

If $L = 0$, then Theorem 4.1 reduces to Corollary 2.9 given in [7].

4.2 Fixed point with partial order

Let $(X, d, \preceq)$ be a partially ordered metric space. Define the graph G by

$$E(G) = \bigl\{(x,y) \in X \times X : x \preceq y\bigr\}.$$

For the graph condition (i) in Theorem 4.1 means that T is nondecreasing with respect to this order [18]. From Theorem 4.1 we derive the following important results in partially ordered metric spaces.

Theorem 4.2 *Let $(X, d, \preceq)$ be a partially ordered metric space and let T be a self-mapping on X. Suppose that the following assertions hold*:

(i) *T is a nondecreasing map.*

(ii) *There exists $x_0 \in X$ such that $x_0 \preceq Tx_0$.*

(iii) *There exist a number $\tau > 0$, $L \geq 0$, and $F \in \mathcal{F}$ such that*

$$\begin{aligned}
&d(Tx,Ty) > 0 \quad \Rightarrow \quad \tau + F\bigl(d(Tx,Ty)\bigr) \leq F\bigl[d(x,y) + Ld(y,Tx)\bigr] \quad \textit{and} \\
&d(Tx,Ty) > 0 \quad \Rightarrow \quad \tau + F\bigl(d(Tx,Ty)\bigr) \leq F\bigl[d(x,y) + Ld(x,Ty)\bigr]
\end{aligned}$$

for all $x, y \in X$ with $x \preceq y$.

(iv) *Either for any sequence $\{x_n\} \subset X$ and $x \in X$ with*

$$\begin{aligned}
&x_n \to x, \quad \textit{as } n \to \infty \quad \textit{and} \\
&x_n \preceq x_{n+1}, \quad \forall n \in \mathbb{N} \cup \{0\}, \quad \textit{we have} \quad x_n \preceq x,
\end{aligned}$$

or T is continuous. Then T has a fixed point.

Corollary 4.1 [19] *Let $(X, d, \preceq)$ be a partially ordered complete metric space and let $T : X \to X$ be a continuous, nondecreasing self-mapping such that $x_0 \preceq Tx_0$ for some $x_0 \in X$.*

Assume that

$$d(Tx, Ty) \leq rd(x, y)$$

holds for all $x, y \in X$ *with* $x \preceq y$ *where* $0 \leq r < 1$. *Then* T *has a fixed point*.

5 Some applications to fractional calculus

First, let us recall some basic definitions of fractional calculus (see in [20–22]). For a continuous function $g : [0, \infty) \to \mathbb{R}$, the Caputo derivative of fractional order β is defined as

$$^{C}D^{\beta}\big(g(t)\big) = \frac{1}{\Gamma(n-\beta)}\int_0^t (t-s)^{n-\beta-1} g^n(s)\,ds \quad \big(n-1 < \beta < n, n = [\beta]+1\big)$$

where $[\beta]$ denotes the integer part of the real number β and Γ is a gamma function.

In this section, we present an application of Theorem 3.2 in establishing the existence of solutions for a nonlinear fractional differential equation:

$$^{C}D^{\beta}\big(x(t)\big) + f\big(t, x(t)\big) = 0 \quad (0 \leq t \leq 1, \beta < 1) \tag{5.1}$$

via the boundary conditions $x(0) = 0 = x(1)$, where $x \in C([0,1], \mathbb{R})$ ($C([0,1], \mathbb{R})$ is the set of all continuous functions from $[0,1]$ into $\mathbb{R}$), $^{C}D^{\beta}$ denotes the Caputo fractional derivative of order β, and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous function (see [23]). Recall that the Green function associated to the problem (5.1) is given by

$$G(t,s) = \begin{cases} (t(1-s))^{\alpha-1} - (t-s)^{\alpha-1} & \text{if } 0 \leq s \leq t \leq 1, \\ \frac{(t(1-s))^{\alpha-1}}{\Gamma(\alpha)} & \text{if } 0 \leq t \leq s \leq 1. \end{cases}$$

Now, we prove the following existence theorem.

Theorem 5.1 *Consider the nonlinear fractional differential equation* (5.1). *Let* $\zeta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *be a given function. Suppose that the following conditions hold*:

(i) $|f(t,a) - f(t,b)| \leq e^{-\tau}|a-b|$ $(\tau > 0)$ *for all* $t \in [0,1]$ *and* $a, b \in \mathbb{R}$ *with* $\zeta(a,b) \geq 0$.

(ii) *There exists* $x_0 \in C([0,1], \mathbb{R})$ *such that* $\zeta(x_0(t), \int_0^1 Tx_0(t)\,dt) \geq 0$ *for all* $t \in [0,1]$ *where* $T : C([0,1], \mathbb{R}) \to C([0,1], \mathbb{R})$ *is defined by*

$$Tx(t) = \int_0^1 G(t,s) f\big(s, x(s)\big)\,ds$$

for all $t \in [0,1]$.

(iii) *For each* $t \in [0,1]$ *and* $x, y \in C([0,1], \mathbb{R})$, $\zeta(x(t), y(t)) > 0$ *implies* $\zeta(Tx(t), Ty(t)) > 0$.

(iv) *For each* $t \in [0,1]$, *if* $\{x_n\}$ *is a sequence in* $C([0,1], \mathbb{R})$ *such that* $x_n \to x$ *in* $C([0,1], \mathbb{R})$ *and* $\zeta(x_n(t), x_{n+1}(t)) > 0$ *for all* $n \in \mathbb{N}$, *then* $\zeta(x_n(t), x(t)) > 0$ *for all* $n \in \mathbb{N}$.

Then the problem (5.1) *has at least one solution*.

Proof First of all, we let $X = C([0,1], \mathbb{R})$. It is well known that X is a Banach space endowed with the supremum norm $\|x\|_\infty = \sup_{t\in[0,1]} |x(t)|$ for all $x \in X$. It is easy to see that $x \in X$ is a solution of (5.1) if and only if $x \in X$ is a solution of the equation $x(t) = \int_0^1 G(t,s)f(s,x(s))\,ds$ for all $t \in [0,1]$. Then the problem (5.1) is equivalent to finding $x^* \in X$ which is a fixed point of T.

Now let $x, y \in X$ such that $\zeta(x(t), y(t)) \geq 0$ for all $t \in [0,1]$. By (i) we have

$$\begin{aligned}
\left|Tx(t) - Ty(t)\right| &= \left|\int_0^1 G(t,s)\bigl[f\bigl(s,x(s)\bigr) - f\bigl(s,y(s)\bigr)\bigr]\,ds\right| \\
&\leq \int_0^1 G(t,s)\bigl|f\bigl(s,x(s)\bigr) - f\bigl(s,y(s)\bigr)\bigr|\,ds \\
&\leq \int_0^1 G(t,s) \cdot e^{-\tau}\bigl(\bigl|x(s) - y(s)\bigr|\bigr)\,ds \\
&\leq e^{-\tau}\|x - y\|_\infty \sup_{t\in I}\int_0^1 G(t,s)\,ds \\
&\leq e^{-\tau}\|x - y\|_\infty.
\end{aligned}$$

Thus for each $x, y \in X$, with $\zeta(x(t), y(t)) > 0$ for all $t \in [0,1]$ we have

$$\|Tx - Ty\|_\infty \leq e^{-\tau}\|x - y\|_\infty \quad \text{or} \quad d(Tx, Ty) \leq e^{-\tau} d(x,y).$$

By passing through a logarithm, we write $\ln d(Tx, Ty) \leq \ln(e^{-\tau} d(x,y))$ and hence $\tau + \ln d(Tx, Ty) \leq \ln d(x,y)$.

Now consider the function $F : \mathbb{R}^+ \to \mathbb{R}$ defined by $F(u) = \ln u$ for each $u \in X$, then $F \in \mathcal{F}$. Also, define $\alpha : X \times X \to \{-\infty\} \cup (0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 1 & \text{if } \zeta(x(t), y(t)) > 0, t \in [0,1], \\ -\infty & \text{otherwise.} \end{cases}$$

Therefore,

$$\begin{aligned}
&\tau + \alpha(x,y)F\bigl(d(Tx,Ty)\bigr) \leq F\bigl(d(x,y)\bigr) \leq F\bigl[d(x,y) + Ld(y,Tx)\bigr] \quad \text{and} \\
&\tau + \alpha(x,y)F\bigl(d(Tx,Ty)\bigr) \leq F\bigl(d(x,y)\bigr) \leq F\bigl[d(x,y) + Ld(x,Ty)\bigr]
\end{aligned}$$

for all $x, y \in X$ with $d(Tx, Ty) > 0$ and $L \geq 0$. This implies that T is an α-type almost-F-contraction. From (ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Next, by using (iii), we get the following assertions holding for all $x, y \in X$

$$\begin{aligned}
\alpha(x,y) \geq 1 \quad &\Longrightarrow \quad \zeta\bigl(x(t), y(t)\bigr) > 0 \quad \text{for all } t \in [0,1] \\
&\Longrightarrow \quad \zeta\bigl(Tx(t), Ty(t)\bigr) > 0 \quad \text{for all } t \in [0,1] \\
&\Longrightarrow \quad \alpha(Tx, Ty) \geq 1;
\end{aligned}$$

hence T is α-admissible. Finally, from condition (iv) in the hypothesis, condition (iii) of Theorem 3.1 holds. Therefore, as an application of Theorem 3.1 we conclude to the existence of $x^* \in X$ such that $x^* = Tx^*$ and so x^* is a solution of the problem (5.1). This completes the proof. □

6 Conclusions

In the present work we introduced the new concepts of an α-type almost-F-contraction and an α-type F-Suzuki contraction, which are generalizations of the concepts given in [14, 15]. Next, we established some fixed point theorems for these contractions. Further, the attached examples and an application to a nonlinear fractional differential equation illustrate the usability of the obtained results.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this article. All authors read and approved final manuscript.

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Acknowledgements

The authors thank the referees for many perceptive comments that led to improvement of the work.

Received: 11 September 2015 Accepted: 3 December 2015 Published online: 04 January 2016

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