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On common fixed points in modular vector spaces

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Abstract

In this work, we discuss the concept of Banach operator pairs in modular vector spaces. We prove the existence of common fixed points for these type of operators which satisfy a modular continuity in modular compact sets. On the basis of our result, we are able to give an analog of DeMarr's common fixed point theorem for a family of symmetric Banach operator pairs in modular vector spaces.

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1 Introduction

In recent years, there was an surge of interest in the study of electrorheological fluids, sometimes referred to as 'smart fluids' (for instance lithium polymetachrylate). For these fluids, modeling with sufficient accuracy using classical Lebesgue and Sobolev spaces, L^p and $W^{1,p}$, where p is a fixed constant is not adequate, rather the exponent p should be able to vary [1–3]. One of the most interesting problem discussed in these spaces is the Dirichlet energy problem [4, 5]. One way to discuss this problem is to convert the energy functional, defined by a modular, to a problem which involves a convoluted and complicated norm (the Luxemburg norm). The modular approach is natural and has not been used.

The purpose of this paper is to discuss the existence of common fixed points of mappings defined on subsets of modular vector spaces, as introduced by Nakano [6] which are natural generalizations of many classical function spaces. The common fixed point problem of a pair of commuting mappings was investigated as early as the first fixed point results were proved [7, 8]. This problem becomes more challenging in view of the historically significant and negatively settled question that a pair of commuting continuous self-mappings defined on $[0,1]$ may not have a common fixed point [9]. Over the years, many mathematicians have tried to find weaker forms of commutativity that imply the existence of a common fixed point for a pair of self-mappings. Weakly compatible mappings [10] and Banach operator pairs [11–17] were introduced and provided generalizing results in metric fixed point theory for single-valued mappings. In this work, we discuss some of these results for multi-valued in modular vector spaces.

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A good reference for metric fixed point theory is the book [18]. For modular spaces, the interested reader may consult the books [19–21].

2 Preliminaries

Modular vector spaces have been studied in [6, 21].

Definition 2.1 [6] Let X be a real vector space. A functional $\rho : X \to [0,+\infty]$ is called a modular if

(1) $\rho(x) = 0$ if and only if $x = 0$;
(2) $\rho(x) = \rho(-x)$;
(3) $\rho(\alpha x + (1-\alpha)y) \leq \rho(x) + \rho(y)$, for any $x, y \in X$ and $\alpha \in [0,1]$.

If we replace (3) by

(4) $\rho(\alpha x + (1-\alpha)y) \leq \alpha\rho(x) + (1-\alpha)\rho(y)$, for any $x, y \in X$ and $\alpha \in [0,1]$,

then ρ is called a convex modular.

The concept of modular finds its roots in the work of Nakano [6] who expanded the earlier ideas of Orlicz and Birnbaum who tried to generalize the classical function spaces of the Lebesgue type L^p. Orlicz and Birnbaum's ideas consisted in considering spaces of functions with some growth properties different from the power type growth. In other words, they considered the space:

$$L^{\varphi} = \left\{ f : \mathbb{R} \to \mathbb{R}; \exists \lambda > 0 \text{ such that } \int_{\mathbb{R}} \varphi\left(\lambda \left|f(x)\right|\right) dx < \infty \right\},$$

where $\varphi : [0,\infty] \to [0,\infty]$ was assumed to be a convex function increasing to infinity, *i.e.*, the function which to some extent behaves similar to power functions $\varphi(t) = t^p$. The functional $\rho : L^{\varphi} \to [0,+\infty]$ defined by

$$\rho(f) = \int_{\mathbb{R}} \varphi\left(\left|f(x)\right|\right) dx,$$

is a modular on L^{φ}.

The classical book [20] contains many examples of modular function spaces. The interested reader may also consult the most recent book [19].

Definition 2.2 Let X be a vector space and ρ a convex modular defined on X. The set

$$X_{\rho} = \left\{ x \in X; \lim_{\lambda \to 0+} \rho(\lambda x) = 0 \right\}$$

is a vector subspace of X known as the associated modular vector space. On it we define the Luxemburg norm

$$\|x\|_{\rho} = \inf\left\{ \lambda; \rho\left(\frac{x}{\lambda}\right) \leq 1 \right\},$$

for any $x \in X_{\rho}$.

In the next definition, we give the basic definitions needed throughout this work.

Definition 2.3 Let X be a vector space and ρ a convex modular defined on X.

(1) The sequence $\{x_n\}_{n\in\mathbb{N}}$ in X_ρ is said to be ρ-convergent to $x \in X_\rho$ if and only if $\rho(x_n - x) \to 0$, as $n \to \infty$. x will be called the ρ-limit of $\{x_n\}$.
(2) The sequence $\{x_n\}_{n\in N}$ in X_ρ is said to be ρ-Cauchy if $\rho(x_m - x_n) \to 0$, as $m, n \to \infty$.
(3) A subset M of X_ρ is said to be ρ-closed if the ρ-limit of a ρ-convergent sequence of M always belong to M.
(4) A subset M of X_ρ is said to be ρ-complete if any ρ-Cauchy sequence in M is a ρ-convergent sequence and its ρ-limit is in M.
(5) A subset M of X_ρ is said to be ρ-bounded if we have

$$\delta_\rho(M) = \sup\bigl\{\rho(x-y); x, y \in M\bigr\} < \infty.$$

(6) A subset M of X_ρ is said to be sequentially ρ-compact if for any $\{x_n\}$ in M there exists a subsequence $\{x_{n_k}\}$ and $x \in M$ such that $\rho(x_{n_k} - x) \to 0$.
(7) A nonempty subset K of X_ρ is said to be ρ-compact if for any family $\{A_\alpha; \alpha \in \Gamma\}$ of ρ-closed subsets of X_ρ with $K \cap A_{\alpha_1} \cap \cdots \cap A_{\alpha_n} \neq \emptyset$, for any $\alpha_1, \ldots, \alpha_n \in \Gamma$, we have

$$K \cap \left(\bigcap_{\alpha\in\Gamma} A_\alpha\right) \neq \emptyset.$$

(8) ρ is said to satisfy the Fatou property if and only if for any sequences $\{x_n\}$ and $\{y_n\}$ in X_ρ which ρ-converge, respectively, to x and y, we have

$$\rho(x-y) \leq \liminf_{n\to\infty} \rho(x_n - y_n).$$

In general if $\lim_{n\to\infty} \rho(\lambda(x_n - x)) = 0$, for some $\lambda > 0$, then we may not have $\lim_{n\to\infty} \rho(\lambda(x_n - x)) = 0$, for all $\lambda > 0$. Therefore, as it is done in modular function spaces, we will say that ρ satisfies Δ_2-condition if this is the case, *i.e.* $\lim_{n\to\infty} \rho(\lambda(x_n - x)) = 0$, for some $\lambda > 0$ implies $\lim_{n\to\infty} \rho(\lambda(x_n - x)) = 0$, for all $\lambda > 0$. In particular, we have

$$\lim_{n\to\infty} \|x_n - x\|_\rho = 0 \quad \text{if and only if} \quad \lim_{n\to\infty} \rho\bigl(\lambda(x_n - x)\bigr) = 0, \quad \text{for all } \lambda > 0,$$

for any $\{x_n\} \in X_\rho$ and $x \in X_\rho$. In other words, we see that ρ-convergence and $\|\cdot\|_\rho$ convergence are equivalent if and only if the modular ρ satisfies the Δ_2-condition.

In the next section, we will prove Brouwer's fixed point theorem in modular vector spaces and obtain a common fixed point result as was done in [22]. The following definition is therefore needed.

Definition 2.4 Let X be a vector space and ρ a convex modular defined on X. Let $C \subset X_\rho$ be nonempty and ρ-closed. Let $T : C \to X_\rho$ be a map. We will say that T is ρ-continuous if $\{T(x_n)\}$ ρ-converges to $T(x)$ if $\{x_n\}$ ρ-converges to x. Moreover, we will say T is strongly ρ-continuous if T is ρ-continuous and

$$\liminf_{n\to\infty} \rho\bigl(y - T(x_n)\bigr) = \rho\bigl(y - T(x)\bigr),$$

for any $\{x_n\} \subset C$ such that $\{x_n\}$ ρ-converges to x and for any $y \in C$. A point $x \in C$ is called a fixed point of T if $T(x) = x$. The set of fixed points of T is denoted by $\mathrm{Fix}(T)$.

We now introduce the concept of Banach operator pairs [11, 17] in modular vector spaces.

Definition 2.5 Let X be a vector space and ρ a convex modular defined on X. Let $C \subset X_\rho$ be nonempty. Let $S, T : C \to C$ be two self-mappings. The ordered pair (T, S) is said to be a Banach operator pair whenever the set $\mathrm{Fix}(T)$ is S-invariant, *i.e.*,

$$S\big(\mathrm{Fix}(T)\big) \subseteq \mathrm{Fix}(T).$$

Note that if S and T commute, *i.e.*, $S \circ T = T \circ S$, then the ordered pairs (S, T) and (T, S) are Banach operator pairs. Therefore the concept of Banach operator pairs is seen as a weakening of the commutativity condition.

3 Common fixed points for Banach operators pairs

We prove the Brouwer fixed point theorem [23] in modular vector spaces via the theorem of Knaster-Kuratowski-Mazurkiewicz (KKM) [24].

Let X be a vector space and ρ a convex modular defined on X. Let $C \subset X_\rho$ be nonempty. The set of all subsets of C is denoted 2^C. The notation $\mathrm{conv}(A)$ describes the convex hull of A, while $\overline{\mathrm{conv}}_\rho(A)$ describes the smallest ρ-closed convex subset of X_ρ which contains A. Recall that a family $\{A_\alpha \subset X_0; \alpha \in \Gamma\}$ is said to have the finite intersection property if the intersection of each finite subfamily is not empty. Throughout this section, we will assume that ρ is finite, *i.e.*, $\rho(x) < +\infty$, for any $x \in X$, and ρ satisfies the Fatou property.

Definition 3.1 Let X be a vector space and ρ a convex modular defined on X. Let $C \subset X_\rho$ be nonempty. A multi-valued mapping $G : C \to 2^{X_\rho}$ is called a Knaster-Kuratowski-Mazurkiewicz mapping (in short KKM-mapping) if

$$\mathrm{conv}\big(\{x_1, \dots, x_n\}\big) \subset \bigcup_{i \in [1,n]} G(x_i),$$

for any $f_1, \dots, f_n \in C$.

A subset $A \subset X_\rho$ is called finitely ρ-closed if for every $x_1, x_2, \dots, x_n \in X_\rho$, the set $\overline{\mathrm{conv}}_\rho(\{x_1, \dots, x_n\}) \cap A$ is ρ-closed.

We are now ready to prove the following result.

Theorem 3.1 *Let X be a vector space and ρ a convex modular defined on X. Let $C \subset X_\rho$ be nonempty, and $G : C \to 2^{X_\rho}$ be a KKM-mapping such that for any $x \in C$, $G(x)$ is nonempty and finitely ρ-closed. Then the family $\{G(x); x \in C\}$ has the finite intersection property.*

Proof Assume not, *i.e.* there exist $x_1, \dots, x_n \in C$ such that $\bigcap_{1 \leq i \leq n} G(x_i) = \emptyset$. Set $L = \overline{\mathrm{conv}}_\rho(\{x_i\})$ in X_ρ. Our assumptions imply that $L \cap G(x_i)$ is ρ-closed for every $i = 1, 2, \dots, n$. Since ρ-closedness implies closedness for the Luxemburg norm $\|\cdot\|_\rho$, $L \cap G(x_i)$ is closed for $\|\cdot\|_\rho$, for any $i \in \{1, \dots, n\}$. Thus for every $x \in L$, there exists i_0 such that x does not belong to $L \cap G(x_{i_0})$ since $L \cap (\bigcap_{1 \leq i \leq n} G(x_i)) = \emptyset$. Hence

$$d\big(x, L \cap G(x_{i_0})\big) = \inf\big\{\|x - y\|_\rho; y \in L \cap G(x_{i_0})\big\} > 0,$$

because $L \cap G(x_{i_0})$ is closed. We use the function

$$\alpha(x) = \sum_{1 \le i \le n} d\big(x, L \cap G(x_i)\big) > 0,$$

where $x \in K = \operatorname{conv}\{x_1, \dots, x_n\}$, to define the map $T : K \to K$ by

$$T(x) = \frac{1}{\alpha(x)} \sum_{1 \le i \le n} d\big(x, L \cap G(x_i)\big) x_i.$$

It is obvious that T is a continuous map. Using the compactness of the convex subset K of $(X_\rho, \|\cdot\|_\rho)$, we can use Brouwer's theorem to ensure the existence of a fixed point $x_0 \in K$ of T, *i.e.* $T(x_0) = x_0$. Set

$$I = \big\{i; d\big(x_0, L \cap G(x_i)\big) \neq 0\big\}.$$

Clearly, we have

$$x_0 = \frac{1}{\alpha(x_0)} \sum_{i \in I} d\big(x_0, L \cap G(x_i)\big) x_i.$$

Hence $x_0 \notin \bigcup_{i \in I} G(x_i)$ and $x_0 \in \operatorname{conv}(\{x_i; i \in I\})$ contradict the assumption

$$\operatorname{conv}\big(\{x_i; i \in I\}\big) \subset \bigcup_{i \in I} G(x_i). \qquad \square$$

As an immediate consequence of Theorem 3.1, we obtain the following result.

Theorem 3.2 *Let X be a vector space and ρ a convex modular defined on X. Let $C \subset X_\rho$ be nonempty, and $G : C \to 2^{X_\rho}$ be a KKM-mapping such that for any $x \in C$, $G(x)$ is nonempty and ρ-closed. Assume there exists $f_0 \in C$ such that $G(f_0)$ is ρ-compact. Then we have $\bigcap_{x \in C} G(x) \neq \emptyset$.*

The following lemma is needed in proving the analog of Ky Fan fixed point result [24] in modular vector spaces.

Lemma 3.1 *Let X be a vector space and ρ a convex modular defined on X. Let $K \subset X_\rho$ be a nonempty convex, ρ-compact subset. Let $T : K \to X_\rho$ be strongly ρ-continuous. Then there exists $x_0 \in K$ such that*

$$\rho\big(x_0 - T(x_0)\big) = \inf_{x \in K} \rho\big(x - T(x_0)\big).$$

Proof Let $G : K \to 2^K \subset 2^{X_\rho}$ be defined by

$$G(y) = \big\{x \in K; \rho\big(x - T(x)\big) \le \rho\big(y - T(x)\big)\big\}.$$

Since T is strongly ρ-continuous and ρ is assumed to satisfy the Fatou property, we have

$$\rho\big(x - T(x)\big) \le \liminf_{n \to \infty} \rho\big(x_n - T(x_n)\big) \le \liminf_{n \to \infty} \rho\big(y - T(x_n)\big) = \rho\big(y - T(x)\big),$$

for any sequence $\{x_n\}$ in $G(y)$ which ρ-converges to x. Hence $G(y)$ is ρ-closed for any $y \in K$. Next, assume that G is not a KKM-mapping. Then there exists $\{y_1,\ldots,y_n\} \subset K$ and $x \in \operatorname{conv}(\{y_i\})$ such that $x \notin \bigcup_{1\leq i\leq n} G(y_i)$. This clearly implies

$$\rho\bigl(y_i - T(x)\bigr) < \rho\bigl(x - T(x)\bigr), \quad \text{for } i = 1,\ldots,n.$$

Let $\varepsilon > 0$ be such that $\rho(y_i - T(x)) \leq \rho(x - T(x)) - \varepsilon$, for $i = 1,2,\ldots,n$. Using the convexity of ρ, we get

$$\rho\bigl(y - T(x)\bigr) \leq \rho\bigl(x - T(x)\bigr) - \varepsilon,$$

for any $y \in \operatorname{conv}(\{y_i\})$. Since $x \in \operatorname{conv}(\{y_i\})$, we get $\rho(x, T(x)) \leq \rho(x, T(x)) - \varepsilon$. This is a contradiction which implies that G is a KKM-mapping. Since K is ρ-compact, we deduce that $G(y)$ is also ρ-compact for any $y \in K$. Using Theorem 3.2, there exists x_0 in $\bigcap_{y\in K} G(y)$. Hence $\rho(x_0 - T(x_0)) \leq \rho(y - T(x_0))$, for any $y \in K$, which implies

$$\rho\bigl(x_0 - T(x_0)\bigr) = \inf_{y\in K} \rho\bigl(y - T(x_0)\bigr). \qquad \square$$

Before we state the Ky Fan fixed point theorem [24] in modular vector spaces, we recall the definition of modular balls. Let $x \in X_\rho$ and $r \geq 0$. The modular ball $B_\rho(x,r)$ is defined by

$$B_\rho(x,r) = \bigl\{y \in X; \rho(x - y) \leq r\bigr\}.$$

Since ρ is convex and satisfies the Fatou property, the modular balls are convex and ρ-closed.

Theorem 3.3 *Let X be a vector space and ρ a convex modular defined on X. Let $K \subset X_\rho$ be a nonempty convex, ρ-compact subset. Let $T : K \to X_\rho$ be strongly ρ-continuous such that for any $x \in K$, with $x \neq T(x)$, there exists $\alpha \in (0,1)$ such that*

$$K \cap B_\rho\bigl(x, \alpha\rho\bigl(x - T(x)\bigr)\bigr) \cap B_\rho\bigl(T(x), (1-\alpha)\rho\bigl(x - T(x)\bigr)\bigr) \neq \emptyset. \tag{3.1}$$

Then T has a fixed point.

Proof By Lemma 3.1, there exists $x_0 \in K$ such that

$$\rho\bigl(x_0 - T(x_0)\bigr) = \inf_{x\in K} \rho\bigl(x - T(x_0)\bigr).$$

Assume that x_0 is not a fixed point of T, *i.e.*, $x_0 \neq T(x_0)$. Then there exists $\alpha \in (0,1)$ such that

$$K_0 = K \cap B_\rho\bigl(x_0, \alpha\rho\bigl(x_0 - T(x_0)\bigr)\bigr) \cap B_\rho\bigl(T(x_0), (1-\alpha)\rho\bigl(x_0 - T(x_0)\bigr)\bigr) \neq \emptyset.$$

Let $y \in K_0$. Then $\rho(y - T(x_0)) \leq (1 - \alpha)\rho(x_0 - T(x_0))$. This implies a contradiction to the property satisfied by x_0. Therefore we must have $T(x_0) = x_0$, *i.e.*, x_0 is a fixed point of T. $\square$

Note that if $T(K) \subset K$, then T satisfies the condition (3.1). The following theorem is an analog to Brouwer's fixed point theorem in modular vector spaces.

Theorem 3.4 *Let X be a vector space and ρ a convex modular defined on X. Let $K \subset X_\rho$ be a nonempty convex, ρ-compact subset. Let $T : K \to K$ be strongly ρ-continuous. Then* $\mathrm{Fix}(T)$ *is a nonempty ρ-compact subset.*

Proof The existence of a fixed point of T is a direct consequence of Theorem 3.3. So $\mathrm{Fix}(T)$ is nonempty. Next we show that $\mathrm{Fix}(T)$ is ρ-compact. Let us prove that $\mathrm{Fix}(T)$ is ρ-closed. Let $\{x_n\}$ be a sequence in $\mathrm{Fix}(T)$ such that $\{x_n\}$ ρ-converges to x. Since T is ρ-continuous, $\{T(x_n)\}$ ρ-converges to $T(x)$. Since $T(x_n) = x_n$, we see that $\{x_n\}$ ρ-converges to x and $T(x)$. Using the uniqueness of the ρ-limit, we get $T(x) = x$, *i.e.*, $x \in \mathrm{Fix}(T)$. Since K is ρ-compact and $\mathrm{Fix}(T)$ is ρ-closed, we conclude that $\mathrm{Fix}(T)$ is ρ-compact. □

In order to prove our main result of this section, we need the following definition.

Definition 3.2 Let X be a vector space and ρ a convex modular defined on X. Let $C \subset X_\rho$ be a nonempty subset. A mapping $T : C \to C$ is said to be an R-map if $\mathrm{Fix}(T)$ is a ρ-continuous retraction of C, *i.e.*, there exists a ρ-continuous mapping $R : C \to \mathrm{Fix}(T)$ such that $R \circ R = R$. Such a mapping is known as a retraction.

Next we discuss our first common fixed point result of this work.

Theorem 3.5 *Let X be a vector space and ρ a convex modular defined on X. Let $K \subset X_\rho$ be a nonempty convex, ρ-compact subset. Let $S, T : K \to K$ be two self-mappings. Assume that S and T are strongly ρ-continuous such that (S, T) is a Banach operator pair. If we assume that T is an R-map, then* $\mathrm{Fix}(T) \cap \mathrm{Fix}(S)$ *is nonempty and ρ-compact.*

Proof Theorem 3.4 implies that $\mathrm{Fix}(T)$ is not empty and ρ-compact. Since T is an R-map, there exists a retraction $R : K \to \mathrm{Fix}(T)$ which is ρ-continuous. Since (S, T) is a Banach pair of operators, $S(\mathrm{Fix}(T)) \subset \mathrm{Fix}(T)$. It is clear that $S \circ R : K \to K$ is strongly ρ-continuous. To see this, let $\{x_n\} \subset K$ be ρ-convergent to x. Then $\{R(x_n)\}$ ρ-converges to $R(x)$ since R is ρ-continuous. Using the strong ρ-continuity of S, we have

$$\liminf_{n\to\infty} \rho\bigl(y - S\bigl(R(x_n)\bigr)\bigr) = \rho\bigl(y - S\bigl(R(x)\bigr)\bigr),$$

for any $y \in K$, *i.e.*, $S \circ R$ is strongly ρ-continuous. Theorem 3.4 implies that $\mathrm{Fix}(S \circ R)$ is nonempty and ρ-compact. Moreover, if $x \in \mathrm{Fix}(S \circ R)$, then $S \circ R(x) = S(R(x)) = x \in \mathrm{Fix}(T)$ because $S \circ R(K) \subset \mathrm{Fix}(T)$. In particular, we have $R(x) = x$. Hence $S(x) = x$, *i.e.* $x \in \mathrm{Fix}(T) \cap \mathrm{Fix}(S)$. It is easy to see that $\mathrm{Fix}(T) \cap \mathrm{Fix}(S) = \mathrm{Fix}(S \circ R)$. Hence $\mathrm{Fix}(T) \cap \mathrm{Fix}(S)$ is nonempty and ρ-compact. □

Next we extend the conclusion of Theorem 3.5 to a family of Banach operator mappings. The following concept will be needed.

Definition 3.3 Let X be a vector space and ρ a convex modular defined on X. Let $C \subset X_\rho$ be a nonempty subset. Let $S, T : C \to C$ be two self-mappings. (S, T) is called a symmetric

Banach operator pair if (S,T) and (T,S) are Banach operator pairs, *i.e.*,

$$T\big(\mathrm{Fix}(S)\big)\subseteq \mathrm{Fix}(S) \quad \text{and} \quad S\big(\mathrm{Fix}(T)\big)\subseteq \mathrm{Fix}(T).$$

Recall that the set of common fixed points of a family of mappings $\mathcal{F}$ is the set $\mathrm{Fix}(\mathcal{F}) = \bigcap_{T\in\mathcal{F}}\mathrm{Fix}(T)$. The main result of this work states:

Theorem 3.6 *Let X be a vector space and ρ a convex modular defined on X. Let $K\subset X_\rho$ be nonempty convex and ρ-compact. Let $\mathcal{F}$ be a family of self-mappings defined on K such that any map in $\mathcal{F}$ is a strongly ρ-continuous R-map. Assume that any two mappings in $\mathcal{F}$ form a symmetric Banach operator pair. Then* $\mathrm{Fix}(\mathcal{F})$ *is a nonempty ρ-compact subset.*

Proof Theorem 3.5 implies that $\mathrm{Fix}(T_1)\cap\cdots\cap\mathrm{Fix}(T_n)$ is a nonempty ρ-compact subset of K, for any $T_1,T_2,\ldots,T_n$ in $\mathcal{F}$. Therefore any finite family of the subsets $\{\mathrm{Fix}(T); T\in\mathcal{F}\}$ has a nonempty intersection. Since these sets are all ρ-closed and K is ρ-compact, we conclude that $\mathrm{Fix}(\mathcal{F}) = \bigcap_{T\in\mathcal{F}}\mathrm{Fix}(T)$ is not empty and is ρ-closed. Therefore $\mathrm{Fix}(\mathcal{F})$ is ρ-compact. □

As commuting mappings are symmetric Banach operator pairs, we obtain an analog to DeMarr's common fixed point theorem [8] in modular spaces as follows.

Corollary 3.1 *Let X be a vector space and ρ a convex modular defined on X. Let $K\subset X_\rho$ be a nonempty convex, ρ-compact subset. Let $\mathcal{F}$ be a family of commuting self-mappings defined on K such that any map in $\mathcal{F}$ is a strongly ρ-continuous R-map. Then* $\mathrm{Fix}(\mathcal{F})$ *is nonempty and ρ-compact.*

Remark 3.1 Since convexity plays a crucial role in the proof of Brouwer's fixed point theorem, an analog of this profound theorem does not exist in metric spaces. In order to obtain an extension of this fundamental fixed point result, one needs to use some kind of convexity therein. This is the case for example of hyperconvex metric spaces [25]. So to obtain an extension of all the results obtained in this work for modular metric spaces [26–28], we will have to introduce something like hyperbolicity in modular metric spaces. This concept will be new and has not been introduced or investigated so far.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the manuscript.

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