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Fixed point theorems for Meir-Keeler type mappings in M-metric spaces with applications

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Abstract
In this paper, we establish some fixed point theorems for a Meir-Keeler type contraction in M-metric spaces via Gupta-Saxena type contraction. Also, we extend and improve very recent results in fixed point theory.

MSC: Primary 47H10; secondary 54H25

Keywords: fixed point; partial metric space; M-metric space

1 Introduction and preliminaries

Ekeland formulated a variational principle that is the foundation of modern variational calculus, having applications in many branches of mathematics, including optimization and fixed point theory [1] and applications in nonlinear analysis, since it entails the existence of approximate solutions of minimization problems for a lower semi-continuous function that is bounded from below on complete metric spaces. Also, Ekeland's variational principle is a fruitful tool in simplifying and unifying the proofs of already known theorems and has many generalizations; see Borwein and Zhu [2].

Matthews in 1994 [3] introduced a partial metric space and proved the contraction principle of Banach in this new framework. Afterward, by several mathematicians many fixed point theorems were founded in partial metric spaces. Recently Haghi *et al.* [4] published a paper which stated that we should 'be careful on partial metric fixed point results' along with very some results therein. They showed that fixed point generalizations to partial metric spaces can be obtained from the corresponding results in metric spaces.

In 2014, Asadi *et al.* [5] introduced the M-metric space, which extends the p-metric space and certain fixed point theorems obtained therein.

In this paper, we establish some of the fixed point theorem for a Meir-Keeler type contraction in M-metric spaces via a Gupta-Saxena type contraction. Also, we extend and improve very recent results in fixed point theory.

Definition 1.1 ([3], [6], Definition 1.1) A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$:

(p1) $p(x,x) = p(y,y) = p(x,y) \iff x = y$,

(p2) $p(x,x) \leq p(x,y)$,

(p3) $p(x,y) = p(y,x)$,
(p4) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.
A partial metric space is a pair (X,p) such that X is a nonempty set and p is a partial metric on X.

Notation The following notations are useful in the sequel:
(i) $m_{xy} := \min\{m(x,x), m(y,y)\} = m(x,x) \vee m(y,y)$,
(ii) $M_{xy} := \max\{m(x,x), m(y,y)\} = m(x,x) \wedge m(y,y)$.

Now we want to extend Definition 1.1 as follows.

Definition 1.2 Let X be a non-empty set. A function $m : X \times X \to \mathbb{R}^+$ is called a m-metric if the following conditions are satisfied:
(m1) $m(x,x) = m(y,y) = m(x,y) \iff x = y$,
(m2) $m_{xy} \leq m(x,y)$,
(m3) $m(x,y) = m(y,x)$,
(m4) $(m(x,y) - m_{xy}) \leq (m(x,z) - m_{xz}) + (m(z,y) - m_{zy})$.
Then the pair (X,m) is called an M-metric space.

According to the above, our definition of the condition (p1) in the definition [3] changes to (m1) and (p2) for $p(x,x)$ is expressed by just $p(y,y) = 0$; we may have $p(y,y) \neq 0$, so we improved that condition by replacing it by $\min\{p(x,x), p(y,y)\} \leq p(x,y)$, and also we improved the condition (p4) to the form (m4). In the sequel we present an example that holds for the m-metric, but not for the p-metric.

Remark 1.1 For every $x, y \in X$:
(i) $0 \leq M_{xy} + m_{xy} = m(x,x) + m(y,y)$,
(ii) $0 \leq M_{xy} - m_{xy} = |m(x,x) - m(y,y)|$,
(iii) $M_{xy} - m_{xy} \leq (M_{xz} - m_{xz}) + (M_{zy} - m_{zy})$.

The next examples state that m^s and m^w are ordinary metrics.

Example 1.1 Let m be a m-metric. Put:
(i) $m^w(x,y) = m(x,y) - 2m_{xy} + M_{xy}$,
(ii) $m^s(x,y) = m(x,y) - m_{xy}$ when $x \neq y$ and $m^s(x,y) = 0$ if $x = y$.
Then m^w and m^s are ordinary metrics.

Proof If $m^w(x,y) = 0$, then

$$m(x,y) = 2m_{xy} - M_{xy}. \tag{1}$$

But from (1) and $m_{xy} \leq m(x,y)$ we get $m_{xy} = M_{xy} = m(x,x) = m(y,y)$, so by (1) we obtain $m(x,y) = m(x,x) = m(y,y)$, and therefore $x = y$. For the triangle inequality it is enough that we consider Remark 1.1 and (m4). □

In the following example, we present an example of a m-metric which is not a p-metric.

Remark 1.2 For every $x, y \in X$:

(i) $m(x,y) - M_{xy} \leq m^w(x,y) \leq m(x,y) + M_{xy}$,

(ii) $(m(x,y) - M_{xy}) \leq m^s(x,y) \leq m(x,y)$.

Example 1.2 Let $X = \{1,2,3\}$. Define

$$m(1,2) = m(2,1) = m(1,1) = 8,$$

$$m(1,3) = m(3,1) = m(3,2) = m(2,3) = 7, \qquad m(2,2) = 9, \quad \text{and} \quad m(3,3) = 5,$$

so m is an m-metric but m is not a p-metric. Since $m(2,2) \nleq m(1,2)$, m is not a p-metric. If $D(x,y) = m(x,y) - m_{x,y}$ then $m(1,2) = m_{1,2} = 8$ but it means $D(1,2) = 0$, while $1 \neq 2$ means D is not a metric.

Example 1.3 ([5]) Let (X,d) be a metric space, $\phi : [0,\infty) \to [\phi(0),\infty)$ be a one to one and nondecreasing or strictly increasing mapping with $\phi(0)$, defined such that

$$\phi(x+y) \leq \phi(x) + \phi(y) - \phi(0) \quad \forall x, y \geq 0.$$

Then $m(x,y) = \phi(d(x,y))$ is an m-metric.

Example 1.4 Let (X,d) be a metric space. Then $m(x,y) = ad(x,y) + b$ where $a, b > 0$ is an m-metric, because we can put $\phi(t) = at + b$.

Remark 1.3 According to the Example 1.4, by the Banach contraction

$$\exists k \in [0,1), \quad m(Tx,Ty) \leq km(x,y), \quad \text{for all } x, y \in X,$$

we have

$$m(Tx,Ty) = ad(Tx,Ty) + b \leq kad(x,y) + kb \Rightarrow d(Tx,Ty) \leq kd(x,y) + \frac{b(k-1)}{a},$$

which does not imply that we have the ordinary Banach contraction

$$\exists k \in [0,1), \quad d(Tx,Ty) \leq kd(x,y) \quad \text{for all } x, y \in X,$$

for all self-maps T on X. So this states that if the m-metric m and the ordinary metric d even have the same topology, but the Banach contraction of an m-metric, this does not imply the Banach contraction of the ordinary metric d.

Lemma 1.1 ([5]) *Every p-metric is an m-metric.*

2 Topology for *M*-metric space

It is clear that each m-metric p on X generates a T_0 topology τ_m on X. The set

$$\{B_m(x,\varepsilon) : x \in X, \varepsilon > 0\},$$

where

$$B_m(x,\varepsilon) = \{y \in X : m(x,y) < m_{x,y} + \varepsilon\},$$

for all $x \in X$ and $\varepsilon > 0$, forms the base of τ_m.

Definition 2.1 Let (X,m) be an M-metric space. Then:

(1) A sequence $\{x_n\}$ in an M-metric space (X,m) converges to a point $x \in X$ if and only if

$$\lim_{n\to\infty}\left(m(x_n,x) - m_{x_n,x}\right) = 0. \tag{2}$$

(2) A sequence $\{x_n\}$ in an M-metric space (X,m) is called an m-Cauchy sequence if

$$\lim_{n,m\to\infty}\left(m(x_n,x_m) - m_{x_n,x_m}\right) \quad \text{and} \quad \lim_{n,m\to\infty}(M_{x_n,x_m} - m_{x_n,x_m}) \tag{3}$$

in this space exist (and are finite).

(3) An M-metric space (X,m) is said to be complete if every m-Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_m, to a point $x \in X$ such that

$$\left(\lim_{n\to\infty}\left(m(x_n,x) - m_{x_n,x}\right) = 0 \text{ and } \lim_{n\to\infty}(M_{x_n,x} - m_{x_n,x}) = 0\right).$$

Lemma 2.1 *Let (X,m) be an M-metric space. Then*:

(1) *$\{x_n\}$ is a m-Cauchy sequence in (X,m) if and only if it is a Cauchy sequence in the metric space (X,m^w).*

(2) *An M-metric space (X,m) is complete if and only if the metric space (X,m^w) is complete. Furthermore,*

$$\lim_{n\to\infty} m^w(x_n,x) = 0 \iff \left(\lim_{n\to\infty}\left(m(x_n,x) - m_{x_n,x}\right) = 0 \textit{ and } \lim_{n\to\infty}(M_{x_n,x} - m_{x_n,x}) = 0\right).$$

Likewise the above definition holds also for m^s.

Lemma 2.2 *Assume that $x_n \to x$ and $y_n \to y$ as $n \to \infty$ in an M-metric space (X,m). Then*

$$\lim_{n\to\infty}\left(m(x_n,y_n) - m_{x_n,y_n}\right) = m(x,y) - m_{xy}.$$

Proof We have

$$\left|\left(m(x_n,y_n) - m_{x_n,y_n}\right) - \left(m(x,y) - m_{x,y}\right)\right| \le \left(m(x_n,x) - m_{x_n,x}\right) + \left(m(y,y_n) - m_{y,y_n}\right). \qquad \square$$

From Lemma 2.2 we can deduce the following lemma.

Lemma 2.3 *Assume that $x_n \to x$ as $n \to \infty$ in an M-metric space (X,m). Then*

$$\lim_{n\to\infty}\left(m(x_n,y) - m_{x_n,y}\right) = m(x,y) - m_{x,y},$$

for all $y \in X$.

Lemma 2.4 *Assume that $x_n \to x$ and $x_n \to y$ as $n \to \infty$ in an M-metric space (X, m). Then $m(x, y) = m_{xy}$. Further if $m(x, x) = m(y, y)$, then $x = y$.*

Proof By Lemma 2.2 we have

$$0 = \lim_{n\to\infty}\big(m(x_n, x_n) - m_{x_n,x_n}\big) = m(x, y) - m_{xy}.$$ □

Lemma 2.5 *Let $\{x_n\}$ be a sequence in an M-metric space (X, m), such that*

$$\exists r \in [0,1) \quad \text{such that} \quad m(x_{n+1}, x_n) \le rm(x_n, x_{n-1}) \quad \forall n \in \mathbb{N}. \tag{4}$$

Then

(A) $\lim_{n\to\infty} m(x_n, x_{n-1}) = 0$,
(B) $\lim_{n\to\infty} m(x_n, x_n) = 0$,
(C) $\lim_{m,n\to\infty} m_{x_m x_n} = 0$,
(D) $\{x_n\}$ *is an m-Cauchy sequence.*

Proof From (4) we have,

$$m(x_n, x_{n-1}) \le rm(x_{n-1}, x_{n-2}) \le r^2 m(x_{n-2}, x_{n-3}) \le \cdots \le r^n m(x_0, x_1),$$

thus

$$\lim_{n\to\infty} m(x_n, x_{n-1}) = 0,$$

which implies (A).

To prove (B), from (m2) and (A) we have

$$\lim_{n\to\infty} \min\big\{m(x_n, x_n), m(x_{n-1}, x_{n-1})\big\} = \lim_{n\to\infty} m_{x_n x_{n-1}} \le \lim_{n\to\infty} m(x_n, x_{n-1}) = 0.$$

That is, (B) holds.

Clearly, (C) holds, since $\lim_{n\to\infty} m(x_n, x_n) = 0$. □

Theorem 2.1 *The topology τ_m is not Hausdorff.*

Theorem 2.2 *Let (X, m) be a complete M-metric space and $T : X \to X$ be mapping satisfying the following condition*:

$$\exists k \in [0,1) \quad \text{such that} \quad m(Tx, Ty) \le km(x, y) \quad \forall x, y \in X. \tag{5}$$

Then T has a unique fixed point.

Theorem 2.3 *Let (X, m) be a complete M-metric space and $T : X \to X$ be mapping satisfying the following condition*:

$$\exists k \in \left[0, \frac{1}{2}\right) \quad \text{such that} \quad m(Tx, Ty) \le k\big(m(x, Tx) + m(y, Ty)\big) \quad \forall x, y \in X. \tag{6}$$

Then T has a unique fixed point.

3 Main result and fixed point theorems

The following definition is new version of the definition in [7] for an M-metric space.

Definition 3.1 A Meir-Keeler mapping is a mapping $T : M \to M$ on an M-metric space (X, M) such that

$$\forall \varepsilon > 0\ \exists \delta > 0 \text{ such that } \forall x, y \in X \text{ and } \varepsilon \leq m(x, y) < \varepsilon + \delta \Rightarrow m(Tx, Ty) < \varepsilon. \tag{7}$$

Theorem 3.1 *Let (X, m) be a complete M-metric space and let T be a mapping from X into itself satisfying the following condition*:

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x, y \in X\ \varepsilon \leq m(x, y) < \varepsilon + \delta \Rightarrow m(Tx, Ty) < \varepsilon.$$

Then T has a unique fixed point $u \in X$. Moreover, for all $x \in X$, the sequence $\{T_n(x)\}$ converges to u.

Proof We first observe that (7) trivially implies that T is a strict contraction, *i.e.*,

$$x \neq y \Rightarrow m(Tx, Ty) < m(x, y). \tag{8}$$

Let $x_0 \in X$ and $x_n := Tx_{n-1}$, so we have

$$m(x_n, x_{n-1}) = m(Tx_{n-1}, Tx_{n-2}) \leq m(x_{n-1}, x_{n-2}). \tag{9}$$

So the sequence $\{m(x_n, x_{n-1})\}$ is bounded below and decreasing; thus $m(x_n, x_{n-1}) \to m$ for some $m \in \mathbb{R}^+$. Let $m > 0$, therefore $m(x_n, x_{n-1}) \geq m$. On the other hand for $m > 0$ there exists $\delta(m) > 0$ such that

$$m \leq m(x_{n-1}, x_{n-2}) < m + \delta(m) \Rightarrow m(Tx_{n-1}, Tx_{n-2}) = m(x_n, x_{n-1}) < m,$$

which implies that it is contradiction; so $m = 0$, *i.e.*,

$$\lim_{n\to\infty} m(x_n, x_{n+1}) = 0, \tag{10}$$

$$\lim_{n\to\infty} \min\bigl\{m(x_n, x_n), m(x_{n-1}, x_{n-1})\bigr\} = \lim_{n\to\infty} m_{x_n x_{n-1}} \leq \lim_{n\to\infty} m(x_n, x_{n-1}) = 0,$$

and

$$\lim_{m,n\to\infty} m_{x_m x_n} = 0 \quad \text{and} \quad \lim_{m,n\to\infty} M_{x_m x_n} = 0, \tag{11}$$

since, $\lim_{n\to\infty} m(x_n, x_n) = 0$. Now we want to show that $\lim_{m,n\to\infty} m(x_m, x_n) = 0$. Let it be untrue. So for some $\varepsilon > 0$ we have $\limsup_{m,n\to\infty} m(x_m, x_n) > 2\varepsilon$. Also, by hypothesis, there exists a $\delta > 0$, such that

$$\varepsilon \leq m(x, y) < \varepsilon + \delta \Rightarrow m(Tx, Ty) < \varepsilon,$$

which remains true with δ replaced by $\delta' = \min\{\delta, \varepsilon\}$. Now by (10)

$$\exists N > 0\ \forall n \quad \left(n > N \Rightarrow m(x_n, x_{n+1}) < \frac{\delta'}{3}\right),$$

and for $m, n > N$, $m(x_m, x_n) > 2\varepsilon$. This implies, since

$$m(x_n, x_{n+1}) < \varepsilon \quad \text{and} \quad \varepsilon + \delta' < 2\varepsilon < m(x_m, x_n),$$

that there exists i with $m < i < n$ with

$$\varepsilon + \frac{2\delta'}{3} < m(x_m, x_i) - m_{x_m, x_i} < \varepsilon + \delta'. \tag{12}$$

However, for all m and i,

$$\begin{aligned} m(x_m, x_i) - m_{x_m, x_i} &\leq m(x_m, x_{m+1}) - m_{x_{m+1}, x_{i+1}} + m(x_{m+1}, x_{i+1}) - m_{x_{m+1}, x_{i+1}} \\ &\quad + m(x_{i+1}, x_i) - m_{x_{i+1}, x_i} \\ &\leq m(x_m, x_{m+1}) + m(x_{m+1}, x_{i+1}) + m(x_{i+1}, x_i) \\ &\leq \frac{\delta'}{3} + \varepsilon + \frac{\delta'}{3}, \end{aligned}$$

which contradicts (12). So by (11) and $\lim_{m,n\to\infty} m(x_m, x_n) = 0$ we see that the sequence $\{x_n\}$ is a Cauchy sequence and by completeness of X, $x_n \to x^*$ in m for some $x^* \in X$, *i.e.*,

$$\lim_{n\to\infty} \bigl(m(x_n, x^*) - m_{x_n, x^*}\bigr) = 0. \tag{13}$$

But $m_{x_n, x^*} \to 0$ because $m(x_n, x_n) \to 0$ so $m(x_n, x^*) \to 0$. Thus, by hypothesis, $m(Tx_n, Tx^*) \leq m(x_n, x^*) \to 0$. Hence by (m2) $m_{Tx_n, Tx^*} \leq m(Tx_n, Tx^*) \to 0$, so by (2) $Tx_n \to Tx^*$.

Equation (10) implies that $m(x_n, Tx_n) \to 0$. Since $m_{x_n, Tx_n} \to 0$, by Lemma 2.2, we get $m(x^*, Tx^*) = m_{x^*, Tx^*}$.

On the other hand, by Lemma 2.2 and

$$Tx_{n-1} = x_n \to x^* \quad \text{and also} \quad x_{n+1} = Tx_n \to Tx^*,$$

we have

$$\begin{aligned} 0 &= \lim_{n\to\infty} \bigl(m(x_n, Tx_n) - m_{x_n, Tx_n}\bigr) \\ &= \lim_{n\to\infty} \bigl(m(x_n, x_{n-1}) - m_{x_n, Tx_n}\bigr) \\ &= m\bigl(x^*, x^*\bigr) - m_{x^*, Tx^*} \\ &= m\bigl(Tx^*, Tx^*\bigr) - m_{x^*, Tx^*} \end{aligned}$$

thus $m(x^*, x^*) = m_{x^*, Tx^*} = m(Tx^*, Tx^*)$ and since

$$m\bigl(x^*, Tx^*\bigr) = m_{x^*, Tx^*} = m\bigl(Tx^*, Tx^*\bigr) = m\bigl(x^*, x^*\bigr),$$

now by (m1) $x^* = Tx^*$. Uniqueness by the contraction (8) is clear. □

Put

$$C(x,y) = m(x,y) + \frac{(1+m(x,Tx))m(y,Ty)}{1+m(x,y)} + \frac{m(x,Tx)m(y,Ty)}{m(x,y)}.$$

Theorem 3.2 *Let (X,m) be a complete M-metric space and let T be a continuous mapping from X into itself satisfying the following condition*:

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x,y \in X\ \varepsilon \leq kC(x,y) < \varepsilon + \delta \Rightarrow m(Tx,Ty) < \varepsilon, \tag{14}$$

for some $0 < k < \frac{1}{3}$. Then T has a unique fixed point $u \in X$. Moreover, for all $x \in X$, the sequence $\{T_n(x)\}$ converges to u.

Proof We first observe that (14) trivially implies that T is a strict contraction, *i.e.*,

$$x \neq y \Rightarrow m(Tx,Ty) < kC(x,y). \tag{15}$$

Let $x_0 \in X$ and $x_n := Tx_{n-1}$ so we have

$$\begin{aligned} C(x_{n-1},x_n) &= m(x_{n-1},x_n) + \frac{(1+m(x_{n-1},x_n))m(x_n,x_{n+1})}{1+m(x_{n-1},x_n)} + \frac{m(x_{n-1},x_n)m(x_n,x_{n+1})}{m(x_{n-1},x_n)} \\ &\leq k\bigl(m(x_{n-1},x_n) + 2m(x_n,x_{n+1})\bigr), \\ m(x_n,x_{n+1}) &= m(Tx_{n-1},Tx_n) \\ &\leq kC(x_{n-1},x_n) \\ &\leq k\bigl(m(x_{n-1},x_n) + 2m(x_n,x_{n+1})\bigr), \end{aligned}$$

therefore

$$m(x_n,x_{n+1}) \leq rm(x_{n-1},x_n), \tag{16}$$

where $r = \frac{k}{1-2k} < 1$. Now by Lemma 2.5, $\{x_n\}$ is a Cauchy sequence, and by completeness of X, $Tx_{n-1} = x_n \to x^*$ in m for some $x^* \in X$. Since T is a continuous mapping, so $x_n = Tx_{n-1} \to Tx^*$, in m now by Lemma 2.4 we find

$$\begin{aligned} &m\bigl(x^*,Tx^*\bigr) = m_{x^*,Tx^*}, \\ &0 = \lim_{n\to\infty}\bigl(m(x_n,Tx_n) - m_{x_n,Tx_n}\bigr) = m\bigl(x^*,x^*\bigr) - m_{x^*,Tx^*} = m\bigl(Tx^*,Tx^*\bigr) - m_{x^*,Tx^*}, \end{aligned}$$

by Lemma 2.2 and

$$m\bigl(x^*,Tx^*\bigr) = m_{x^*,Tx^*} = m\bigl(Tx^*,Tx^*\bigr) = m\bigl(x^*,x^*\bigr).$$

So $x^* = Tx^*$. Uniqueness by the contraction (15) is clear. □

Corollary 3.1 (Gupta and Saxena [8]) *Let (X,d) be a complete metric space and T be a continuous mapping from X into itself. Assume that T satisfies*

$$\forall x,y \in X\ x \neq y\ d(Tx,Ty) \leq kC(x,y),$$

where $k \in (0, \frac{1}{3})$ is a constant. Then T has a unique fixed point $u \in X$. Moreover, for all $x \in X$, the sequence $\{T_n(x)\}$ converges to u.

4 Applications

In this section, after an idea of Samet *et al.* [9], we shall state an integral version of the Gupta-Saxena result.

Theorem 4.1 *Let (X, m) be an M-metric space and let T be a self-mapping defined on X. Assume that there exists a function $\varphi : [0, \infty) \to [0, \infty)$ satisfying the following*:

(1) $\varphi(0) = 0$ *and* $t > 0 \Rightarrow \varphi(t) > 0$;

(2) φ *is nondecreasing and right continuous*;

(3) *for every* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that*

$$\varepsilon \leq \varphi\bigl(kC(x,y)\bigr) < \varepsilon + \delta \Rightarrow \varphi\bigl(m(Tx, Ty)\bigr) < \varepsilon, \tag{17}$$

for some $0 < k < \frac{1}{3}$ *and for all* $x, y \in X$ *with* $x \neq y$.

Then (14) *is satisfied*.

Proof Fix $\varepsilon > 0$, so $\varphi(\varepsilon) > 0$. Hence by (17) there exists $\delta_1 > 0$ such that

$$\forall x, y \in X\ x \neq y\ \varphi(\varepsilon) \leq \varphi\bigl(kC(x,y)\bigr) < \varphi(\varepsilon) + \delta_1 \Rightarrow \varphi\bigl(m(Tx, Ty)\bigr) < \varphi(\varepsilon). \tag{18}$$

According to the right continuity of φ

$$\exists \delta > 0\ \varphi(\varepsilon + \delta_1) < \varphi(\varepsilon) + \delta.$$

Now for $x, y \in X$ with $x \neq y$ and fixed

$$\varepsilon \leq kC(x,y) < \varepsilon + \delta, \tag{19}$$

since φ is a nondecreasing mapping, we have

$$\varphi(\varepsilon) \leq \varphi\bigl(kC(x,y)\bigr) < \varphi(\varepsilon + \delta_1) < \varphi(\varepsilon) + \delta.$$

So we get

$$\varphi\bigl(m(Tx, Ty)\bigr) < \varphi(\varepsilon),$$

which implies that $m(Tx, Ty) < \varepsilon$. □

Corollary 4.1 *Let (X, m) be an M-metric space and let T be a self-mapping defined on X. Let $h : [0, \infty) \to [0, \infty)$ be a locally integrable function such that*

(1) $t > 0 \Rightarrow \int_0^t h(s)\, ds > 0$;

(2) *for every* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that*

$$\frac{1}{k}\varepsilon \leq \int_0^{C(x,y)} h(s)\, ds < \frac{1}{k}\varepsilon + \delta \Rightarrow \int_0^{\frac{1}{k}m(Tx,Ty)} h(s)\, ds < \frac{1}{k}\varepsilon, \tag{20}$$

for some $0 < k < \frac{1}{3}$ *and for all* $x, y \in X$ *with* $x \neq y$.

Then (14) *is satisfied*.

Competing interests
The author declares to have no competing interests.

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