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General iterative methods for monotone mappings and pseudocontractive mappings related to optimization problems

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Abstract

In this paper, we introduce two general iterative methods for a certain optimization problem of which the constrained set is the common set of the solution set of the variational inequality problem for a continuous monotone mapping and the fixed point set of a continuous pseudocontractive mapping in a Hilbert space. Under some control conditions, we establish the strong convergence of the proposed methods to a common element of the solution set and the fixed point set, which is the unique solution of a certain optimization problem. As a direct consequence, we obtain the unique minimum-norm common point of the solution set and the fixed point set.

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1 Introduction

Let H be a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the induced norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H, and let $S: C \to C$ be a self-mapping on C. We denote by $\mathrm{Fix}(S)$ the set of fixed points of S and by P_C the metric projection of H onto C.

A mapping F of C into H is called *monotone* if

$$\langle x - y, Fx - Fy\rangle \geq 0, \quad \forall x, y \in C.$$

A mapping F of C into H is called α*-inverse-strongly monotone* (see [1, 2]) if there exists a positive real number α such that

$$\langle x - y, Fx - Fy\rangle \geq \alpha\|Fx - Fy\|^2, \quad \forall x, y \in C.$$

If F is an α-inverse-strongly monotone mapping of C into H, then it is obvious that F is $\frac{1}{\alpha}$-Lipschitz continuous, that is, $\|Fx - Fy\| \leq \frac{1}{\alpha}\|x - y\|$ for all $x, y \in C$. Clearly, the class of monotone mappings includes the class of α-inverse-strongly monotone mappings.

An operator A is said to be *strongly positive* on H if there exists a constant $\overline{\gamma} > 0$ such that

$$\langle Ax, x\rangle \geq \overline{\gamma}\|x\|^2, \quad \forall x \in H.$$

A mapping F of C into H is called $\overline{\gamma}$-strongly monotone if there exists a positive real number $\overline{\gamma}$ such that

$$\langle x - y, Fx - Fy\rangle \geq \overline{\gamma}\|x - y\|^2, \quad \forall x, y \in C.$$

Clearly, the class of monotone mappings includes the class of strongly positive mappings.

Let F be a nonlinear mapping of C into H. The variational inequality problem is to find $u \in C$ such that

$$\langle v - u, Fu\rangle \geq 0, \quad \forall v \in C. \tag{1.1}$$

We denote the set of solutions of the variational inequality problem (1.1) by $\mathrm{VI}(C, F)$. The variational inequality problem has been extensively studied in the literature; see [2–6] and the references therein.

The class of pseudocontractive mappings is one of the most important classes of mappings among nonlinear mappings. We recall that a mapping $T : C \to H$ is said to be *pseudocontractive* if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \bigl\|(I - T)x - (I - T)y\bigr\|^2, \quad \forall x, y \in C,$$

and T is said to be *k-strictly pseudocontractive* if there exists a constant $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\bigl\|(I - T)x - (I - T)y\bigr\|^2, \quad \forall x, y \in C,$$

where I is the identity mapping. Note that the class of k-strictly pseudocontractive mappings includes the class of nonexpansive mappings as a subclass. That is, T is nonexpansive (*i.e.*, $\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in C$) if and only if T is 0-strictly pseudocontractive. Clearly, the class of pseudocontractive mappings includes the class of strictly pseudocontractive mappings as a subclass, and the class of k-strictly pseudocontractive mappings falls into the one between the class of nonexpansive mappings and the class of pseudocontractive mappings. Moreover, this inclusion is strict due to an example in [7] (see also Example 5.7.1 and Example 5.7.2 in [8]). Recently, many authors have been devoting the studies to the problems of finding fixed points for pseudocontractive mappings; see, for example, [9–15] and the references therein.

The following optimization problem has been studied extensively by many authors:

$$\min_{x \in \Omega} \frac{\mu}{2}\langle Ax, x\rangle + \frac{1}{2}\|x - u\|^2 - h(x),$$

where $\Omega = \bigcap_{i=1}^{\infty} C_i$, $C_1, C_2, \ldots$ are infinitely many closed convex subsets of H such that $\bigcap_{i=1}^{\infty} C_i \neq \emptyset$; $u \in H$; $\mu \geq 0$ is a real number; A is a strongly positive bounded linear self-adjoint operator on H; and h is a potential function for γf (*i.e.*, $h' = \gamma f$ for $\gamma > 0$ and a function f on H). For this kind of minimization problems, see, for example, Bauschke and Borwein [16], Combettes [17], Deutsch and Yamada [18], Jung [19], and Xu [20, 21] when $\Omega = \bigcap_{i=1}^N C_i$ and $h(x) = \langle x, b\rangle$ for a given point b in H.

Iterative methods for nonexpansive mappings and strictly pseudocontractive mappings have recently been applied to solve the optimization problem, where the constraint set is the fixed point set of the mapping; see, for instance, [6, 10, 11, 18, 22–25] and the references therein. Some iterative methods for equilibrium problems, variational inequality problems, and fixed point problems to solve optimization problem, where the constraint set is the common set of the solution set of the problems and the fixed point set of the mappings, were also investigated by many authors recently; see, for instance, [26, 27] and the references therein. We can refer to [28] for certain iterative methods for the integral boundary value problems with causal operators, and we can refer to [29] for iterative methods for solving certain random operator equations.

In particular, in 2006, combining Moudafi's method [30] with Xu's method [21], Marino and Xu [24] introduced the following general iterative method for a nonexpansive mapping S:

$$x_{n+1} = \alpha_n \gamma f x_n + (I - \alpha_n A) S x_n, \quad \forall n \geq 0, \tag{1.2}$$

where $\gamma > 0$ and f is a contractive mapping on H. Under well-known control conditions on the sequence $\{\alpha_n\} \subset [0,1]$, they proved the strong convergence of the sequence $\{x_n\}$ generated by (1.2) to a point $\widetilde{x} \in \operatorname{Fix}(S)$, which is the unique solution of the variational inequality

$$\bigl\langle (A - \gamma f)\widetilde{x}, \widetilde{x} - p\bigr\rangle \leq 0, \quad \forall p \in \operatorname{Fix}(S),$$

which is the optimality condition for the optimization problem

$$\min_{x \in \operatorname{Fix}(S)} \frac{1}{2}\langle Ax, x\rangle - h(x),$$

where h is a potential function for γf. Very recently, Jung [23] proposed the following general iterative method for a k-strictly pseudocontractive mapping T for some $0 \leq k < 1$:

$$x_{n+1} = \alpha_n (u + \gamma f x_n) + \bigl(I - \alpha_n (I + \mu A)\bigr) P_C S x_n, \quad \forall n \geq 0, \tag{1.3}$$

where $u \in C$; $\mu \geq 0$ is a real number; and $S : C \to H$ is a mapping defined by $Sx = kx + (1-k)Tx$. Under different control conditions on the sequence $\{\alpha_n\} \subset [0,1]$ and the sequence $\{x_n\}$ generated by (1.3), he showed the strong convergence of the sequence $\{x_n\}$ to a point $\widetilde{x} \in \operatorname{Fix}(T)$, which is the unique solution of the optimization problem

$$\min_{x \in \operatorname{Fix}(T)} \frac{\mu}{2}\langle Ax, x\rangle + \frac{1}{2}\|x - u\|^2 - h(x),$$

where h is a potential function for γf.

On the other hand, in order to study the variational inequality problem (1.1) coupled with the fixed point problem, many authors have introduced some iterative methods for finding an element of $\mathrm{VI}(C,F) \cap \mathrm{Fix}(S)$, where F is an α-inverse-strongly monotone mapping and S is a nonexpansive mapping; see [1, 31–34] and the references therein. Some iterative methods for finding an element of $\mathrm{VI}(C,F) \cap \mathrm{Fix}(T)$ were also presented by many authors, where F is a continuous monotone mapping and T is a continuous pseudocontractive mapping; see [35–37] and the references therein. In the case that E is a Banach space with the dual E^*, we can refer to [38] for iterative methods for finding an element of $\mathrm{VI}(C,F) \cap \mathrm{Fix}(T)$, where $F : C \to E^*$ is an α-inverse-strongly monotone mapping and T is a relatively weak nonexpansive mapping, and we can refer to [39] for iterative methods for finding an element of $\bigcap_{i=1}^{N} \mathrm{Fix}(T_i) \cap \mathrm{VI}(C,F)$, where F is an α-inverse-strongly accretive mapping and T_i, $i = 1,\ldots,N$, are k_i-strictly pseudocontractive mappings. And we can consult [40] for iterative methods for finding a common element of $\mathrm{VI}(C,F_1) \cap \mathrm{VI}(C,F_2)$, where $F_1, F_2 : C \to E^*$ are two continuous monotone mappings.

Recently, researchers have also invented some iterative methods for finding the minimum norm element in the solution set of certain problems (for instance, variational inequality problem, minimization problem, split feasibility problem, *etc.*) and the fixed point set of nonlinear mappings (for instance, nonexpansive mapping, strictly pseudocontractive mapping, Lipschitzian pseudocontractive mapping, *etc.*); see, for instance, [41–43] and the references therein.

In this paper, as a continuation of study for the above-mentioned optimization problems, we consider the following optimization problem of which the constrained set is $\mathrm{VI}(C,F) \cap \mathrm{Fix}(T)$:

$$\min_{x \in \mathrm{VI}(C,F) \cap \mathrm{Fix}(T)} \frac{\mu}{2}\langle Ax, x\rangle + \frac{1}{2}\|x-u\|^2 - h(x), \tag{1.4}$$

where F is a continuous monotone mapping; T is a continuous pseudocontractive mapping $u \in C$; $\mu \geq 0$ is a real number; and h is a potential function for γf when f is a contractive mapping and $\gamma > 0$. We present two general iterative methods for solving the optimization problem (1.4). First, we introduce an implicit general iterative method. Consequently, by discretizing the continuous implicit method, we provide an explicit general iterative method. Under some control conditions, we show the strong convergence of the proposed methods to an element of $\mathrm{VI}(C,F) \cap \mathrm{Fix}(T)$, which is the unique solution of the optimization problem (1.4). As special cases, we obtain two iterative methods which converge strongly to the minimum norm point of $\mathrm{VI}(C,F) \cap \mathrm{Fix}(T)$. Our results unify, complement, develop, and improve upon the corresponding results of Jung [22, 23], Yao *et al.* [27], and some recent results in the literature.

2 Preliminaries and lemmas

Let H be a real Hilbert space, and let C be a nonempty closed convex subset of H. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. $x_n \to x$ implies that $\{x_n\}$ converges strongly to x.

For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that

$$\|x - P_C(x)\| \leq \|x-y\|, \quad \forall y \in C.$$

P_C is called the *metric projection* of H onto C. It is well known that P_C is nonexpansive and is characterized by the properties

$$u = P_C(x) \iff \langle x - u, u - y\rangle \geq 0, \quad \forall x \in H, y \in C. \tag{2.1}$$

In a Hilbert space H, the following equality holds:

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\langle x, y\rangle, \quad \forall x, y \in H. \tag{2.2}$$

We need the following lemmas for the proof of our main results.

Lemma 2.1 *In a real Hilbert space H, there holds the following inequality*:

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in H.$$

Lemma 2.2 ([20]) *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1 - w_n)s_n + w_n\delta_n + \nu_n, \quad \forall n \geq 0,$$

where $\{w_n\}$, $\{\delta_n\}$, and $\{\nu_n\}$ satisfy the following conditions:

(i) $\{w_n\} \subset [0,1]$ *and* $\sum_{n=0}^{\infty} w_n = \infty$ *or, equivalently,* $\prod_{n=0}^{\infty}(1 - w_n) = 0$;

(ii) $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=0}^{\infty} w_n|\delta_n| < \infty$;

(iii) $\nu_n \geq 0$ $(n \geq 0)$, $\sum_{n=0}^{\infty} \nu_n < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

The following lemmas can be easily proven, and therefore, we omit the proofs.

Lemma 2.3 *Let H be a real Hilbert space, and let $A : H \to H$ be a strongly positive bounded linear operator with a constant $\overline{\gamma} > 0$. Let $f : H \to H$ be a contractive mapping with a constant $k \in (0,1)$. Let $\mu \geq 0$ and $0 < \gamma < \frac{1+\mu\overline{\gamma}}{k}$. Then*

$$\bigl\langle x - y, \bigl((I + \mu A) - \gamma f\bigr)x - \bigl((I + \mu A) - \gamma f\bigr)y\bigr\rangle \geq (1 + \mu\overline{\gamma} - \gamma k)\|x - y\|^2, \quad \forall x, y \in H.$$

That is, $(I + \mu A) - \gamma f$ is strongly monotone with a constant $1 + \mu\overline{\gamma} - \gamma k$.

Lemma 2.4 ([24]) *Let $\mu > 0$, and let $A : H \to H$ be a strongly positive bounded linear self-adjoint operator on a Hilbert space H with a constant $\overline{\gamma} > 0$. Let $0 < \xi \leq (1 + \mu\|A\|)^{-1}$. Then $\|I - \xi(I + \mu A)\| < 1 - \xi(1 + \mu\overline{\gamma})$.*

The following lemmas are Lemma 2.3 and Lemma 2.4 of Zegeye [44], respectively.

Lemma 2.5 ([44]) *Let C be a closed convex subset of a real Hilbert space H. Let $F : C \to H$ be a continuous monotone mapping. Then, for $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$\langle y - z, Fz\rangle + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C.$$

For $r > 0$ and $x \in H$, define $F_r : H \to C$ by

$$F_r x = \left\{ z \in C : \langle y - z, Fz \rangle + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \forall y \in C \right\}.$$

Then the following hold:

(i) *F_r is single-valued*;

(ii) *F_r is firmly nonexpansive, that is,*

$$\| F_r x - F_r y \|^2 \leq \langle x - y, F_r x - F_r y \rangle, \quad \forall x, y \in H;$$

(iii) $\mathrm{Fix}(F_r) = \mathrm{VI}(C, F)$;

(iv) $\mathrm{VI}(C, F)$ *is a closed convex subset of C.*

Lemma 2.6 ([44]) *Let C be a closed convex subset of a real Hilbert space H. Let $T : C \to H$ be a continuous pseudocontractive mapping. Then, for $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$\langle y - z, Tz \rangle - \frac{1}{r} \langle y - z, (1 + r)z - x \rangle \leq 0, \quad \forall y \in C.$$

For $r > 0$ and $x \in H$, define $T_r : H \to C$ by

$$T_r x = \left\{ z \in C : \langle y - z, Tz \rangle - \frac{1}{r} \langle y - z, (1 + r)z - x \rangle \leq 0, \forall y \in C \right\}.$$

Then the following hold:

(i) *T_r is single-valued*;

(ii) *T_r is firmly nonexpansive, that is,*

$$\| T_r x - T_r y \|^2 \leq \langle x - y, T_r x - T_r y \rangle, \quad \forall x, y \in H;$$

(iii) $\mathrm{Fix}(T_r) = \mathrm{Fix}(T)$;

(iv) $\mathrm{Fix}(T)$ *is a closed convex subset of C.*

The following lemma can be found in [26, 45].

Lemma 2.7 *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $g : C \to \mathbb{R} \cup \{\infty\}$ be a proper lower semicontinuous differentiable convex function. If x^* is a solution of the minimization problem*

$$g(x^*) = \inf_{x \in C} g(x),$$

then

$$\langle g'(x^*), p - x^* \rangle \geq 0, \quad p \in C.$$

In particular, if x^* solves the optimization problem

$$\min_{x \in C} \frac{\mu}{2} \langle Ax, x \rangle + \frac{1}{2} \| x - u \|^2 - h(x),$$

where A is a bounded linear self-adjoint operator on H, then

$$\langle u + (\gamma f - (I + \mu A))x^*, p - x^*\rangle \le 0, \quad p \in C,$$

where h is a potential function of γf.

3 Main results

Throughout the rest of this paper, we always assume the following:

- H is a real Hilbert space;
- C is a nonempty closed subspace subset of H;
- $A : C \to C$ is a strongly positive linear bounded self-adjoint operator with a constant $\overline{\gamma} > 0$;
- $f : C \to C$ is a contractive mapping with a constant $k \in (0,1)$;
- Constants $\mu \ge 0$ and $0 < \gamma < \frac{1+\mu\overline{\gamma}}{k}$;
- $F : C \to H$ is a continuous monotone mapping;
- $\mathrm{VI}(C,F)$ is the set of the variational inequality problem (1.1) for F;
- $T : C \to C$ is a continuous pseudocontractive mapping such that $\mathrm{VI}(C,F) \cap \mathrm{Fix}(T) \ne \emptyset$;
- $F_{r_t} : H \to C$ is a mapping defined by

$$F_{r_t}x = \left\{z \in C : \langle y - z, Fz\rangle + \frac{1}{r_t}\langle y - z, z - x\rangle \ge 0, \forall y \in C\right\}$$

  for $r_t \in (0,\infty)$, $t \in (0,1)$, and $\liminf_{t\to 0} r_t > 0$;
- $T_{r_t} : H \to C$ is a mapping defined by

$$T_{r_t}x = \left\{z \in C : \langle y - z, Tz\rangle - \frac{1}{r_t}\langle y - z, (1 + r_t)z - x\rangle \le 0, \forall y \in C\right\}$$

  for $r_t \in (0,\infty)$, $t \in (0,1)$, and $\liminf_{t\to 0} r_t > 0$;
- $F_{r_n} : H \to C$ is a mapping defined by

$$F_{r_n}x = \left\{z \in C : \langle y - z, Fz\rangle + \frac{1}{r_n}\langle y - z, z - x\rangle \ge 0, \forall y \in C\right\}$$

  for $r_n \in (0,\infty)$ and $\liminf_{n\to\infty} r_n > 0$;
- $T_{r_n} : H \to C$ is a mapping defined by

$$T_{r_n}x = \left\{z \in C : \langle y - z, Tz\rangle - \frac{1}{r_n}\langle y - z, (1 + r_n)z - x\rangle \le 0, \forall y \in C\right\}$$

  for $r_n \in (0,\infty)$ and $\liminf_{n\to\infty} r_n > 0$;
- $u \in C$.

By Lemma 2.5 and Lemma 2.6, we note that F_{r_t}, T_{r_t}, F_{r_n}, and T_{r_n} are nonexpansive, $\mathrm{Fix}(F_{r_n}) = \mathrm{VI}(C,F) = \mathrm{Fix}(F_{r_t})$, and $\mathrm{Fix}(T_{r_n}) = \mathrm{Fix}(T) = \mathrm{Fix}(T_{r_t})$.

In this section, first, we introduce the following general iterative method that generates a net $\{x_t\}_{t\in(0,\min\{1,\frac{1}{1+\mu\|A\|}\})}$ in an implicit way:

$$x_t = t(u + \gamma f x_t) + (I - t(I + \mu A))T_{r_t}F_{r_t}x_t. \tag{3.1}$$

Now, for $t \in (0, \min\{1, \frac{1}{1+\mu\|A\|}\})$, consider a mapping $Q_t : C \to C$ defined by

$$Q_t x = t(u + \gamma fx) + \bigl(I - t(I + \mu A)\bigr) T_{r_t} F_{r_t} x, \quad \forall x \in C.$$

It is easy to see that Q_t is a contractive mapping with a constant $1 - t(1 + \mu\overline{\gamma} - \gamma k)$. Indeed, since $T_{r_t} F_{r_t}$ is nonexpansive, by Lemma 2.4, we have

$$\begin{aligned} \|Q_t x - Q_t y\| &\le \bigl\| \bigl(I - t(I + \mu A)\bigr) T_{r_t} F_{r_t} x - \bigl(I - t(I + \mu A)\bigr) T_{r_t} F_{r_t} y \bigr\| \\ &\quad + t \bigl\| (u + \gamma fx) - (u + \gamma fy) \bigr\| \\ &\le \bigl(1 - t(1 + \mu\overline{\gamma})\bigr) \|x - y\| + t\gamma k \|x - y\| \\ &= \bigl(1 - t(1 + \mu\overline{\gamma} - \gamma k)\bigr) \|x - y\|. \end{aligned}$$

Since $0 < t < \min\{1, \frac{1}{1+\mu\|A\|}\}$, it follows that

$$0 < 1 - t(1 + \mu\overline{\gamma} - \gamma k) < 1.$$

Hence Q_t is a contractive mapping. By the Banach contraction principle, Q_t has a unique fixed point, denoted by x_t, which uniquely solves the fixed point equation (3.1).

We summarize the basic properties of $\{x_t\}$.

Proposition 3.1 *Let* $\{x_t\}$ *be defined via* (3.1). *Then*
(i) $\{x_t\}$ *is bounded for* $t \in (0, \min\{1, \frac{1}{1+\mu\|A\|}\})$;
(ii) $\lim_{t\to 0} \|x_t - T_{r_t} F_{r_t} x_t\| = 0$;
(iii) $x_t : (0, \min\{1, \frac{1}{1+\mu\|A\|}\}) \to H$ *is locally Lipschitzian, provided* $r_t : (0, \min\{1, \frac{1}{1+\mu\|A\|}\}) \to (0, \infty)$ *is locally Lipschitzian*;
(iv) x_t *defines a continuous path from* $(0, \min\{1, \frac{1}{1+\mu\|A\|}\})$ *into* H, *provided* $r_t : (0, \min\{1, \frac{1}{1+\mu\|A\|}\}) \to (0, \infty)$ *is continuous.*

Proof (i) Let $z_t = F_{r_t} x_t$, and let $u_t = T_{r_t} z_t$. Let $p \in \mathrm{VI}(C, F) \cap \mathrm{Fix}(T)$. Then $p = F_{r_t} p$ by Lemma 2.5(iii) and $p = T_{r_t} p$ $(= Tp)$ by Lemma 2.6(iii), and from the nonexpansivity of T_{r_t} and F_{r_t}, it follows that

$$\|u_t - p\| = \|T_{r_t} z_t - T_{r_t} p\| \le \|z_t - p\|, \tag{3.2}$$

and

$$\|z_t - p\| = \|F_{r_t} x_t - F_{r_t} p\| \le \|x_t - p\|. \tag{3.3}$$

Let $\overline{A} = I + \mu A$. By (3.2) and (3.3), we have

$$\begin{aligned} \|x_t - p\| &= \bigl\| t(u + \gamma f x_t) + (I - t\overline{A}) T_{r_t} z_t - p \bigr\| \\ &= \bigl\| (I - t\overline{A}) T_t z_t - (I - t\overline{A}) T_{r_t} p + t\gamma f x_t - t\gamma fp + t(u + \gamma f - \overline{A})p \bigr\| \\ &\le \bigl\| (I - t\overline{A}) T_{r_t} z_t - (I - t\overline{A}) T_{r_t} p \bigr\| + t\gamma k \|x_t - p\| \\ &\quad + t\bigl(\|u\| + \bigl\|(\gamma f - \overline{A})p\bigr\|\bigr) \end{aligned}$$

$$\begin{aligned}
&\le \bigl(1 - t(1 + \mu\overline{\gamma})\bigr)\|z_t - p\| + t\gamma k\|x_t - p\| + t\bigl(\|u\| + \bigl\|(\gamma f - \overline{A})p\bigr\|\bigr) \\
&\le \bigl(1 - t(1 + \mu\overline{\gamma})\bigr)\|x_t - p\| + t\gamma k\|x_t - p\| + t\bigl(\|u\| + \bigl\|(\gamma f - \overline{A})p\bigr\|\bigr) \\
&= \bigl(1 - t(1 + \mu\overline{\gamma} - \gamma k)\bigr)\|x_t - p\| + t\bigl(\|u\| + \bigl\|(\gamma f - \overline{A})p\bigr\|\bigr).
\end{aligned}$$

So, it follows that

$$\|x_t - p\| \le \frac{\|u\| + \|(\gamma f - \overline{A})p\|}{1 + \mu\overline{\gamma} - \gamma k}.$$

Hence $\{x_t\}$ is bounded and so are $\{z_t\} = \{F_{r_t}x_t\}$, $\{u_t\} = \{T_{r_t}z_t\}$, and $\{\overline{A}T_{r_t}z_t\} = \{\overline{A}T_{r_t}F_{r_t}x_t\}$, and $\{fx_t\}$.

(ii) Let $z_t = F_{r_t}x_t$. By the definition of $\{x_t\}$ and the boundedness of $\{fx_t\}$ and $\{\overline{A}T_{r_t}z_t\}$ in (i), we have

$$\begin{aligned}
\|x_t - T_{r_t}F_{r_t}x_t\| &= t\bigl\|\overline{A}T_{r_t}F_{r_t}x_t - (u + \gamma fx_t)\bigr\| \\
&\le t\bigl(\|\overline{A}T_{r_t}z_t\| + \|u\| + \gamma\|fx_t\|\bigr) \to 0 \quad \text{as } t \to 0.
\end{aligned}$$

(iii) Let $t, t_0 \in (0, \min\{1, \frac{1}{1+\mu\|A\|}\})$, and let $z_t = F_{r_t}x_t$ and $z_{t_0} = F_{r_{t_0}}x_{t_0}$. Let $u_t = T_{r_t}z_t$ and $u_{t_0} = T_{r_{t_0}}z_{t_0}$. Then we get

$$\langle y - u_{t_0}, Tu_{t_0}\rangle - \frac{1}{r_{t_0}}\bigl\langle y - u_{t_0}, (1 + r_{t_0})u_{t_0} - z_{t_0}\bigr\rangle \le 0, \quad \forall y \in C, \tag{3.4}$$

and

$$\langle y - u_t, Tu_t\rangle - \frac{1}{r_t}\bigl\langle y - u_t, (1 + r_t)u_t - z_t\bigr\rangle \le 0, \quad \forall y \in C. \tag{3.5}$$

Putting $y = u_t$ in (3.4) and $y = u_{t_0}$ in (3.5), we obtain

$$\langle u_t - u_{t_0}, Tu_{t_0}\rangle - \frac{1}{r_{t_0}}\bigl\langle u_t - u_{t_0}, (1 + r_{t_0})u_{t_0} - z_{t_0}\bigr\rangle \le 0, \tag{3.6}$$

and

$$\langle u_{t_0} - u_t, Tu_t\rangle - \frac{1}{r_t}\bigl\langle u_{t_0} - u_t, (1 + r_t)u_t - z_t\bigr\rangle \le 0. \tag{3.7}$$

Adding up (3.6) and (3.7), we have

$$\langle u_t - u_{t_0}, Tu_{t_0} - Tu_t\rangle - \biggl\langle u_t - u_{t_0}, \frac{(1 + r_{t_0})u_{t_0} - z_{t_0}}{r_{t_0}} - \frac{(1 + r_t)u_t - z_t}{r_t}\biggr\rangle \le 0,$$

which implies that

$$\bigl\langle u_t - u_{t_0}, (u_t - Tu_t) - (u_{t_0} - Tu_{t_0})\bigr\rangle - \biggl\langle u_t - u_{t_0}, \frac{u_{t_0} - z_{t_0}}{r_{t_0}} - \frac{u_t - z_t}{r_t}\biggr\rangle \le 0.$$

Now, using the fact that T is pseudocontractive, we get

$$\biggl\langle u_t - u_{t_0}, \frac{u_{t_0} - z_{t_0}}{r_{t_0}} - \frac{u_t - z_t}{r_t}\biggr\rangle \ge 0,$$

and hence

$$\left\langle u_t - u_{t_0}, u_{t_0} - u_t + u_t - z_{t_0} - \frac{r_{t_0}}{r_t}(u_t - z_t)\right\rangle \geq 0. \tag{3.8}$$

Without loss of generality, let us assume that there exists a real number $r_t > b > 0$ for $t \in (0, \min\{1, \frac{1}{1+\mu\|A\|}\})$. Then, by (3.8), we have

$$\begin{aligned}\|u_t - u_{t_0}\|^2 &\leq \left\langle u_t - u_{t_0}, z_t - z_{t_0} + \left(1 - \frac{r_{t_0}}{r_t}\right)(u_t - z_t)\right\rangle \\ &\leq \|u_t - u_{t_0}\| \left\{ \|z_t - z_{t_0}\| + \left|1 - \frac{r_{t_0}}{r_t}\right| \|u_t - z_t\| \right\}.\end{aligned} \tag{3.9}$$

Hence, from (3.9) we obtain

$$\begin{aligned}\|u_t - u_{t_0}\| &\leq \|z_t - z_{t_0}\| + \frac{1}{r_t}|r_t - r_{t_0}| \|u_t - z_t\| \\ &\leq \|z_t - z_{t_0}\| + \frac{1}{b}|r_t - r_{t_0}|L,\end{aligned} \tag{3.10}$$

where $L = \sup\{\|u_t - z_t : t \in (0, \min\{1, \frac{1}{1+\mu\|A\|}\})\|\}$.

Moreover, since $z_t = F_{r_t} x_t$ and $z_{t_0} = F_{r_{t_0}} x_{t_0}$, we get

$$\langle y - z_t, F z_t\rangle + \frac{1}{r_t}\langle y - z_t, z_t - x_t\rangle \geq 0, \quad \forall y \in C, \tag{3.11}$$

and

$$\langle y - z_{t_0}, F z_{t_0}\rangle + \frac{1}{r_{t_0}}\langle y - z_{t_0}, z_{t_0} - x_{t_0}\rangle \geq 0, \quad \forall y \in C. \tag{3.12}$$

Putting $y = z_{t_0}$ in (3.11) and $y = z_t$ in (3.12), we obtain

$$\langle z_{t_0} - z_t, F z_t\rangle + \frac{1}{r_t}\langle z_{t_0} - z_t, z_t - x_t\rangle \geq 0, \tag{3.13}$$

and

$$\langle z_t - z_{t_0}, F z_{t_0}\rangle + \frac{1}{r_{t_0}}\langle z_t - z_{t_0}, z_{t_0} - x_{t_0}\rangle \geq 0. \tag{3.14}$$

Adding up (3.13) and (3.14), we have

$$-\langle z_t - z_{t_0}, F z_t - F z_{t_0}\rangle + \left\langle z_{t_0} - z_t, \frac{z_t - x_t}{r_t} - \frac{z_{t_0} - x_{t_0}}{r_{t_0}}\right\rangle \geq 0.$$

Since F is monotone, we get

$$\left\langle z_{t_0} - z_t, \frac{z_t - x_t}{r_t} - \frac{z_{t_0} - x_{t_0}}{r_{t_0}}\right\rangle \geq 0,$$

and hence

$$\left\langle z_t - z_{t_0}, z_{t_0} - z_t + z_t - x_{t_0} - \frac{r_{t_0}}{r_t}(z_t - x_t)\right\rangle \geq 0. \tag{3.15}$$

Then, using the method in (3.8) and (3.9), from (3.15) we have

$$\begin{aligned}\|z_t - z_{t_0}\|^2 &\leq \left\langle z_t - z_{t_0}, z_t - x_t + x_t - x_{t_0} - \frac{r_{t_0}}{r_t}(z_t - x_t)\right\rangle \\ &= \left\langle z_t - z_{t_0}, x_t - x_{t_0} + \left(1 - \frac{r_{t_0}}{r_t}\right)(z_t - x_t)\right\rangle \\ &\leq \|z_t - z_{t_0}\|\left\{\|x_t - x_{t_0}\| + \frac{1}{b}|r_t - r_{t_0}|\|z_t - x_t\|\right\}.\end{aligned}$$

This implies that

$$\|z_t - z_{t_0}\| \leq \|x_t - x_{t_0}\| + \frac{1}{b}|r_t - r_{t_0}|M, \tag{3.16}$$

where $M = \sup\{\|z_t - x_t\| : t \in (0, \min\{1, \frac{1}{1+\mu\|A\|}\})\|\}$. Combining (3.10) with (3.16), we get

$$\|u_t - u_{t_0}\| = \|T_{r_t}z_t - T_{r_{t_0}}z_{t_0}\| \leq \|x_t - x_{t_0}\| + \frac{1}{b}|r_t - r_{t_0}|(L + M). \tag{3.17}$$

Now, using (3.17), we calculate

$$\begin{aligned}&\|x_t - x_{t_0}\| \\ &\quad = \left\|(I - t\overline{A})T_{r_t}F_{r_t}x_t + t(u + \gamma fx_t) - (I - t_0\overline{A})T_{r_{t_0}}F_{r_{t_0}}x_{t_0} - t_0(u + \gamma fx_{t_0})\right\| \\ &\quad \leq \left\|(I - t\overline{A})T_{r_t}z_t - (I - t_0\overline{A})T_{r_t}z_t\right\| + \left\|(I - t_0\overline{A})T_{r_t}z_t - (I - t_0\overline{A})T_{r_{t_0}}z_{t_0}\right\| \\ &\qquad + \gamma|t - t_0|\|fx_t\| + |t - t_0|\|u\| + \gamma t_0\|fx_t - fx_{t_0}\| \\ &\quad \leq |t - t_0|\|\overline{A}\|\|T_{r_t}z_t\| + \big(1 - t_0(1 + \mu\overline{\gamma})\big)\|T_{r_t}z_t - T_{r_{t_0}}z_{t_0}\| \\ &\qquad + \gamma|t - t_0|\|fx_t\| + |t - t_0|\|u\| + t_0\gamma k\|x_t - x_{t_0}\| \\ &\quad \leq |t - t_0|\|\overline{A}\|\|T_{r_t}z_t\| + \big(1 - t_0(1 + \mu\overline{\gamma})\big)\left[\|x_t - x_{t_0}\| + \frac{1}{b}|r_t - r_{t_0}|(L + M)\right] \\ &\qquad + \gamma|t - t_0|\|fx_t\| + |t - t_0|\|u\| + t_0\gamma k\|x_t - x_{t_0}\|.\end{aligned}$$

This implies that

$$\begin{aligned}\|x_t - x_{t_0}\| &\leq \frac{\|\overline{A}\|\|T_{r_t}z_t\| + \gamma\|fx_t\| + \|u\|}{t_0(1 + \mu\overline{\gamma} - \gamma k)}|t - t_0| \\ &\quad + \frac{(1 - t_0(1 + \mu\overline{\gamma}))\frac{1}{b}(L + M)}{t_0(1 + \mu\overline{\gamma} - \gamma k)}|r_t - r_{t_0}|.\end{aligned}$$

Since $r_t : (0, \min\{1, \frac{1}{1+\mu\|A\|}\}) \to (0, \infty)$ is locally Lipschitzian, x_t is also locally Lipschitzian.

(iv) From the last inequality in (iii), the result follows immediately. □

We prove the following theorem for strong convergence of the net $\{x_t\}$ as $t \to 0$, which guarantees the existence of solutions of the optimization problem (1.4).

Theorem 3.1 *Let the net* $\{x_t\}$ *be defined via* (3.1). *Then* x_t *converges strongly to a point* $\widetilde{x} \in \mathrm{VI}(C,F) \cap \mathrm{Fix}(T)$ *as* $t \to 0$, *which solves the variational inequality*

$$\bigl\langle u + \bigl(\gamma f - (I + \mu A)\bigr)\widetilde{x}, p - \widetilde{x}\bigr\rangle \le 0, \quad \forall p \in \mathrm{VI}(C,F) \cap \mathrm{Fix}(T). \tag{3.18}$$

This $\widetilde{x}$ *is the unique solution of the optimization problem* (1.4).

Proof We first show the uniqueness of a solution of the variational inequality (3.18), which is indeed a consequence of the strong monotonicity of $(I + \mu A) - \gamma f$. In fact, since A is a strongly positive bounded linear operator with a constant $\overline{\gamma} > 0$, we know from Lemma 2.3 that $I + \mu A - \gamma f$ is strongly monotone with a constant $1 + \mu\overline{\gamma} - \gamma k \in (0,1)$. Suppose that $\widetilde{x} \in \mathrm{VI}(C,F) \cap \mathrm{Fix}(T)$ and $\widehat{x} \in \mathrm{VI}(C,F) \cap \mathrm{Fix}(T)$ both are solutions to (3.18). Then we have

$$\bigl\langle u + \bigl(\gamma f - (I + \mu A)\bigr)\widetilde{x}, \widetilde{x} - \widehat{x}\bigr\rangle \le 0 \tag{3.19}$$

and

$$\bigl\langle u + \bigl(\gamma f - (I + \mu A)\bigr)\widehat{x}, \widehat{x} - \widetilde{x}\bigr\rangle \le 0. \tag{3.20}$$

Adding up (3.19) and (3.20) yields

$$\bigl\langle \bigl((I + \mu A) - \gamma f\bigr)\widetilde{x} - \bigl((I + \mu A) - \gamma f\bigr)\widehat{x}, \widetilde{x} - \widehat{x}\bigr\rangle \le 0.$$

The strong monotonicity of $(I + \mu A) - \gamma f$ (Lemma 2.3) implies that $\widetilde{x} = \widehat{x}$ and the uniqueness is proved.

Next, we prove that $x_t \to \widetilde{x}$ as $t \to 0$. Let $\overline{A} = (I + \mu A)$, and let $z_t = F_{r_t} x_t$. Observing $\mathrm{Fix}(T) = \mathrm{Fix}(T_{r_t})$ (by Lemma 2.6(iii)) and $\mathrm{Fix}(F_{r_t}) = \mathrm{VI}(C,F)$ (by Lemma 2.5(iii)), from (3.1) we write, for given $p \in \mathrm{VI}(C,F) \cap \mathrm{Fix}(T)$,

$$\begin{aligned} x_t - p &= (I - t\overline{A})T_{r_t} z_t - (I - t\overline{A})T_{r_t} p + t\gamma(fx_t - fp) + t\bigl(u + (\gamma f - \overline{A})p\bigr) \\ &= (I - t\overline{A})(T_{r_t} z_t - T_{r_t} p) + t\gamma(fx_t - fp) + t\bigl(u + (\gamma f - \overline{A})p\bigr) \end{aligned}$$

to derive that

$$\begin{aligned} \|x_t - p\|^2 &= \bigl\langle (I - t\overline{A})(T_{r_t} z_t - T_{r_t} p), x_t - p\bigr\rangle + t\gamma\langle fx_t - fp, x_t - p\rangle \\ &\quad + t\bigl\langle u + (\gamma f - \overline{A})p, x_t - p\bigr\rangle \\ &\le \bigl(1 - t(1 + \mu\overline{\gamma})\bigr)\|z_t - p\|\|x_t - p\| + t\gamma k\|x_t - p\|^2 \\ &\quad + t\bigl\langle u + (\gamma f - \overline{A})p, x_t - p\bigr\rangle \\ &\le \bigl(1 - t(1 + \mu\overline{\gamma})\bigr)\|x_t - p\|^2 + t\gamma k\|x_t - p\|^2 \\ &\quad + t\bigl\langle u + (\gamma f - \overline{A})p, x_t - p\bigr\rangle. \end{aligned}$$

Therefore we have

$$\|x_t - p\|^2 \le \frac{1}{1 + \mu\overline{\gamma} - \gamma k}\big\langle u + (\gamma f - \overline{A})p, x_t - p\big\rangle. \tag{3.21}$$

Since $\{x_t\}$ is bounded as $t \to 0$ (by Proposition 3.1(i)), there exists a subsequence $\{t_n\}$ in $(0, \min\{1, \frac{1}{1+\mu\|A\|}\})$ such that $t_n \to 0$ and $x_{t_n} \rightharpoonup x^*$. First of all, we prove that $x^* \in \mathrm{VI}(C,F) \cap \mathrm{Fix}(T)$. To this end, we divide its proof into four steps.

Step 1. We show that $\lim_{n\to\infty} \|x_{t_n} - z_{t_n}\| = 0$, where $z_{t_n} = F_{r_{t_n}} x_{t_n}$. To show this, let $p \in \mathrm{VI}(C,F) \cap \mathrm{Fix}(T)$. Since $p = F_{r_{t_n}} p$, from (2.2) we deduce

$$\begin{aligned}
\|z_{t_n} - p\|^2 &= \|F_{r_{t_n}} x_{t_n} - F_{r_{t_n}} p\|^2 \\
&\le \langle z_{t_n} - p, x_{t_n} - p\rangle \\
&= \frac{1}{2}\big[\|x_{t_n} - p\|^2 + \|z_{t_n} - p\|^2 - \|x_{t_n} - z_{t_n}\|^2\big],
\end{aligned}$$

and hence

$$\|z_{t_n} - p\|^2 \le \|x_{t_n} - p\|^2 - \|x_{t_n} - z_{t_n}\|^2.$$

Thus, from (3.2) we have

$$\|T_{r_n} z_{t_n} - p\|^2 \le \|z_{t_n} - p\|^2 \le \|x_{t_n} - p\|^2 - \|x_{t_n} - z_{t_n}\|^2.$$

This implies

$$\begin{aligned}
\|x_{t_n} - z_{t_n}\|^2 &\le \|x_{t_n} - p\|^2 - \|T_{r_n} z_{t_n} - p\|^2 \\
&\le \big(\|x_{t_n} - p\| + \|T_{r_n} z_{t_n} - p\|\big)\big(\|x_{t_n} - p\| - \|T_{r_n} z_{t_n} - p\|\big) \\
&\le \big(\|x_{t_n} - p\| + \|T_{r_n} z_{t_n} - p\|\big)\|x_{t_n} - T_{r_n} z_{t_n}\| \\
&= \big(\|x_{t_n} - p\| + \|T_{r_n} z_{t_n} - p\|\big)\|x_{t_n} - T_{r_n} F_{r_n} x_{t_n}\|.
\end{aligned}$$

Since $t_n \to 0$ and $\|x_{t_n} - T_{r_n} F_{r_n} x_{t_n}\| \to 0$ by Proposition 3.1(ii), we get $\|x_{t_n} - z_{t_n}\| \to 0$ by the boundedness of $\{x_t\}$ and $\{T_{r_t} z_t\}$.

Step 2. We show that $\lim_{n\to\infty} \|u_{t_n} - z_{t_n}\| = 0$, where $u_{t_n} = T_{r_{t_n}} z_{t_n}$. Indeed, from Proposition 3.1(ii) and Step 1 it follows that

$$\|u_{t_n} - z_{t_n}\| \le \|u_{t_n} - x_{t_n}\| + \|x_{t_n} - z_{t_n}\| \to 0 \quad (\text{as } n \to \infty).$$

Step 3. We show that $x^* \in \mathrm{VI}(C,F)$. In fact, from the definition of $z_{t_n} = F_{r_{t_n}} x_{t_n}$ we have

$$\langle y - z_{t_n}, F z_{t_n}\rangle + \left\langle y - z_{t_n}, \frac{z_{t_n} - x_{t_n}}{r_{t_n}}\right\rangle \ge 0, \quad \forall y \in C. \tag{3.22}$$

Set $w_t = tv + (1-t)x^*$ for all $t \in (0,1]$ and $v \in C$. Then $w_t \in C$. From (3.22) it follows that

$$\begin{aligned}\langle w_t - z_{t_n}, Fw_t\rangle &\geq \langle w_t - z_{t_n}, Fw_t\rangle - \langle w_t - z_{t_n}, Fz_{t_n}\rangle - \left\langle w_t - z_{t_n}, \frac{z_{t_n} - x_{t_n}}{r_{t_n}}\right\rangle \\ &= \langle w_t - z_{t_n}, Fw_t - Fz_{t_n}\rangle - \left\langle w_t - z_{t_n}, \frac{z_{t_n} - x_{t_n}}{r_{t_n}}\right\rangle. \end{aligned} \tag{3.23}$$

By Step 1, we have $\frac{z_{t_n}-x_{t_n}}{r_{t_n}} \to 0$ as $n \to \infty$. Moreover, since $x_{t_n} \rightharpoonup x^*$, by Step 1, we have $z_{t_n} \rightharpoonup x^*$ as $n \to \infty$. Since F is monotone, we also have that $\langle w_t - z_{t_n}, Fw_t - Fz_{t_n}\rangle \geq 0$. Thus, from (3.23) it follows that

$$0 \leq \lim_{n\to\infty} \langle w_t - z_{t_n}, Fw_t\rangle = \langle w_t - x^*, Fw_t\rangle,$$

and hence

$$\langle v - x^*, Fw_t\rangle \geq 0, \quad \forall v \in C.$$

If $t \to 0$, the continuity of F yields that

$$\langle v - x^*, Fx^*\rangle \geq 0, \quad \forall v \in C.$$

This implies that $x^* \in \mathrm{VI}(C,F)$.

Step 4. We show that $x^* \in \mathrm{Fix}(T)$. In fact, from the definition of $u_{t_n} = T_{r_{t_n}} z_{t_n}$, we have

$$\langle y - u_{t_n}, Tu_{t_n}\rangle - \frac{1}{r_{t_n}}\langle y - u_{t_n}, (1 + r_{t_n})u_{t_n} - z_{t_n}\rangle \leq 0, \quad \forall y \in C. \tag{3.24}$$

Put $w_t = tv + (1-t)x^*$ for all $t \in (0,1]$ and $v \in C$. Then $w_t \in C$, and from (3.24) and pseudocontractivity of T it follows that

$$\begin{aligned}\langle u_{t_n} - w_t, Tw_t\rangle &\geq \langle u_{t_n} - w_t, Tw_t\rangle + \langle w_t - u_{t_n}, Tu_{t_n}\rangle \\ &\quad - \frac{1}{r_{t_n}}\langle w_t - u_{t_n}, (1 + r_{t_n})u_{t_n} - z_{t_n}\rangle \\ &= -\langle w_t - u_{t_n}, Tw_t - Tu_{t_n}\rangle - \frac{1}{r_{t_n}}\langle w_t - u_{t_n}, u_{t_n} - z_{t_n}\rangle \\ &\quad - \langle w_t - u_{t_n}, u_{t_n}\rangle \\ &\geq -\|w_t - u_{t_n}\|^2 - \frac{1}{r_{t_n}}\langle w_t - u_{t_n}, u_{t_n} - z_{t_n}\rangle - \langle w_t - u_{t_n}, u_{t_n}\rangle \\ &= -\langle w_t - u_{t_n}, w_t\rangle - \left\langle w_t - u_{t_n}, \frac{u_{t_n} - z_{t_n}}{r_{t_n}}\right\rangle. \end{aligned} \tag{3.25}$$

By Step 2, we get $\frac{u_{t_n}-z_{t_n}}{r_{t_n}} \to 0$ as $n \to \infty$. Moreover, since $x_{t_n} \rightharpoonup x^*$, by Step 1 and Step 2, we have $u_{t_n} \rightharpoonup x^*$ as $n \to \infty$. Therefore, from (3.25), as $n \to \infty$, it follows that

$$\langle x^* - w_t, Tw_t\rangle \geq \langle x^* - w_t, w_t\rangle,$$

and hence

$$-\langle v - x^*, Tw_t \rangle \geq -\langle v - x^*, w_t \rangle, \quad \forall v \in C.$$

Letting $t \to 0$ and using the fact that T is continuous, we get

$$-\langle v - x^*, Tx^* \rangle \geq -\langle v - x^*, x^* \rangle, \quad \forall v \in C.$$

Now, let $v = Tx^*$. Then we obtain $x^* = Tx^*$ and hence $x^* \in \mathrm{Fix}(T)$. Therefore, $x^* \in \mathrm{VI}(C, F) \cap \mathrm{Fix}(T)$.

Now, we substitute x^* for p in (3.21) to obtain

$$\|x_{t_n} - x^*\|^2 \leq \frac{1}{1 + \mu\overline{\gamma} - \gamma k} \langle u + (\gamma f - \overline{A})x^*, x_{t_n} - x^* \rangle. \tag{3.26}$$

Note that $x_{t_n} \rightharpoonup x^*$ and $\lim_{n\to\infty} t_n = 0$. These facts and inequality (3.26) imply that $x_{t_n} \to x^*$ strongly.

Finally, we prove that x^* is a solution of the variational inequality (3.18). In fact, putting x_{t_n} in place of x_t in (3.21) and taking the limit as $t_n \to 0$, we obtain

$$\|x^* - p\|^2 \leq \frac{1}{1 + \mu\overline{\gamma} - \gamma k} \langle u + (\gamma f - \overline{A})p, x^* - p \rangle, \quad \forall p \in \mathrm{VI}(C, F) \cap \mathrm{Fix}(T).$$

In particular, x^* solves the following variational inequality:

$$x^* \in \mathrm{VI}(C, F) \cap \mathrm{Fix}(T), \quad \langle u + (\gamma f - \overline{A})p, p - x^* \rangle \leq 0, \quad \forall p \in \mathrm{VI}(C, F) \cap \mathrm{Fix}(T),$$

or the equivalent dual variational inequality (see [46])

$$x^* \in \mathrm{VI}(C, F) \cap \mathrm{Fix}(T), \quad \langle u + (\gamma f - \overline{A})x^*, p - x^* \rangle \leq 0, \quad \forall p \in \mathrm{VI}(C, F) \cap \mathrm{Fix}(T).$$

That is, $x^* \in \mathrm{VI}(C, F) \cap \mathrm{Fix}(T)$ is a solution of the variational inequality (3.18); hence $x^* = \widetilde{x}$ by uniqueness. In summary, we have shown that each cluster point of $\{x_t\}$ (at $t \to 0$) equals $\widetilde{x}$. Therefore $x_t \to \widetilde{x}$ as $t \to 0$. By (3.18) and Lemma 2.7, we deduce immediately the desired result. This completes the proof. □

If we take $\mu = 0$, $u = 0$ and $f \equiv 0$ in Theorem 3.1, then we have the following corollary.

Corollary 3.1 *Let* $\{x_t\}$ *be defined by*

$$x_t = (I - t)T_{r_t} F_{r_t} x_t.$$

Then $\{x_t\}$ *converges strongly as* $t \to 0$ *to a point* $\widetilde{x} \in \mathrm{VI}(C, F) \cap \mathrm{Fix}(T)$, *which is the minimum norm point of* $\mathrm{VI}(C, F) \cap \mathrm{Fix}(T)$.

Taking $T \equiv I$ in Theorem 3.1, we have the following corollary.

Corollary 3.2 *Let* $\{x_t\}$ *be defined by*

$$x_t = t(u + \gamma f x_t) + \big(I - t(I + \mu A)\big)F_{r_t}x_t.$$

Then $\{x_t\}$ *converges strongly as* $t \to 0$ *to a point* $\widetilde{x} \in \mathrm{VI}(C,F)$, *where is the unique solution of the optimization problem*

$$\min_{x\in \mathrm{VI}(C,F)} \frac{\mu}{2}\langle Ax, x\rangle + \frac{1}{2}\|x - u\|^2 - h(x). \tag{3.27}$$

Proof If $T \equiv I$, then T_r in Lemma 2.6 is the identity mapping. Thus the result follows from Theorem 3.1. □

Taking $F \equiv 0$ in Theorem 3.1, we get the following corollary.

Corollary 3.3 *Let* $\{x_t\}$ *be defined by*

$$x_t = t(u + \gamma f x_t) + \big(I - t(I + \mu A)\big)T_{r_t}x_t.$$

Then $\{x_t\}$ *converges strongly as* $t \to 0$ *to a point* $\widetilde{x} \in \mathrm{Fix}(T)$, *where is the unique solution of the optimization problem*

$$\min_{x\in \mathrm{Fix}(T)} \frac{\mu}{2}\langle Ax, x\rangle + \frac{1}{2}\|x - u\|^2 - h(x). \tag{3.28}$$

Proof If $F \equiv 0$, then F_r in Lemma 2.5 is the identity mapping. Thus the result follows from Theorem 3.1. □

If, in Theorem 3.1, we take $C \equiv H$, then we obtain the following corollary.

Corollary 3.4 *Let* $T : H \to H$ *be a continuous pseudocontractive mapping, and let* $F : H \to H$ *be a continuous monotone mapping. Let* $\{x_t\}$ *be defined by* (3.1). *Then* $\{x_t\}$ *converges strongly as* $t \to 0$ *to a point* $\widetilde{x} \in F^{-1}(0) \cap \mathrm{Fix}(T)$, *which is the unique solution of the optimization problem*

$$\min_{x\in F^{-1}(0)\cap \mathrm{Fix}(T)} \frac{\mu}{2}\langle Ax, x\rangle + \frac{1}{2}\|x - u\|^2 - h(x). \tag{3.29}$$

Proof Since $D(F) = H$, we have $\mathrm{VI}(H,F) = F^{-1}(0)$. So, by Theorem 3.1, we obtain the desired result. □

Now, we propose the following general iterative method which generates a sequence in an explicit way:

$$x_{n+1} = \alpha_n(u + \gamma f x_n) + \big(I - \alpha_n(I + \mu A)\big)T_{r_n}F_{r_n}x_n, \quad \forall n \ge 0, \tag{3.30}$$

where $x_0 \in H$ is an arbitrary initial guess; $\{\alpha_n\} \in [0,1]$ and $\{r_n\} \subset (0,\infty)$; and we establish the strong convergence of this sequence to a point $\widetilde{x} \in \mathrm{VI}(C,F) \cap \mathrm{Fix}(T)$, which is the unique solution of the optimization problem (1.4).

Theorem 3.2 *Let $\{x_n\}$ be the sequence generated by the explicit scheme* (3.30). *Let $\{\alpha_n\}$ and $\{r_n\} \subset (0,\infty)$ satisfy the following conditions*:

(C1) $\{\alpha_n\} \subset [0,1]$ *and* $\alpha_n \to 0$ *as* $n \to \infty$;

(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(C3) $|\alpha_{n+1} - \alpha_n| \le o(\alpha_{n+1}) + \sigma_n$, $\sum_{n=0}^{\infty} \sigma_n < \infty$ (*the perturbed control condition*);

(C4) $\liminf_{n\to\infty} r_n > 0$ *and* $\sum_{n=0}^{\infty} |r_{n+1} - r_n| < \infty$.

Then $\{x_n\}$ converges strongly to a point $\widetilde{x} \in \mathrm{VI}(C,F) \cap \mathrm{Fix}(T)$, which is the unique solution of the variational inequality (3.18). *This $\widetilde{x}$ is the unique solution of the optimization problem* (1.4).

Proof First, note that from condition (C1), without loss of generality, we assume that $\alpha_n(1 + \mu\overline{\gamma} - \gamma k) < 1$ and $\frac{2\alpha_n(1+\mu\overline{\gamma}-\gamma k)}{1-\alpha_n\gamma k} < 1$ for all $n \ge 0$. Let $\widetilde{x} \in \mathrm{VI}(C,F) \cap \mathrm{Fix}(T)$ be the unique solution of the variational inequality (3.18). (The existence of $\widetilde{x}$ follows from Theorem 3.1.)

From now on, we put $\overline{A} = I + \mu A$, $z_n = F_{r_n}x_n$ and $u_n = T_{r_n}z_n$. Let $p \in \mathrm{VI}(C,F) \cap \mathrm{Fix}(T)$. Then $p = T_{r_n}p$ by Lemma 2.6(iii) and $p = F_{r_n}p$ by Lemma 2.5(iii). Moreover, from the nonexpansivity of F_{r_n} it follows that

$$\|z_n - p\| = \|F_{r_n}x_n - F_{r_n}p\| \le \|x_n - p\|. \tag{3.31}$$

We divide the proof into several steps as follows.

Step 1. We show that $\{x_n\}$ is bounded. First of all, by (3.31), we deduce

$$\begin{aligned}
&\|x_{n+1} - p\| \\
&\quad = \bigl\|\alpha_n(u + \gamma fx_n) + (I - \alpha_n\overline{A})T_{r_n}z_n - p\bigr\| \\
&\quad = \bigl\|(I - \alpha_n\overline{A})T_{r_n}z_n - (I - \alpha_n\overline{A})T_{r_n}p + \alpha_n\gamma(fx_n - fp) + \alpha_n\bigl(u + (\gamma f - \overline{A})p\bigr)\bigr\| \\
&\quad \le \bigl\|(I - \alpha_n\overline{A})T_{r_n}z_n - (I - \alpha_n\overline{A})T_{r_n}p\bigr\| + \alpha_n\gamma k\|x_n - p\| + \alpha_n\bigl(\|u\| + \bigl\|(\gamma f - \overline{A})p\bigr\|\bigr) \\
&\quad \le \bigl(1 - \alpha_n(1 + \mu\overline{\gamma})\bigr)\|z_n - p\| + \alpha_n\gamma k\|x_n - p\| + \alpha_n\bigl(\|u\| + \bigl\|(\gamma f - \overline{A})p\bigr\|\bigr) \\
&\quad \le \bigl(1 - \alpha_n(1 + \mu\overline{\gamma})\bigr)\|x_n - p\| + \alpha_n\gamma k\|x_n - p\| + \alpha_n\bigl(\|u\| + \bigl\|(\gamma f - \overline{A})p\bigr\|\bigr) \\
&\quad = \bigl(1 - \alpha_n(1 + \mu\overline{\gamma} - \gamma k)\bigr)\|x_n - p\| + \alpha_n\bigl(\|u\| + \bigl\|(\gamma f - \overline{A})p\bigr\|\bigr) \\
&\quad \le \max\left\{\|x_n - p\|, \frac{\|u\| + \|(\gamma f - \overline{A})p\|}{1 + \mu\overline{\gamma} - \gamma k}\right\}.
\end{aligned}$$

By induction, we derive

$$\|x_n - p\| \le \max\left\{\|x_0 - p\|, \frac{\|u\| + \|(\gamma f - \overline{A})p\|}{1 + \mu\overline{\gamma} - \gamma k}\right\}, \quad \forall n \ge 0.$$

This implies that $\{x_n\}$ is bounded and so are $\{z_n\} = \{F_{r_n}x_n\}$, $\{u_n\} = \{T_{r_n}z_n\}$, $\{fx_n\}$, and $\{\overline{A}T_{r_n}z_n\}$. As a consequence, with the control condition (C1), we get

$$\|x_{n+1} - T_{r_n}z_n\| \le \alpha_n\bigl(\|u\| + \gamma\|fx_n - \overline{A}T_{r_n}z_n\|\bigr) \to 0 \quad (n \to \infty). \tag{3.32}$$

Step 2. We show that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. In fact, by using the same method as in the proof of Proposition 3.1(iii) together with $z_n = F_{r_n}x_n$, $z_{n-1} = F_{r_{n-1}}x_{n-1}$, $u_n = T_{r_n}z_n$, and

$u_{n-1} = T_{r_{n-1}} z_{n-1}$ instead of $z_t = F_{r_t} x_t$, $z_{t_0} = F_{r_{t_0}} x_{t_0}$, $u_t = T_{r_t} z_t$, and $u_{t_0} = T_{r_{t_0}} z_{t_0}$, respectively, we have

$$\|T_{r_n} z_n - T_{r_{n-1}} z_{n-1}\| \leq \|x_n - x_{n-1}\| + \frac{1}{b}|r_n - r_{n-1}|(M_1 + M_2), \tag{3.33}$$

where $M_1 = \sup\{\|u_n - z_n\| : n \geq 0\}$, $M_2 = \sup\{\|z_n - x_n\| : n \geq 0\}$, and $r_n > b > 0$, $n \geq 0$ for some b. Thus, by (3.33) and Lemma 2.4, we derive

$$\begin{aligned}
&\|x_{n+1} - x_n\| \\
&\quad = \big\|\alpha_n(u + \gamma f x_n) + (I - \alpha_n \overline{A}) T_{r_n} z_n \\
&\qquad - \alpha_{n-1}(u + \gamma f x_{n-1}) - (I - \alpha_{n-1}\overline{A}) T_{r_{n-1}} z_{n-1}\big\| \\
&\quad \leq \big\|(I - \alpha_n \overline{A})(T_{r_n} z_n - T_{r_{n-1}} z_{n-1})\big\| + |\alpha_n - \alpha_{n-1}| \|\overline{A}\| \|T_{r_{n-1}} z_{n-1}\| \\
&\qquad + \alpha_n \gamma \|f x_n - f x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|u\| \\
&\quad \leq \big(1 - \alpha_n(1 + \mu\overline{\gamma})\big) \|T_{r_n} z_n - T_{r_{n-1}} z_{n-1}\| \\
&\qquad + \alpha_n \gamma k \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \big(\|\overline{A}\| \|T_{r_{n-1}} z_{n-1}\| + \|u\|\big) \\
&\quad \leq \big(1 - \alpha_n(1 + \mu\overline{\gamma})\big) \left[\|x_n - x_{n-1}\| + \frac{1}{b}|r_n - r_{n-1}|(M_1 + M_2)\right] \\
&\qquad + \alpha_n \gamma k \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| M_3 \\
&\quad \leq \big(1 - \alpha_n(1 + \mu\overline{\gamma} - \gamma k)\big) \|x_n - x_{n-1}\| \\
&\qquad + |\alpha_n - \alpha_{n-1}| M_3 + \frac{1}{b}|r_n - r_{n-1}|(M_1 + M_2) \\
&\quad \leq \big(1 - \alpha_n(1 + \mu\overline{\gamma} - \gamma k)\big) \|x_n - x_{n-1}\| \\
&\qquad + \big(o(\alpha_n) + \sigma_{n-1}\big) M_3 + \frac{1}{b}|r_n - r_{n-1}|(M_1 + M_2),
\end{aligned} \tag{3.34}$$

where $M_3 = \sup\{\|\overline{A}\| \|T_{r_n} z_n\| + \|u\| : n \geq 0\}$. By taking $s_{n+1} = \|x_{n+1} - x_n\|$, $w_n = \alpha_n(1 + \mu\overline{\gamma} - \gamma k)$, $w_n \delta_n = M_3 o(\alpha_n)$ and $\nu_n = \sigma_{n-1} M_3 + \frac{1}{b}|r_n - r_{n-1}|(M_1 + M_2)$, from (3.34) we deduce

$$s_{n+1} \leq (1 - w_n) s_n + w_n \delta_n + \nu_n.$$

Hence, by conditions (C2), (C3), (C4) and Lemma 2.2, we obtain

$$\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0.$$

Step 3. We show that $\lim_{n\to\infty} \|x_n - z_n\| = 0$. By taking x_n and z_n instead of x_{t_n} and z_{t_n} in Step 1 of the proof of Theorem 3.1, the result follows from Step 1 in the proof of Theorem 3.1, (3.32) and Step 2.

Step 4. We show that $\lim_{n\to\infty} \|x_n - u_n\| = 0$, where $u_n = T_{r_n} z_n$. In fact, from (3.32) and Step 2, we have

$$\begin{aligned}
\|x_n - u_n\| &= \|x_n - T_{r_n} z_n\| \\
&\leq \|x_n - x_{n+1}\| + \|x_{n+1} - T_{r_n} z_n\| \to 0 \quad (\text{as } n \to \infty).
\end{aligned}$$

Step 5. We show that $\lim_{n\to\infty}\|u_n - z_n\| = 0$, where $u_n = T_{r_n}z_n$. In fact, from Step 3 and Step 4, we have

$$\|u_n - z_n\| \le \|u_n - x_n\| + \|x_n - z_n\| \to 0 \quad (\text{as } n \to \infty).$$

Step 6. We show that $\limsup_{n\to\infty}\langle u + (\gamma f - \overline{A})\widetilde{x}, x_n - \widetilde{x}\rangle \le 0$. To this end, take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\bigl\langle u + (\gamma f - \overline{A})\widetilde{x}, x_n - \widetilde{x}\bigr\rangle = \lim_{k\to\infty}\bigl\langle u + (\gamma f - \overline{A})\widetilde{x}, x_{n_k} - \widetilde{x}\bigr\rangle.$$

Without loss of generality, we may assume that $x_{n_k} \rightharpoonup p$. Take x_{n_k} and z_{n_k} in place of x_{t_n} and z_{t_n} in Step 3 and Step 4 of the proof of Theorem 3.1. Then, from Step 3 and Step 4 in the proof of Theorem 3.1 along with Step 5, we derive $p \in \mathrm{VI}(C,F) \cap \mathrm{Fix}(T)$. Hence, from (3.18) we conclude

$$\begin{aligned}\limsup_{n\to\infty}\bigl\langle u + (\gamma f - \overline{A})\widetilde{x}, x_n - \widetilde{x}\bigr\rangle &= \lim_{k\to\infty}\bigl\langle u + (\gamma f - \overline{A})\widetilde{x}, x_{n_k} - \widetilde{x}\bigr\rangle \\ &= \bigl\langle u + (\gamma f - \overline{A})\widetilde{x}, p - \widetilde{x}\bigr\rangle \le 0.\end{aligned}$$

Step 7. We show that $\lim_{n\to\infty}\|x_n - \widetilde{x}\| = 0$. Note that $\widetilde{x} \in \mathrm{VI}(C,F) \cap \mathrm{Fix}(T)$. Let $z_n = F_{r_n}x_n$. By (3.30), $\widetilde{x} = F_{r_n}\widetilde{x}$, and $\widetilde{x} = T_{r_n}\widetilde{x}$, we deduce

$$x_{n+1} - \widetilde{x} = (I - \alpha_n\overline{A})(T_{r_n}z_n - T_{r_n}\widetilde{x}) + \alpha_n\gamma(fx_n - f\widetilde{x}) + \alpha_n\bigl(u + (\gamma f - \overline{A})\widetilde{x}\bigr).$$

Applying Lemma 2.1 and Lemma 2.4, we obtain

$$\begin{aligned}&\|x_{n+1} - \widetilde{x}\|^2 \\ &\quad= \bigl\|(I - \alpha_n\overline{A})(T_{r_n}z_n - T_{r_n}\widetilde{x}) + \alpha_n\gamma(fx_n - f\widetilde{x}) + \alpha_n\bigl(u + (\gamma f - \overline{A})\widetilde{x}\bigr)\bigr\|^2 \\ &\quad\le \bigl\|(I - \alpha_n\overline{A})(T_{r_n}z_n - T_{r_n}\widetilde{x})\bigr\|^2 + 2\alpha_n\gamma\langle fx_n - f\widetilde{x}, x_{n+1} - \widetilde{x}\rangle \\ &\qquad+ 2\alpha_n\bigl\langle u + (\gamma f - \overline{A})\widetilde{x}, x_{n+1} - \widetilde{x}\bigr\rangle \\ &\quad\le \bigl(1 - \alpha_n(1 + \mu\overline{\gamma})\bigr)^2\|z_n - \widetilde{x}\|^2 + 2\alpha_n\gamma k\|x_n - \widetilde{x}\|\|x_{n+1} - \widetilde{x}\| \\ &\qquad+ 2\alpha_n\bigl\langle u + (\gamma f - \overline{A})\widetilde{x}, x_{n+1} - \widetilde{x}\bigr\rangle \\ &\quad\le \bigl(1 - \alpha_n(1 + \mu\overline{\gamma})\bigr)^2\|x_n - \widetilde{x}\|^2 + \alpha_n\gamma k\bigl(\|x_n - \widetilde{x}\|^2 + \|x_{n+1} - \widetilde{x}\|^2\bigr) \\ &\qquad+ 2\alpha_n\bigl\langle u + (\gamma f - \overline{A})\widetilde{x}, x_{n+1} - \widetilde{x}\bigr\rangle \\ &\quad\le \bigl(1 - 2\alpha_n(1 + \mu\overline{\gamma}) + \alpha_n\gamma k\bigr)\|x_n - \widetilde{x}\|^2 + \alpha_n^2(1 + \mu\overline{\gamma})^2\|x_n - \widetilde{x}\|^2 + \alpha_n\gamma k\|x_{n+1} - \widetilde{x}\|^2 \\ &\qquad+ 2\alpha_n\bigl\langle u + (\gamma f - \overline{A})\widetilde{x}, x_{n+1} - \widetilde{x}\bigr\rangle. \end{aligned} \tag{3.35}$$

It then follows from (3.35) that

$$\begin{aligned}\|x_{n+1} - \widetilde{x}\|^2 \le{}& \frac{1 - 2\alpha_n(1 + \mu\overline{\gamma}) + \alpha_n\gamma k}{1 - \alpha_n\gamma k}\|x_n - \widetilde{x}\|^2 \\ &+ \frac{\alpha_n^2(1 + \mu\overline{\gamma})^2}{1 - \alpha_n\gamma k}\|x_n - \widetilde{x}\|^2 + \frac{2\alpha_n}{1 - \alpha_n\gamma k}\bigl\langle u + (\gamma f - \overline{A})\widetilde{x}, x_{n+1} - \widetilde{x}\bigr\rangle\end{aligned}$$

$$\begin{aligned}
&= \left(1 - \frac{2\alpha_n(1+\mu\overline{\gamma}-\gamma k)}{1-\alpha_n\gamma k}\right)\|x_n - \widetilde{x}\|^2 \\
&\quad + \frac{\alpha_n^2(1+\mu\overline{\gamma})^2}{1-\alpha_n\gamma k}\|x_n - \widetilde{x}\|^2 + \frac{2\alpha_n}{1-\alpha_n\gamma k}\langle u + (\gamma f - \overline{A})\widetilde{x}, x_{n+1} - \widetilde{x}\rangle \\
&\le (1-w_n)\|x_n - \widetilde{x}\|^2 + w_n\delta_n,
\end{aligned}$$

where

$$\begin{aligned}
w_n &= \frac{2\alpha_n(1+\mu\overline{\gamma}-\gamma k)}{1-\alpha_n\gamma k} \quad \text{and} \\
\delta_n &= \frac{1}{2(1+\mu\overline{\gamma}-\gamma k)}\left[\alpha_n(1+\mu\overline{\gamma})^2 M_4 + 2\langle u + (\gamma f - \overline{A})\widetilde{x}, x_{n+1} - \widetilde{x}\rangle\right],
\end{aligned}$$

where $M_4 = \sup\{\|x_n - \widetilde{x}\|^2 : n \ge 0\}$. It can be easily seen from conditions (C1) and (C2) and Step 6 that $w_n \to 0$, $\sum_{n=0}^\infty w_n = \infty$ and $\limsup_{n\to\infty} \delta_n \le 0$. From Lemma 2.2 with $\nu_n = 0$, we conclude that $\lim_{n\to\infty}\|x_n - \widetilde{x}\| = 0$. This completes the proof. □

If we take $\mu = 0$, $u = 0$ and $f \equiv 0$ in Theorem 3.2, then we have the following corollary.

Corollary 3.5 *Let $\{x_n\}$ be defined by*

$$x_{n+1} = (1-\alpha_n)T_{r_n}F_{r_n}x_n.$$

Assume that the sequences $\{\alpha_n\}$ and $\{r_n\}$ satisfy conditions (C1)-(C4) *in Theorem* 3.2. *Then $\{x_n\}$ converges strongly to a point $\widetilde{x} \in \mathrm{VI}(C,F) \cap \mathrm{Fix}(T)$, which is the minimum norm point of* $\mathrm{VI}(C,F) \cap \mathrm{Fix}(T)$.

Taking $T \equiv I$ in Theorem 3.2, we have the following corollary.

Corollary 3.6 *Let $\{x_n\}$ be generated by the following iterative scheme*:

$$x_{n+1} = \alpha_n(u + \gamma f x_n) + \bigl(I - \alpha_n(I + \mu A)\bigr)F_{r_n}x_n, \quad \forall n \ge 0.$$

Assume that the sequences $\{\alpha_n\}$ and $\{r_n\}$ satisfy conditions (C1)-(C4) *in Theorem* 3.2. *Then $\{x_n\}$ converges strongly to a point $\widetilde{x} \in \mathrm{VI}(C,F)$, which is the unique solution of the optimization problem* (3.27).

Taking $F \equiv 0$ in Theorem 3.2, we get the following corollary.

Corollary 3.7 *Let $\{x_n\}$ be generated by the following iterative scheme*:

$$x_{n+1} = \alpha_n(u + \gamma f x_n) + \bigl(I - \alpha_n(I + \mu A)\bigr)T_{r_n}x_n, \quad \forall n \ge 0.$$

Assume that the sequences $\{\alpha_n\}$ and $\{r_n\}$ satisfy conditions (C1)-(C4) *in Theorem* 3.2. *Then $\{x_n\}$ converges strongly to a point $\widetilde{x} \in \mathrm{Fix}(T)$, which is the unique solution of the optimization problem* (3.28).

Taking $C \equiv H$, we have the following corollary.

Corollary 3.8 *Let $T : H \to H$ be a continuous pseudocontractive mapping, and let $F : H \to H$ be a continuous monotone mapping. Let $\{x_n\}$ be generated by* (3.30). *Assume that the sequences $\{\alpha_n\}$ and $\{r_n\}$ satisfy conditions* (C1)-(C4) *in Theorem* 3.2. *Then $\{x_n\}$ converges strongly to a point $\widetilde{x} \in F^{-1}(0) \cap \mathrm{Fix}(T)$, which is the unique solution of the optimization problem* (3.29).

Remark 3.1 (1) For finding an element of $\mathrm{VI}(C,F) \cap \mathrm{Fix}(T)$, $\mathrm{VI}(C,F)$, $\mathrm{Fix}(T)$, and $F^{-1}(0) \cap \mathrm{Fix}(T)$, which is the unique solution of the optimization problems (1.4), (3.27), (3.28), and (3.29), respectively, where T is a continuous pseudocontractive mapping and F is a continuous monotone mapping, our results are new ones different from previous those introduced by several authors. Consequently, our results supplement, develop, and improve upon the corresponding results given recently by several authors in this direction (for example, see [35–37] for $\mathrm{VI}(C,F) \cap \mathrm{Fix}(T)$ in the case of a continuous monotone mapping F and a continuous pseudocontractive mapping T; see [1, 31–34] for $\mathrm{VI}(C,F) \cap \mathrm{Fix}(S)$ in the case of an α-inverse-strongly monotone mapping F and a nonexpansive mapping S; see [11, 22, 23] for $\mathrm{Fix}(T)$ of a strictly pseudocontractive mapping T; and see [24, 27] for $\mathrm{Fix}(S)$ of a nonexpansive mapping S).

(2) As in Corollary 3.1 and Corollary 3.5, from Corollaries 3.2, 3.3, 3.4, 3.6, 3.7, and 3.8, we can obtain the minimum norm point of $\mathrm{VI}(C,F)$, $\mathrm{Fix}(T)$, and $F^{-1}(0) \cap \mathrm{Fix}(T)$ for the continuous monotone mapping F and the continuous pseudocontractive mapping T, respectively.

(3) We can replace the perturbed control condition $|\alpha_{n+1} - \alpha_n| \le o(\alpha_{n+1}) + \sigma_n$, $\sum_{n=0}^{\infty} \sigma_n < \infty$ on the control parameter $\{\alpha_n\}$ in (C3) of Theorem 3.2 by the following conditions [20, 21]:

(a) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$; or

(b) $\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}} = 1$ or, equivalently, $\lim_{n\to\infty} \frac{\alpha_n - \alpha_{n+1}}{\alpha_{n+1}} = 0$.

Competing interests
The author declares to have no competing interests.

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