Yang and Kong *Fixed Point Theory and Applications* (2015) 2015:172
DOI 10.1186/s13663-015-0425-z

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

Stability and convergence of a new composite implicit iterative sequence in Banach spaces

Liping Yang* and Weiming Kong

*Correspondence:
yanglping2003@126.com
School of Applied Mathematics, Guangdong University of Technology, Guangzhou, 510520, China

Abstract
The purpose of this paper is to study stability and strong convergence of asymptotically pseudocontractive mappings by using a new composite implicit iteration process in an arbitrary real Banach space. The results in this paper improve and extend the corresponding results in the literature.

MSC: 47H09; 47H10; 47J20

Keywords: implicit iteration process; strictly pseudocontractive mappings; asymptotically pseudocontractive mappings; almost *T*-stability; weak *T*-stability

1 Introduction

Throughout this paper we assume that E is an arbitrary real Banach space and E^* denotes the dual space of E. The normalized duality map $J : E \to 2^{E^*}$ is defined by

$$Jx := \{u^* \in E^* : \langle x, u^* \rangle = \|x\|^2; \|u^*\| = \|x\|\},$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing between elements of E and E^*. If E^* is strictly convex, then J is single-valued.

We first recall some definitions and conclusions.

Definition 1.1 Let $T : D(T) \subset E \to E$ be a mapping.

(1) T is said to be asymptotically nonexpansive (see [1]) if there exists a sequence $\{k_n\} \subset [1, \infty)$, $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \le k_n \|x - y\|, \quad \forall x, y \in D(T), n \ge 1;$$

(2) T is said to be asymptotically pseudocontractive (see [2]) with sequence $\{k_n\} \subset [0, \infty)$, if and only if $\lim_{n\to\infty} k_n = 1$, for all $n \ge 1$, $x, y \in D(T)$ and there exists $j(x - y) \in J(x - y)$ such that

$$\langle T^n x - T^n y, j(x - y)\rangle \le k_n \|x - y\|^2;$$

Springer

© 2015 Yang and Kong. This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (http://creativecommons.org/licenses/by/4.0/), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.

(3) T is said to be strictly asymptotically pseudocontractive with sequence $\{k_n\} \subset [0,\infty)$, if and only if $\lim_{n\to\infty} k_n = k \in (0,1)$, for all $n \geq 1$, $x, y \in D(T)$ and there exists $j(x-y) \in J(x-y)$ such that

$$\langle T^n x - T^n y, j(x-y)\rangle \leq k_n\|x-y\|^2;$$

(4) T is said to be uniformly L-Lipschitzian (see [3]) if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \leq L\|x-y\| \quad \text{for all } x, y \in D(T), n \geq 1.$$

It is easy to see that every asymptotically nonexpansive mapping is uniformly L-Lipschitzian and asymptotically pseudocontractive. In [4], Rhoades constructed an example to show that the class of asymptotically pseudocontractive mappings properly contains the class of asymptotically nonexpansive mappings.

The class of asymptotically pseudocontractive mappings has been studied by several authors (see [2, 4–7] and the references cited therein) by using the modified Mann iteration process (see [8]) and the modified Ishikawa iteration process (see [9]). Schu [5] proved the following theorem.

Theorem SC *Let H be a Hilbert space, $K \subset H$ nonempty bounded closed convex, $L > 0$, $T : K \to K$ completely continuous, uniformly L-Lipschitzian and asymptotically pseudocontractive with sequence $\{k_n\} \subset [1,\infty)$, $q_n = 2k_n - 1$, $\forall n \geq 1$, $\sum_{n=1}^{\infty}(q_n^2 - 1) < \infty$. $\{\alpha_n\} \subset [0,1]$, $\epsilon < \alpha_n \leq b$, $\forall n \geq 1$, and some $b \in (0, L^{-2}[(1+L^2)^{1/2} - 1])$, $x_1 \in K$, for all $n \geq 1$, define*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n. \tag{1.1}$$

Then $\{x_n\}$ converges to some fixed point of T.

The recursion formula (1.1) is a modification of the well-known Mann iteration process (see, *e.g.*, [8]).

Recently, Chang [6] extended Theorem SC to real uniformly smooth Banach spaces and proved the following theorem.

Theorem CH *Let E be a real uniformly smooth Banach space, K a nonempty bounded closed convex subset of E, $T : K \to K$ be an asymptotically pseudocontractive mapping with sequence $\{k_n\} \subset [1,\infty)$, $\lim_{n\to\infty} k_n = 1$, and $F(T) = \{x \in K : Tx = x\} \neq \emptyset$. Let $\{\alpha_n\} \subset [0,1]$ satisfy the following conditions*: (i) $\lim_{n\to\infty}\alpha_n = 0$, (ii) $\sum_{n=1}^{\infty}\alpha_n = \infty$. *For an arbitrary $x_0 \in K$, let $\{x_n\}$ be iteratively defined by* (1.1). *If there exists a strictly increasing function $\phi : [0,\infty) \to [0,\infty)$, $\phi(0) = 0$ such that $\langle T^n x_n - x^*, j(x_n - x^*)\rangle \leq k_n\|x_n - x^*\|^2 - \phi(\|x_n - x^*\|)$, $\forall n \geq 1$, then $x_n \to x^* \in F(T)$.*

Let K be nonempty closed convex subset of E and $\{T_i\}_{i=1}^r$ be a finite family of nonexpansive mappings from K into itself (*i.e.*, $\|T_i x - T_i y\| \leq \|x-y\|$ for $x, y \in K$ and $i = 1, 2, \ldots, r$). In 2001, Xu and Ori [10] introduced the following implicit iteration process. For an arbitrary

$x_0 \in K$ and $\{\alpha_n\}_{n=1}^{\infty} \subset [0,1)$, the sequence $\{x_n\}_{n=1}^{\infty}$ is generated as follows:

$$\begin{cases} x_1 = (1-\alpha_1)x_0 + \alpha_1 T_1 x_1, \\ x_2 = (1-\alpha_2)x_1 + \alpha_2 T_2 x_2, \\ \vdots \\ x_r = (1-\alpha_r)x_{r-1} + \alpha_r T_r x_r, \\ x_{r+1} = (1-\alpha_{r+1})x_r + \alpha_{r+1} T_1 x_{r+1}, \\ \vdots \end{cases}$$

The scheme can be expressed in compact form by

$$x_n = (1-\alpha_n)x_{n-1} + \alpha_n T_{n(\bmod r)} x_n, \quad n \geq 1. \tag{1.2}$$

Using this iteration, they proved that the sequence $\{x_n\}$ converges weakly to a common fixed point of the finite family of nonexpansive mappings $\{T_i\}_{i=1}^r$ in Hilbert spaces under certain conditions. Since then, the construction of fixed points for nonexpansive mappings and strictly pseudocontractive mappings and some other mappings via the implicit iterative algorithm has been extensively investigated by many authors (see, *e.g.*, [10–16] and the references cited therein). An implicit process is generally desirable when no explicit scheme is available. Such a process is generally used as a 'tool' to establish the convergence of an explicit scheme.

In 2006, Chang *et al.* [13] introduced another implicit iteration process with error. In the sense of [13], the implicit iteration process with errors for a finite family of asymptotically nonexpansive mappings $\{T_i\}_{i=1}^r$ is generated from an arbitrary $x_0 \in K$ by

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n) T_{i(n)}^{k(n)} x_n + u_n, \quad \forall n \geq 1, \tag{1.3}$$

where $n = (k(n)-1)r + i(n)$ with $i(n) \in \{1,2,\ldots,r\}$ and $k(n) \in \mathbb{N}$ (the positive integer set) and $k(n) \to \infty$, as $n \to \infty$. $\{\alpha_n\}_{n=1}^{\infty}$ is a suitable sequence in $[0,1]$ and $\{u_n\} \subset K$ is such that $\sum_{n=1}^{\infty} \|u_n\| < \infty$. They extended the results of [10] from Hilbert spaces to more general uniformly convex Banach spaces and from nonexpansive mappings to asymptotically nonexpansive mappings.

It is clear that even if K is a nonempty convex subset of E and $\{u_n\} \subset K$ is such that $\sum_{n=1}^{\infty} \|u_n\| < \infty$, then the implicit iterative sequence with errors in the sense of [13] need not be well defined, *i.e.*, $\{x_n\}_{n=1}^{\infty}$ may fail to be in K. More precisely, the conditions imposed on the error terms are not compatible with the randomness of the occurrence of errors.

In [16], Thakur proposed another modified composite implicit iteration process for a finite family of asymptotically nonexpansive mappings as follows:

$$\begin{cases} x_n = (1-\alpha_n)x_{n-1} + \alpha_n T_{i(n)}^{k(n)} y_n, \\ y_n = (1-\beta_n)x_{n-1} + \beta_n T_{i(n)}^{k(n)} x_n, \end{cases} \quad \{\alpha_n\}, \{\beta_n\} \subset [0,1], \forall n \geq 1,$$

where $n = (k-1)N + i$, $i = i(n) \in \{1,2,\ldots,N\}$, $k = k(n) \geq 1$ is some positive integer and $k(n) \to \infty$ as $n \to \infty$.

Inspired and motivated by these facts, we introduce a new modified composite implicit iteration process for an asymptotically pseudocontractive mappings as follows:

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n y_n, & \{\alpha_n\},\{\beta_n\} \subset [0,1], \forall n \geq 1. \\ y_n = (1-\beta_n)x_n + \beta_n T^n x_{n+1}, \end{cases} \tag{1.4}$$

Unlike iteration methods with errors of [13], our iteration process (1.4) is always well defined, that is, $\{x_n\}$ is always in K if K is convex subset of E.

If $\{\beta_n\} = \{0\}$ for all $n \geq 1$, (1.4) becomes the explicit form as follows:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n. \tag{1.5}$$

Equation (1.5) is the modified Mann iterative process (see, *e.g.*, [8]).

Stability results established in metric space, normed linear space, and Banach space settings are available in the literature. There are several authors whose contributions are of colossal value in the study of stability of the fixed point iterative procedures: Imoru and Olatinwo [17], Olatinwo and Postolache [18], Akewe and Okeke [19].

Harder and Hicks [20] mentioned that the study of the stability of iterative schemes is useful for both theoretical and numerical investigations. Consequently, several authors have studied the stability of iterative schemes for various types of nonlinear mappings (see, *e.g.*, [20–25] and the references cited therein).

The purpose of this paper is to study the stability and convergence of the composite implicit iterative sequence for an asymptotically pseudocontractive mapping in arbitrary real Banach spaces.

Let K be a nonempty convex subset of E and $T: K \to K$ be a mapping, $x_0 \in K$ and $\{x_n\} \subset K$ defined by

$$x_{n+1} = f(T, x_n), \tag{1.6}$$

where f is a continuous mapping. Suppose that $F(T) \neq \emptyset$ and $x_n \to p \in F(T)$. Let $\{s_n\}$ be any bounded sequence in K and $\{\varepsilon_n\}$ a sequence in $[0,\infty)$ defined by

$$\varepsilon_n = \| s_{n+1} - f(T, s_n) \|, \quad n \geq 0. \tag{1.7}$$

If $\varepsilon_n \to 0$ implies that $s_n \to p$, then the iterative sequence $\{x_n\}$ defined by (1.6) is said to be *T-stable*. If $\sum_{n=1}^{\infty} \varepsilon_n < \infty$ implies that $s_n \to p$, then the sequence $\{x_n\}$ defined by (1.6) is said to be *almost T-stable*. An example in [25] represents an iterative sequence which is almost T-stable but not T-stable.

2 Preliminaries

In the sequel, $F(T) = \{x \in K : Tx = x\}$ denotes the set of fixed points of T. We give the stability definition for the sequence $\{x_n\}$ defined by (1.4).

Definition 2.1 Let $\{x_n\}$ be the sequence defined by (1.4) such that $x_n \to p \in F(T)$. Let $\{s_n\}$ be any bounded sequence in K. Define a sequence $\{\varepsilon_n\}$ by

$$\begin{cases} \varepsilon_n = \|s_{n+1} - (1-\alpha_n)s_n - \alpha_n T^n z_n\|, & n \geq 1. \\ z_n = (1-\beta_n)s_n + \beta_n T^n s_{n+1}, \end{cases} \tag{2.1}$$

If $\varepsilon_n \to 0$ implies that $s_n \to p$, then the sequence $\{x_n\}$ is said to be *T-stable*. If $\varepsilon_n/\alpha_n \to 0$ implies that $s_n \to p$, then the sequence $\{x_n\}$ is said to be *weakly T-stable*. If $\sum_{n=1}^{\infty} \varepsilon_n < \infty$ implies that $s_n \to p$, then the sequence $\{x_n\}$ is said to be *almost T-stable*.

The following lemmas will be needed in proving our main results.

Lemma 2.2 (Lemma 2 of [26]) *Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be nonnegative real sequences satisfying the following conditions*:

$$a_{n+1} \leq (1 - t_n)a_n + b_n t_n + c_n, \quad \forall n \geq 0,$$

where $\{t_n\}$ is a sequence in $[0,1]$ such that $\sum_{n=1}^{\infty} t_n = \infty$, $\lim_{n\to\infty} b_n = 0$ and $\sum_{n=1}^{\infty} c_n < \infty$. Then $a_n \to 0$ $(n \to \infty)$.

We denote $\Phi := \{\phi \mid \phi : [0,\infty) \to [0,\infty)$ be a nondecreasing function such that $\phi(t) = 0$ if and only if $t = 0\}$.

Lemma 2.3 (Lemma 2.1 of [27]) *Let $\{\theta_n\}$ be a sequence of nonnegative real numbers, $\{\lambda_n\}$ be a real sequence satisfying*

$$0 \leq \lambda_n \leq 1, \quad \sum_{n=0}^{\infty} \lambda_n = \infty,$$

and let $\phi \in \Phi$. If there exists a positive integer n_0 such that

$$\theta_{n+1}^q \leq \theta_n^q - \lambda_n \phi(\theta_{n+1}) + \lambda_n \sigma_n$$

for some $q > 1$, all $n \geq n_0$ and $\lim_{n\to\infty} \sigma_n = 0$, then $\lim_{n\to\infty} \theta_n = 0$.

Lemma 2.4 (Lemma 1.1 of [28]) *Let E be a real Banach space, $T : E \to E$ a mapping, and λ any positive real number. Then for any $x, y \in E$ and $k > 0$,*

$$\|x - y\| \leq \bigl\|x - y + \lambda\bigl[(I - T - kI)x - (I - T - kI)y\bigr]\bigr\|$$

whenever

$$\bigl\langle (I - T - kI)x - (I - T - kI)y, j(x - y)\bigr\rangle \geq 0, \quad \forall j(x - y) \in J(x - y).$$

Lemma 2.5 *Let E be a normed linear space then for all $x, y \in E$ and for all $j(x + y) \in J(x + y)$, the following inequality holds*:

$$\|x + y\|^2 \leq \|x\|^2 + 2\bigl\langle y, j(x + y)\bigr\rangle.$$

First we give two auxiliary lemmas.

Lemma 2.6 *Let K be a nonempty convex subset of an arbitrary real Banach space E and $T : K \to K$ an asymptotically pseudocontractive mapping with $\{k_n\} \subset [0,\infty)$ and let $F(T) \neq \emptyset$.*

If $\{x_n\}$ *is the iterative sequence defined by* (1.4) *and* $p \in F(T)$, *then*

$$\|x_n - p\| \leq (1 + k_n\alpha_n)\big(1 - \alpha_n + \alpha_n^2\big)\|x_{n-1} - p\| + \alpha_n b_n,$$

where

$$b_n = \big\| T^n x_{n+1} - T^n y_n \big\| + \alpha_n(k_n + 1)\big\|x_n - T^n y_n\big\|. \tag{2.2}$$

Proof Since T is an asymptotically pseudocontractive mapping with $\{k_n\} \subset [0,\infty)$, there exists $j(x_{n+1} - p) \in J(x_{n+1} - p)$ for $x_{n+1} \in K$ and $p \in F(T)$ such that

$$\big\langle T^n x_{n+1} - p, j(x_{n+1} - p)\big\rangle \leq k_n\|x_{n+1} - p\|^2.$$

Then we have

$$\big\langle \big(I - T^n + (k_n - 1)I\big)x_{n+1} - \big(I - T^n + (k_n - 1)I\big)p, j(x_{n+1} - p)\big\rangle \geq 0.$$

Thus it follows from Lemma 2.4 that

$$\|x_{n+1} - p\| \leq \left\| x_{n+1} - p + \frac{\alpha_n}{1 + \alpha_n}\big\{\big[I - T^n + (k_n - 1)I\big]x_{n+1} - \big[I - T^n + (k_n - 1)I\big]p\big\}\right\|. \tag{2.3}$$

It follows from (1.4) that

$$\begin{aligned} x_n &= x_{n+1} + \alpha_n x_n - \alpha_n T^n y_n \\ &= (1 + \alpha_n)x_{n+1} + \alpha_n\big[I - T^n + (k_n - 1)I\big]x_{n+1} + \alpha_n\big(T^n x_{n+1} - T^n y_n\big) \\ &\quad - \alpha_n k_n x_n + \alpha_n^2(k_n + 1)\big(x_n - T^n y_n\big). \end{aligned} \tag{2.4}$$

Since $p \in F(T)$, we have

$$p = (1 + \alpha_n)p + \alpha_n\big[I - T^n + (k_n - 1)I\big]p - \alpha_n k_n p. \tag{2.5}$$

From (2.4) and (2.5), we have

$$\begin{aligned} \|x_n - p\| \geq (1 + \alpha_n)\bigg\| x_{n+1} - p + \frac{\alpha_n}{1 + \alpha_n}\big[\big(I - T^n + (k_n - 1)I\big)x_{n+1} \\ - \big(I - T^n + (k_n - 1)I\big)p\big]\bigg\| - \alpha_n k_n\|x_n - p\| \\ - \alpha_n\big\|T^n x_{n+1} - T^n y_n\big\| - \alpha_n^2(k_n + 1)\big\|x_n - T^n y_n\big\|. \end{aligned} \tag{2.6}$$

Therefore from (2.3) we have

$$\begin{aligned} \|x_n - p\| &\geq (1 + \alpha_n)\|x_{n+1} - p\| - \alpha_n k_n\|x_n - p\| \\ &\quad - \alpha_n\big\|T^n x_{n+1} - T^n y_n\big\| - \alpha_n^2(k_n + 1)\big\|x_n - T^n y_n\big\|, \end{aligned}$$

i.e.,

$$(1+\alpha_n)\|x_{n+1}-p\| \le (1+\alpha_n k_n)\|x_n-p\| + \alpha_n \left\| T^n x_{n+1} - T^n y_n \right\| + \alpha_n^2 (k_n+1) \left\| x_n - T^n y_n \right\|. \tag{2.7}$$

Since $(1+\alpha_n)^{-1} \le 1$ and $(1+\alpha_n)^{-1} \le 1-\alpha_n+\alpha_n^2$, from (2.7), we have

$$\|x_{n+1}-p\| \le (1+\alpha_n k_n)\bigl(1-\alpha_n+\alpha_n^2\bigr)\|x_n-p\| + \alpha_n b_n, \tag{2.8}$$

where $b_n = \|T^n x_{n+1} - T^n y_n\| + \alpha_n(k_n+1)\|x_n - T^n y_n\|$. This completes the proof. □

3 Main results

Theorem 3.1 *Let K be a nonempty convex subset of an arbitrary real Banach space E and $T: K \to K$ be a uniformly Lipschitzian* (*with a Lipschitzian constant $L > 0$*) *and strictly asymptotically pseudocontractive mapping with sequence $\{k_n\} \subset [0,\infty)$, $k_n \to k \in (0,1)$, and let $F(T) \neq \emptyset$. Assume that $\{x_n\}$, $\{y_n\}$ are the sequences defined by* (1.4), *$\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ satisfy the following conditions*:

$$\text{(i) } \alpha_n \to 0, \qquad \beta_n \to 0 \quad (n \to \infty); \qquad \text{(ii) } \sum_{n=1}^{\infty} \alpha_n = \infty.$$

If $\{x_n\}$ is bounded in K, then

(1) *$\{x_n\}$ converges strongly to the unique common fixed point p of T*;

(2) *$\{x_n\}$ is both almost T-stable and weakly T-stable.*

Proof Assume that $p_1, p_2 \in F(T)$. Since T is a strictly asymptotically pseudocontractive mapping, there exists $j(p_1-p_2) \in J(p_1-p_2)$ such that

$$\|p_1-p_2\|^2 = \bigl\langle T^n p_1 - T^n p_2, j(p_1-p_2)\bigr\rangle \le k_n \|p_1-p_2\|^2.$$

Letting $n \to \infty$ we have $\|p_1-p_2\|^2 \le k\|p_1-p_2\|^2$, $k \in (0,1)$. This implies that $p_1 = p_2$.

By the strictly asymptotically pseudocontractive property of T, similar to (2.8), we have

$$\|x_{n+1}-p\| \le (1+\alpha_n k_n)\bigl(1-\alpha_n+\alpha_n^2\bigr)\|x_n-p\| + \alpha_n b_n, \tag{3.1}$$

where $b_n = \|T^n x_{n+1} - T^n y_n\| + \alpha_n(k_n+1)\|x_n - T^n y_n\|$. Since $\alpha_n \to 0$, $k_n \to k \in (0,1)$, and

$$(1+\alpha_n k_n)\bigl(1-\alpha_n+\alpha_n^2\bigr) \le \bigl(1-\alpha_n(1-k_n-\alpha_n)\bigr).$$

Since $\lim_{n\to\infty}(1-k_n-\alpha_n) = 1-k \ge \delta \in (0,1-k)$, there exists a natural number n_1 such that $(1-k_n-\alpha_n) \ge \delta$ for all $n \ge n_1$. Thus we have

$$(1+\alpha_n k_n)\bigl(1-\alpha_n+\alpha_n^2\bigr) \le 1-\delta\alpha_n, \quad \forall n \ge n_1. \tag{3.2}$$

Substituting (3.2) into (3.1) we have

$$\|x_n-p\| \le (1-\delta\alpha_n)\|x_{n-1}-p\| + \alpha_n b_n, \quad \forall n \ge n_1. \tag{3.3}$$

Next we will prove that $b_n \to 0$ $(n \to \infty)$.

Since $T : K \to K$ is uniformly Lipschitzian with a Lipschitzian constant $L > 0$, for all $x, y \in K$, we have

$$\|T^n x_{n+1} - p\| \le L\|x_{n+1} - p\|, \quad \forall n \ge 1.$$

This implies that $\{T^n x_{n+1}\}$ is bounded in K since $\{x_n\}$ is bounded. Similarly, $\{T^n x_n\}$ is bounded sequence in K. It follows from (1.4) that

$$\|y_n - x_n\| = \beta_n \|x_n - T^n x_{n+1}\| \le \beta_n \bigl(\|x_n\| + \|T^n x_{n+1}\|\bigr) \to 0 \quad (n \to \infty). \tag{3.4}$$

This implies that $\{y_n\}$ is bounded. Thus by the uniform Lipschitzianness of T, we have

$$\|T^n y_n\| \le \|T^n y_n - T^n x_n\| + \|T^n x_n\| \le L\|y_n - x_n\| + \|T^n x_n\|.$$

This implies that $\{T^n y_n\}$ is also bounded. From (1.4) and (3.4), we have

$$\begin{aligned}\|x_{n+1} - y_n\| &= \|(x_n - y_n) + \alpha_n(x_n - T^n y_n)\| \\ &\le \|y_n - x_n\| + \alpha_n\bigl(\|x_n\| + \|T^n y_n\|\bigr) \to 0 \quad (n \to \infty).\end{aligned} \tag{3.5}$$

Therefore we have

$$\|T^n x_{n+1} - T^n y_n\| \le L\|x_{n+1} - y_n\| \to 0 \quad (n \to \infty). \tag{3.6}$$

Observing (3.6), $\{x_n\}$, $\{T^n y_n\}$ bounded in K and $\alpha_n \to 0$ $(n \to \infty)$, we know that $b_n \to 0$ $(n \to \infty)$. By Lemma 2.2 and (3.3), we have $x_n \to p$ $(n \to \infty)$. The conclusion (1) is proved.

Next we prove the conclusion (2). For any bounded sequence $\{s_n\} \subset K$ defined by (2.1) and $p \in F(T)$, we have

$$\begin{aligned}\|s_{n+1} - p\| &= \|s_{n+1} - (1 - \alpha_n)s_n - \alpha_n T^n z_n + (1 - \alpha_n)s_n + \alpha_n T^n z_n - p\| \\ &\le \varepsilon_n + \|p_n - p\|,\end{aligned} \tag{3.7}$$

where

$$p_n = (1 - \alpha_n)s_n + \alpha_n T^n z_n, \quad \forall n \ge 1. \tag{3.8}$$

It follows from (3.8) that

$$\begin{aligned}s_n &= p_n + \alpha_n s_n - \alpha_n T^n z_n \\ &= (1 + \alpha_n)p_n + \alpha_n\bigl[I - T^n + (k_n - 1)I\bigr]p_n + \alpha_n\bigl(T^n p_n - T^n z_n\bigr) \\ &\quad + \alpha_n k_n s_n + \alpha_n^2(k_n + 1)\bigl(s_n - T^n z_n\bigr).\end{aligned}$$

By using a similar method to that given in proving (2.5)-(2.8), (3.2), we can prove that

$$\|p_n - p\| \le (1 - \alpha_n\delta)\|s_n - p\| + \alpha_n b_n, \quad \forall n \ge n_1, \tag{3.9}$$

where $b_n = \|T^n p_n - T^n z_n\| + \alpha_n(k_n + 1)\|s_n - T^n z_n\|$. Since T is uniformly Lipschitzian with Lipschitzian constant $L > 0$, we have $\|T^n s_n - p\| \le L\|s_n - p\|$. This implies that the sequence $\{T^n s_n\}$ is bounded since $\{s_n\}$ is bounded in K. By the same method as in proving (3.4)-(3.6), we can prove that the sequence $\{T^n z_n\}$ is bounded and $\|T^n p_n - T^n z_n\| \to 0$ $(n \to \infty)$. Therefore we have $b_n \to 0$ $(n \to \infty)$. Substituting (3.9) into (3.7) we have

$$\|s_{n+1} - p\| \le (1 - \alpha_n\delta)\|s_n - p\| + \alpha_n b_n + \varepsilon_n, \quad \forall n \ge n_1. \tag{3.10}$$

If $\sum_{n=1}^{\infty} \varepsilon_n < \infty$, taking $a_n = \|s_n - p\|$, $t_n = \alpha_n\delta$, $c_n = \varepsilon_n$ in Lemma 2.2, from (3.10), we have $s_n \to p$ $(n \to \infty)$, *i.e.*, $\{x_n\}$ is almost T-stable.

If $\varepsilon_n/\alpha_n \to 0$, taking $a_n = \|s_n - p\|$, $t_n = \alpha_n\delta$, $c_n = 0$ in Lemma 2.2, from (3.10), we have $s_n \to p$ $(n \to \infty)$, *i.e.*, $\{x_n\}$ is weakly T-stable. This completes the proof. □

Example 3.2 Let $E = R = (-\infty, \infty)$ with the usual norm. Take $K = [0,1]$ and define $T : K \to K$ by

$$Tx = \begin{cases} 0 & \text{if } x = 0, \\ \frac{1}{4} & \text{if } x = 1, \\ x - \frac{1}{2^{n+1}} & \text{if } \frac{1}{2^{n+1}} \le x < \frac{1}{2}(\frac{1}{2^{n+1}} + \frac{1}{2^n}), \\ \frac{1}{2^n} - x & \text{if } \frac{1}{2}(\frac{1}{2^{n+1}} + \frac{1}{2^n}) \le x < \frac{1}{2^n} \end{cases}$$

for all $n \ge 0$. Then $F(T) = \{0\}$ and T is not continuous at $x = 1$. We can verify that

$$Tx \le \frac{1}{2}x, \quad x \in K.$$

Thus T^2 is continuous in K and $T^2K \subset [0, 2^{-n}]$ for all $n \ge 1$. Then for any $x \in K$, there exists $j(x - 0) \in J(x - 0)$ satisfying

$$\langle T^n x - T^n 0, j(x - 0)\rangle = T^n x \cdot x \le \frac{1}{2}\|x\|^2$$

for all $n \ge 1$. That is, T is a strictly asymptotically pseudocontractive mapping.

Example 3.3 Let $E = [0,1]$. Define $T : E \to E$ by $Tx = \frac{x}{5}$, where E has the usual norm. Then $F(T) = \{0\}$ and T is a strictly asymptotically pseudocontractive mapping with $k_n = \frac{1}{5}$. Consider the following conditions:

$$\alpha_n = \frac{1}{n+1}, \qquad \beta_n = \frac{1}{n+2}, \quad \forall n \ge 1.$$

Let $\{x_n\}$ be the sequence defined by (1.4). So

$$\begin{cases} x_1 = 0.25, \\ x_{n+1} = (1 - \frac{1}{n+1})x_n + \frac{1}{n+1}T^n y_n, \qquad \forall n \ge 1. \\ y_n = (1 - \frac{1}{n+2})x_n + \frac{1}{n+2}T^n x_{n+1}, \end{cases}$$

We have the results in Table 1.

Table 1 The iteration chart with initial value $x_1 = 0.25$

n	x_n	*n*	x_n
1	0.25	6	0.045313
2	0.134228	7	0.038840
3	0.090397	8	0.033985
4	0.067943	9	0.030209
5	0.054373	10	0.027188

Therefore, the conditions of Theorem 3.1 are fulfilled.

Theorem 3.4 *Let K be a nonempty convex subset of an arbitrary real Banach space E, $T: K \to K$ be a uniformly Lipschitzian (with a Lipschitzian constant $L > 0$) and asymptotically pseudocontractive mapping with sequence $\{k_n\} \subset [1,\infty)$, $\lim_{n\to\infty} k_n = 1$ and let $F(T) \neq \emptyset$ and $p \in F(T)$. Let $\{x_n\}$, $\{y_n\}$ be the sequences defined by (1.4). Assume that $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ satisfy the following conditions:*

$$\text{(i) } \alpha_n \to 0, \qquad \beta_n \to 0 \quad (n \to \infty); \qquad \text{(ii) } \sum_{n=1}^{\infty} \alpha_n = \infty.$$

If $\{x_n\}$ is bounded in K and there exists a strictly increasing function $\phi: [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that

$$\limsup_{n\to\infty} \bigl\{ \bigl\langle T^n x_{n+1} - p, j(x_{n+1} - p) \bigr\rangle - k_n \|x_{n+1} - p\|^2 + \phi\bigl(\|x_{n+1} - p\|\bigr) \bigr\} \leq 0,$$

where $j(x_{n+1} - p) \in J(x_{n+1} - p)$. Then $\{x_n\}$ converges strongly to the fixed point of T.

Proof Since $\{x_n\}$ is bounded in K, then $M = \sup_{n\geq 1}\{\|x_n - p\|\} < \infty$ for $p \in F(T)$. It follows from Lemma 2.5 and (1.4) that there exists $j(x_{n+1} - p) \in J(x_{n+1} - p)$ such that

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \bigl\|(1-\alpha_n)(x_n - p) + \alpha_n\bigl(T^n y_n - p\bigr)\bigr\|^2 \\
&\leq (1-\alpha_n)^2\|x_n - p\|^2 + 2\alpha_n\bigl\langle T^n y_n - p, j(x_{n+1} - p)\bigr\rangle \\
&= (1-\alpha_n)^2\|x_n - p\|^2 + 2\alpha_n\bigl\langle T^n x_{n+1} - p, j(x_{n+1} - p)\bigr\rangle \\
&\quad + 2\alpha_n\bigl\langle T^n y_n - T^n x_{n+1}, j(x_{n+1} - p)\bigr\rangle \\
&\leq (1-\alpha_n)^2\|x_n - p\|^2 + 2\alpha_n\bigl\langle T^n x_{n+1} - p, j(x_{n+1} - p)\bigr\rangle \\
&\quad + 2\alpha_n M\bigl\|T^n y_n - T^n x_{n+1}\bigr\| \\
&= (1-\alpha_n)^2\|x_n - p\|^2 + 2\alpha_n d_n \\
&\quad + 2\alpha_n\bigl[k_n\|x_{n+1} - p\|^2 - \phi\bigl(\|x_{n+1} - p\|\bigr)\bigr] + 2M\alpha_n e_n,
\end{aligned} \tag{3.11}$$

where $d_n = \langle T^n x_{n+1} - p, j(x_{n+1} - p)\rangle - k_n\|x_{n+1} - p\|^2 + \phi(\|x_{n+1} - p\|)$, $e_n = \|T^n y_n - T^n x_{n+1}\|$. From (3.6), we have $e_n \to 0$ as $n \to \infty$. It follows from (3.11) that

$$\begin{aligned}
(1 - 2\alpha_n k_n)\|x_{n+1} - p\|^2 &\leq (1-\alpha_n)^2\|x_n - p\|^2 - 2\alpha_n\phi\bigl(\|x_{n+1} - p\|\bigr) \\
&\quad + 2\alpha_n d_n + 2M\alpha_n e_n.
\end{aligned}$$

Notice that $\lim_{n\to\infty}(1-2\alpha_n k_n) = 1 > 0$, without loss of generality, we assume that $(1-2\alpha_n k_n) > 0$ for all $n \geq 1$. Therefore, we have

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\leq \frac{(1-\alpha_n)^2}{1-2\alpha_n k_n}\|x_n-p\|^2 \\
&\quad - \frac{2\alpha_n}{1-2\alpha_n k_n}\phi\bigl(\|x_{n+1}-p\|\bigr) + \frac{2\alpha_n d_n}{1-2\alpha_n k_n} + \frac{2M\alpha_n e_n}{1-2\alpha_n k_n} \\
&= \|x_n-p\|^2 + \frac{\alpha_n[2(k_n-1)+\alpha_n]}{1-2\alpha_n k_n}\|x_n-p\|^2 \\
&\quad - \frac{2\alpha_n}{1-2\alpha_n k_n}\phi\bigl(\|x_{n+1}-p\|\bigr) + \frac{2\alpha_n d_n}{1-2\alpha_n k_n} + \frac{2M\alpha_n e_n}{1-2\alpha_n k_n} \\
&\leq \|x_n-p\|^2 - 2\alpha_n\phi\bigl(\|x_{n+1}-p\|\bigr) + \frac{\alpha_n[2(k_n-1)+\alpha_n]}{1-2\alpha_n k_n}M^2 \\
&\quad + \frac{2M\alpha_n e_n}{1-2\alpha_n k_n} + \frac{2\alpha_n d_n}{1-2\alpha_n k_n} \quad \bigl(\text{since } 1-2\alpha_n k_n \in (0,1)\bigr) \\
&= \|x_n-p\|^2 - 2\alpha_n\phi\bigl(\|x_{n+1}-p\|\bigr) + \frac{\alpha_n\lambda_n'}{1-2\alpha_n k_n} + \frac{2\alpha_n d_n}{1-2\alpha_n k_n},
\end{aligned} \tag{3.12}$$

where $\lambda_n' = [2(k_n-1)+\alpha_n]M^2 + 2e_n M$. Taking $q = 2$, $\theta_n = \|x_n - p\|$, $\lambda_n = 2\alpha_n$, $\sigma_n = \frac{\lambda_n'+2d_n}{2(1-2\alpha_n k_n)}$ in Lemma 2.3, from (3.12), we have $x_n \to p$ as $n \to \infty$. This completes the proof. □

If $\{\beta_n\} = \{0\}$ for all $n \geq 1$ in (1.4), it follows from Theorem 3.4 that we have the following result.

Theorem 3.5 *Let K be a nonempty convex subset of an arbitrary real Banach space E, $T: K \to K$ be a uniformly Lipschitzian* (*with a Lipschitzian constant $L > 0$*) *and asymptotically pseudocontractive mapping with sequence $\{k_n\} \subset [1,\infty)$, $\lim_{n\to\infty} k_n = 1$ and let $F(T) \neq \emptyset$ and $p \in F(T)$, $\{x_n\}$ be the sequence defined by* (1.5). *Let $\{\alpha_n\} \subset [0,1]$ satisfy the following conditions*: (i) $\lim_{n\to\infty}\alpha_n = 0$, (ii) $\sum_{n=1}^{\infty}\alpha_n = \infty$. *If $\{x_n\}$ is bounded in K and there exists a strictly increasing function $\phi: [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that*

$$\limsup_{n\to\infty}\bigl\{\bigl\langle T^n x_{n+1}-p, j(x_{n+1}-p)\bigr\rangle - k_n\|x_{n+1}-p\|^2 + \phi\bigl(\|x_{n+1}-p\|\bigr)\bigr\} \leq 0,$$

where $j(x_{n+1}-p) \in J(x_{n+1}-p)$, $\forall n \geq 1$, then $x_n \to p$ as $n \to \infty$.

Remark 3.6 Theorem 3.5 extends Theorem CH from real uniformly smooth Banach space to arbitrary real Banach space. The requirement that K be bounded closed imposed in Theorem CH is stronger than the requirement that $\{x_n\}$ be bounded imposed in Theorem 3.5. The condition

$$\bigl\langle T^n x_n - p, j(x_n-p)\bigr\rangle \leq k_n\|x_n-p\|^2 + \phi\bigl(\|x_n-p\|\bigr)$$

in Theorem CH is replaced by

$$\limsup_{n\to\infty}\bigl\{\bigl\langle T^n x_{n+1}-p, j(x_{n+1}-p)\bigr\rangle - k_n\|x_{n+1}-p\|^2 + \phi\bigl(\|x_{n+1}-p\|\bigr)\bigr\} \leq 0$$

in Theorem 3.5.

Remark 3.7 We remark that if the error terms are added in (1.4) and are assumed to be bounded, then the results of this paper still hold. On carefully reading Thakur's work [16], we discovered that there are gaps in the proof of Lemma 2.1 in [16]. In (2.2) and (2.3) of Lemma 2.1, one cannot deduce $\sum_{n=1}^{\infty} \sigma_n < \infty$ from $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$. Thus, his main results would not hold.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Acknowledgements
The authors thank the referees for the comments that helped improve the presentation of this article. This work was supported by the Humanity and Social Science Planning Foundation of Ministry of Education of China (Grant No. 14YJAZH095), the National Natural Science Foundation of China (Grant No. 61374081), the High-level Talents Project in Guangdong Province (Grant No. 2014011), and the Natural Science Foundation of Guangdong Province (Grant Nos. S2013010013034, 2015A030313485).

Received: 3 June 2015 Accepted: 14 September 2015 Published online: 22 September 2015

References
1. Goebel, K, Kirk, WA: A fixed point theorem for asymptotically nonexpansive mappings. Proc. Am. Math. Soc. **35**, 171-174 (1972)
2. Schu, J: On a theorem of C.E. Chidume concerning the iterative approximation of fixed points. Math. Nachr. **153**, 313-319 (1991)
3. Goebel, K, Kirk, WA: A fixed point theorem for transformations whose iterates have uniform Lipschitz constant. Stud. Math. **47**, 135-140 (1973)
4. Rhoades, BE: A comparison of various definition of contractive mappings. Trans. Am. Math. Soc. **226**, 257-290 (1977)
5. Schu, J: Iterative construction of fixed point of asymptotically nonexpansive mappings. J. Math. Anal. Appl. **158**, 407-413 (1991)
6. Chang, SS: Some results for asymptotically pseudocontractive mappings and asymptotically nonexpansive mappings. Proc. Am. Math. Soc. **129**, 845-853 (2000)
7. Ofoedu, EU: Strong convergence theorem for uniformly L-Lipschitzian asymptotically pseudocontractive mapping in real Banach space. J. Math. Anal. Appl. **321**, 722-728 (2006)
8. Mann, WR: Mean value methods in iteration. Proc. Am. Math. Soc. **4**, 506-510 (1953)
9. Ishikawa, S: Fixed point and iteration of a nonexpansive mapping in a Banach space. Proc. Am. Math. Soc. **59**, 65-71 (1976)
10. Xu, HK, Ori, RG: An implicit iteration process for nonexpansive mappings. Numer. Funct. Anal. Optim. **22**(5), 767-773 (2001)
11. Osilike, MO: Implicit iteration process for common fixed points of a finite family of strictly pseudocontractive maps. J. Math. Anal. Appl. **294**, 73-81 (2004)
12. Gu, F: The new composite implicit iterative process with errors for common fixed points of a finite family of strictly pseudocontractive mappings. J. Math. Anal. Appl. **329**, 766-776 (2007)
13. Chang, SS, Tan, KK, Lee, HWJ, Chan, CK: On the convergence of implicit iteration process with errors for a finite family of asymptotically nonexpansive mappings. J. Math. Anal. Appl. **313**, 273-283 (2006)
14. Yang, LP: Convergence of the new composite implicit iteration process with random errors. Nonlinear Anal. TMA **69**, 3591-3600 (2008)
15. Su, Y, Li, S: Composite implicit iteration process for common fixed points of a finite family of strictly pseudocontractive maps. J. Math. Anal. Appl. **320**, 882-891 (2006)
16. Thakur, BS: Weak and strong of composite implicit iteration process. Appl. Math. Comput. **190**, 965-973 (2007)
17. Imoru, CO, Olatinwo, MO: On the stability of Picard and Mann iteration procedures. Carpath. J. Math. **19**(2), 155-160 (2003)
18. Olatinwo, MO, Postolache, M: Stability results for Jungck-type iterative processes in convex metric spaces. Appl. Math. Comput. **218**(12), 6727-6732 (2012)
19. Akewe, H, Okeke, GA: Convergence and stability theorems for the Picard-Mann hybrid iterative scheme for a general class of contractive-like operators. Fixed Point Theory Appl. **2015**, 66 (2015)
20. Harder, AM, Hicks, TL: Stability results for fixed point iteration procedures. Math. Jpn. **33**(5), 693-706 (1988)
21. Zhou, HY, Chang, SS, Cho, YJ: Weak stability of Ishikawa iteration procedures for ϕ-hemicontractive and accretive operators. Appl. Math. Lett. **14**, 949-954 (2001)
22. Stević, S: Stability results for ϕ-strongly pseudocontractive mappings. Yokohama Math. J. **50**, 71-85 (2003)
23. Huang, Z: Weak stability of Mann and Ishikawa iterations with errors for ϕ-hemicontractive operators. Appl. Math. Lett. **20**, 470-475 (2007)
24. Shahzad, N, Zegeye, H: On stability results for ϕ-strongly pseudocontractive mappings. Nonlinear Anal. TMA **64**, 2619-2630 (2006)
25. Osilike, MO: Stability of the Mann and Ishikawa iteration procedures for ϕ-strong pseudocontractions and nonlinear equations of the ϕ-strongly accretive type. J. Math. Anal. Appl. **227**, 319-334 (1998)

26. Liu, LS: Ishikawa and Mann iterative process with errors for nonlinear strongly accretive mappings in Banach spaces. J. Math. Anal. Appl. **194**, 114-125 (1995)
27. Ćirić, LB, Ume, JS: Iterative processes with errors for nonlinear equations. Bull. Aust. Math. Soc. **69**, 117-189 (2005)
28. Kato, T: Nonlinear semigroups and evolution equations. J. Math. Soc. Jpn. **19**, 508-520 (1964)

Submit your manuscript to a SpringerOpen journal and benefit from:

► Convenient online submission
► Rigorous peer review
► Immediate publication on acceptance
► Open access: articles freely available online
► High visibility within the field
► Retaining the copyright to your article

Submit your next manuscript at ► springeropen.com