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Fuzzy fixed point theorems on the complete fuzzy spaces under supremum metric

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Abstract

By using the concept of a class of functions, the \mathcal{R} -functions, we provide some fuzzy fixed point theorems on a space of fuzzy sets equipped with the supremum metric. By presenting a technique of constructing a sequence of successive approximations, we obtain some interesting results that improve many existing results. The related cases are also shown and discussed.

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1 Introduction

By using a natural generalization of the concept of a set, the fuzzy set, which was introduced initially by Zadeh [1], considering mathematical programming problems which are expressed as optimizing some goal function given certain constraints, this be relaxed by means of a subjective gradation. In 1981, Heilpern [2] used the concept of fuzzy sets and introduced a class of fuzzy mappings, which is a generalization of the set-valued mapping, and proved a fixed point theorem for fuzzy contraction mappings in metric linear spaces. It is worth noting that the result announced by Heilpern [2] is a fuzzy extension of the Banach contraction principle. Subsequently, several other authors have studied the existence of fixed points of fuzzy mappings; for example, Estruch and Vidal [3] proved a fixed point theorem for fuzzy contraction mappings over a complete metric space, which is a generalization of the given Heilpern fixed point theorem, and Sedghi *et al.* [4] gave an extended version of the Estruch and Vidal [3] theorem (for more examples, see [5–14]).

Although many kinds of fixed point theorems for fuzzy contraction mappings in complete metric spaces have been studied extensively in recent years, we have to point out that one has given most attention to the class of fuzzy sets with nonempty compact α -cut sets in the metric space X , but little attention to the class of fuzzy sets with nonempty bounded or closed, or even bounded closed, α -cut sets. However, it is well known that all compact sets are bounded closed sets in a general metric space and the converse is not always true.

In 2008, Qui and Shu [12] established the completeness of $\mathcal{CB}(X)$ with respect to the completeness of the metric space X , where $\mathcal{CB}(X)$ denotes the class of fuzzy sets with

nonempty bounded closed α -cut sets equipped with the generalized Hausdorff metric d_∞ , which takes the supremum on the Hausdorff distances between the corresponding α -cut sets. Also they proved the following common fixed point theorem for a family of fuzzy mappings.

Theorem 1.1 [12] *Let (X, d) be a complete metric space and let $\{F_i\}$ be a sequence of self-mappings of $\mathcal{CB}(X)$. If there exists a constant $q \in (0, 1)$ such that for each $\mu_1, \mu_2 \in \mathcal{CB}(X)$, and for arbitrary positive integers i and $j, i \neq j$,*

$$d_\infty(F_i(\mu_1), F_j(\mu_2)) \leq qM_{(i,j)}(\mu_1, \mu_2),$$

where

$$M_{(i,j)}(\mu_1, \mu_2) = \max \left\{ d_\infty(\mu_1, \mu_2), \rho_\infty(\mu_1, F_i(\mu_1)), \rho_\infty(\mu_2, F_j(\mu_2)), \frac{\rho_\infty(\mu_2, F_i(\mu_1)) + \rho_\infty(\mu_1, F_j(\mu_2))}{2} \right\}.$$

Then there exists a $\mu_* \in \mathcal{CB}(X)$ such that $\mu_* \subseteq F_i(\mu_*)$ for all $i \in \mathbb{N}$.

Later, by using the concept of d_∞ metric, Qiu *et al.* [15] proved the following common fixed point theorem, but under the assumption of a compact cut set $\mathcal{C}(X)$ instead of a closed bounded cut set $\mathcal{CB}(X)$.

Theorem 1.2 [15] *Let (X, d) be a compact metric space and let $\{F_i\}_{i=1}^\infty$ be a sequence of self-mappings of $\mathcal{C}(X)$. Let $\Phi : [0, \infty) \rightarrow [0, \infty)$ be a non-decreasing function satisfying the following condition: Φ is continuous from the right and*

$$\sum_{n=1}^\infty \Phi^n(t) < \infty, \quad \text{for all } t > 0,$$

where Φ^n denotes the n th iterative function of Φ . Suppose that for arbitrary positive integers i and $j, i \neq j$,

$$d_\infty(F_i(\mu_1), F_j(\mu_2)) \leq \Phi(M_{(i,j)}(\mu_1, \mu_2)),$$

where $M_{(i,j)}(\mu_1, \mu_2)$ is defined as in Theorem 1.1. Then there exists a $\mu_* \in \mathcal{C}(X)$ such that $\mu_* \subseteq F_i(\mu_*)$ for all $i \in \mathbb{N}$.

Notice that Theorem 1.2 can be used to apply to a larger class of mappings than that of Theorem 1.1. However, after careful consideration, one may see that Theorem 1.2 is relevant when the considered space is a compact metric space instead of a complete metric space, which has been considered in Theorem 1.1.

Based on the above remarks, here we will present an extension of Theorem 1.1, but in a complete metric space setting. In fact, we will use the concept of a class of functions, so-called \mathcal{R} -functions, to show some fixed point theorems for self-mappings of $\mathcal{CB}(X)$ with the supremum metric for fuzzy sets. Of course, our results improve and extend those results which have been presented in [12] and [15].

2 Preliminaries

In this section, we will provide some important basic concepts and useful results. Let (X, d) be a metric space, and let $CB(X)$ be the set of all nonempty bounded closed subsets of X . Recall that the Hausdorff metric is a function H on $CB(X)$ defined by

$$\begin{aligned}
 H(A, B) &= \max \left\{ \sup_{x \in B} d(x, A), \sup_{x \in A} d(x, B) \right\} \\
 &:= \max \{ \rho(B, A), \rho(A, B) \}, \quad \text{for all } A, B \in CB(X),
 \end{aligned}$$

where $\rho(A, B) = \sup_{x \in A} d(x, B)$ is the Hausdorff separation of A from B .

2.1 Fuzzy sets and fuzzy mappings

Let $I = [0, 1]$. A fuzzy set μ of a metric space X is defined by its membership function $\mu(x)$, which is a mapping from X into I . We denote by $\mathfrak{F}(X)$ the set of all fuzzy mappings on X , that is, $\mathfrak{F}(X) := \{ \mu | \mu : X \rightarrow [0, 1] \}$. For any $\alpha \in (0, 1]$, the α -cut of the fuzzy set μ is defined by

$$[\mu]_\alpha = \{ x \in X : \mu(x) \geq \alpha \},$$

where $\alpha \in (0, 1]$, and we separately specify the support $[\mu]_0$ of μ to be the closure of the union of $[\mu]_\alpha$ for $0 < \alpha \leq 1$. We denote by $\mathcal{CB}(X)$ the totality of fuzzy sets $\mu : X \rightarrow I$ for which, for each $\alpha \in I$, the α -cut of μ is a nonempty closed bounded subset of X .

Let $\mu_1, \mu_2 \in \mathfrak{F}(X)$. Then μ_1 is said to be included in μ_2 , denoted by $\mu_1 \subseteq \mu_2$, if and only if $\mu_1(x) \leq \mu_2(x)$ for each $x \in X$. Thus we have $\mu_1 \subseteq \mu_2$ if and only if $[\mu_1]^\alpha \subseteq [\mu_2]^\alpha$ for all $\alpha \in I$. Let X, Y be any underling sets and $\mathfrak{U}, \mathfrak{V}$ are subsets of $\mathfrak{F}(X)$ and $\mathfrak{F}(Y)$, respectively. A mapping $F : \mathfrak{U} \rightarrow \mathfrak{V}$ is said to be a fuzzy mapping, *i.e.*, $F(\mu) \in \mathfrak{V}$ for each $\mu \in \mathfrak{U}$. An element $\mu_* \in \mathfrak{U}$ is said to be a fixed point of a fuzzy self-mapping F on \mathfrak{U} if and only if $\mu_* \subseteq F(\mu_*)$.

The d_∞ -metric (called supremum or generalized Hausdorff metric) is a metric on $\mathcal{CB}(X)$ which is defined as follows:

$$\begin{aligned}
 d_\infty(\mu_1, \mu_2) &= \sup_{0 \leq \alpha \leq 1} H([\mu_1]^\alpha, [\mu_2]^\alpha) \\
 &= \max \{ \rho_\infty(\mu_1, \mu_2), \rho_\infty(\mu_2, \mu_1) \},
 \end{aligned} \tag{2.1}$$

where $\mu_1, \mu_2 \in \mathcal{CB}(X)$, and

$$\rho_\infty(\mu_1, \mu_2) = \sup_{0 \leq \alpha \leq 1} \rho([\mu_1]^\alpha, [\mu_2]^\alpha)$$

is the Hausdorff separation of μ_1 from μ_2 . Notice that the supremum in (2.1) may be not attained, and so it cannot be replaced by a maximum. To clarify this, we include the following example, which can be found in [16].

Example 2.1 Let X be a set of real numbers and $\mu, \nu \in \mathfrak{F}(X)$ be fuzzy subsets of X such that the corresponding level sets are

$$[\mu]^\alpha = [\nu]^\alpha = [0, 1] \quad \text{for } 0 \leq \alpha \leq \frac{1}{2},$$

and

$$[\mu]^\alpha = \{0\}, \quad [v]^\alpha = [0, 2(1 - \alpha)] \quad \text{for } \frac{1}{2} \leq \alpha \leq 1.$$

It follows that

$$H([\mu]^\alpha, [v]^\alpha) = \begin{cases} 0 & \text{for } 0 \leq \alpha \leq \frac{1}{2}, \\ 2(1 - \alpha) & \text{for } \frac{1}{2} \leq \alpha \leq 1. \end{cases}$$

Then $d_\infty(\mu, v) = \sup_{0 \leq \alpha \leq 1} H([\mu]^\alpha, [v]^\alpha) = 1$, but this is not attained.

Note that if $\{\mu_n\}$ be a sequence in $\mathcal{CB}(X)$, then it follows from the definition of d_∞ that $\{\mu_n\}$ converges with respect to the d_∞ -metric if and only if $[\mu_n]^\alpha$ converges uniformly in $\alpha \in I$ with respect to the Hausdorff metric. Further, we know that the metric space $(\mathcal{CB}(X), d_\infty)$ is complete provided (X, d) is complete (see [12]). Here, we collect some useful properties of the d_∞ -metric, which will be used in order to obtain our results.

Lemma 2.2 [12] *Let $\mu_1, \mu_2, \mu_3 \in \mathcal{CB}(X)$. The following items are true:*

- (i) $\rho_\infty(\mu_1, \mu_2) = 0$ if and only if $\mu_1 \subseteq \mu_2$,
- (ii) if $\mu_1 \subseteq \mu_2$, then $\rho_\infty(\mu_1, \mu_3) \leq d_\infty(\mu_2, \mu_3)$,
- (iii) $\rho_\infty(\mu_1, \mu_3) \leq d_\infty(\mu_1, \mu_2) + \rho_\infty(\mu_2, \mu_3)$.

Theorem 2.3 [12] *Let (X, d) be a metric space and $\mu_1, \mu_2 \in \mathcal{CB}(X)$. Then for any $\beta > 1$ and any $\mu_3 \in \mathcal{CB}(X)$ satisfying $\mu_3 \subseteq \mu_1$, there exists a $\mu_4 \in \mathcal{CB}(X)$ such that $\mu_4 \subseteq \mu_2$ and $d_\infty(\mu_3, \mu_4) \leq \beta d_\infty(\mu_1, \mu_2)$.*

2.2 \mathcal{R} -Functions

In this subsection, we will recall an important tool related to our considered class of mappings. A function $\varphi : [0, \infty) \rightarrow [0, 1)$ is said to be an \mathcal{R} -function if

$$\limsup_{s \rightarrow t^+} \varphi(s) < 1 \quad \text{for all } t \in [0, \infty).$$

Note that if $\varphi : [0, \infty) \rightarrow [0, 1)$ is a non-decreasing function or a non-increasing function, then φ is an \mathcal{R} -function. This means the set of \mathcal{R} -functions is a rich class. In [17], Du proved some of the following characterizations for the class of \mathcal{R} -functions.

Theorem 2.4 [17] *Let $\varphi : [0, \infty) \rightarrow [0, 1)$ be a function. Then the following statements are equivalent.*

- (a) φ is an \mathcal{R} -function.
- (b) For any nonincreasing sequence $\{x_n\}_{n \in \mathbb{N}}$ in $[0, \infty)$, we have $0 \leq \sup_{n \in \mathbb{N}} \varphi(x_n) < 1$.

3 Fixed point theorems for fuzzy mappings induced by \mathcal{R} -functions

Now, we are in a position to present our main results.

Theorem 3.1 *Let (X, d) be a complete metric space and let $\{F_i\}_{i=1}^\infty$ be a sequence of fuzzy self-mappings of $\mathcal{CB}(X)$. Assume that there exists an \mathcal{R} -function $\varphi : [0, \infty) \rightarrow [0, 1)$ such*

that for each $\mu_1, \mu_2 \in \mathcal{CB}(X)$, and for arbitrary positive integers i and j , $i \neq j$,

$$d_\infty(F_i(\mu_1), F_j(\mu_2)) \leq \varphi(d_\infty(\mu_1, \mu_2))M_{(i,j)}(\mu_1, \mu_2), \tag{3.1}$$

where $M_{(i,j)}(\mu_1, \mu_2)$ is defined as in Theorem 1.1. Then there exists a $\mu_* \in \mathcal{CB}(X)$ such that $\mu_* \subseteq F_i(\mu_*)$ for all positive integers i .

Proof Let us define a function $k : [0, \infty) \rightarrow [0, 1)$ by

$$k(t) = \frac{1 + \varphi(t)}{2}, \quad \text{for all } t \in [0, \infty).$$

Note that we have $0 \leq \varphi(t) < k(t) < 1$ for all $t \in [0, \infty)$.

We will start by picking a fuzzy set $\mu_0 \in \mathcal{CB}(X)$. We subsequently choose $\mu_1 \subseteq F_1(\mu_0)$ and a positive real number ε_0 such that $\varepsilon_0 \in (\frac{1-k(d_\infty(\mu_0, \mu_1))}{2}, 1 - k(d_\infty(\mu_0, \mu_1)))$. Next, by using this ε_0 , we can find a positive real number β_0 such that $\beta_0 \in (1, \frac{1-\varepsilon_0}{k(d_\infty(\mu_0, \mu_1))})$. Now, by Theorem 2.3, there exists $\mu_2 \in \mathcal{CB}(X)$ such that $\mu_2 \subseteq F_2(\mu_1)$ and

$$d_\infty(\mu_1, \mu_2) \leq \beta_0 d_\infty(F_1(\mu_0), F_2(\mu_1)).$$

Next, let us to choose a positive real number ε_1 such that $\varepsilon_1 \in (\frac{1-k(d_\infty(\mu_1, \mu_2))}{2}, 1 - k(d_\infty(\mu_1, \mu_2)))$, and then pick a positive real number β_1 such that $\beta_1 \in (1, \frac{1-\varepsilon_1}{k(d_\infty(\mu_1, \mu_2))})$. Similarly to the above, by Theorem 2.3, we can find $\mu_3 \in \mathcal{CB}(X)$ such that $\mu_3 \subseteq F_3(\mu_2)$ and

$$d_\infty(\mu_2, \mu_3) \leq \beta_1 d_\infty(F_2(\mu_1), F_3(\mu_2)).$$

By continuing this process, we obtain two sequences of positive real numbers $\{\varepsilon_n\}$, $\{\beta_n\}$ and a sequence $\{\mu_n\}$ in $\mathcal{CB}(X)$ such that

$$\begin{cases} \mu_{n+1} \subseteq F_{n+1}(\mu_n), \\ d_\infty(\mu_{n+1}, \mu_{n+2}) \leq \beta_n d_\infty(F_{n+1}(\mu_n), F_{n+2}(\mu_{n+1})), \\ \varepsilon_n \in (\frac{1-k(d_\infty(\mu_n, \mu_{n+1}))}{2}, 1 - k(d_\infty(\mu_n, \mu_{n+1}))), \\ \beta_n \in (1, \frac{1-\varepsilon_n}{k(d_\infty(\mu_n, \mu_{n+1}))}), \end{cases} \tag{3.2}$$

for each $n \in \mathbb{N}$.

In order to complete the proof, we will divide it into three steps.

Step 1. We show that $\sup_{n \in \mathbb{N}} \beta_n k(d_\infty(\mu_n, \mu_{n+1})) < 1$.

Note that, by Lemma 2.2(iii), we have

$$\rho_\infty(\mu_{n-1}, F_n(\mu_{n-1})) \leq d_\infty(\mu_{n-1}, \mu_n) + \rho_\infty(\mu_n, F_n(\mu_{n-1})),$$

for each $n \in \mathbb{N}$. Subsequently, since $\mu_n \subseteq F_n(\mu_{n-1})$, in view of Lemma 2.2(i) we obtain

$$\rho_\infty(\mu_{n-1}, F_n(\mu_{n-1})) \leq d_\infty(\mu_{n-1}, \mu_n),$$

for each $n \in \mathbb{N}$.

Using this fact, we now derive

$$\begin{aligned}
 M_{(n+1,n+2)}(\mu_n, \mu_{n+1}) &= \max\{d_\infty(\mu_n, \mu_{n+1}), \rho_\infty(\mu_n, F_{n+1}(\mu_n)), \rho_\infty(\mu_{n+1}, F_{n+2}(\mu_{n+1})), \\
 &\quad 1/2[\rho_\infty(\mu_{n+1}, F_{n+1}(\mu_n)) + \rho_\infty(\mu_n, F_{n+2}(\mu_{n+1}))]\} \\
 &\leq \max\{d_\infty(\mu_n, \mu_{n+1}), d_\infty(\mu_n, \mu_{n+1}), d_\infty(\mu_{n+1}, \mu_{n+2}), 1/2[d_\infty(\mu_n, \mu_{n+2}) \\
 &\quad + \rho_\infty(\mu_{n+2}, F_{n+2}(\mu_{n+1}))]\} \\
 &\leq \max\{d_\infty(\mu_n, \mu_{n+1}), d_\infty(\mu_{n+1}, \mu_{n+2}), 1/2[d_\infty(\mu_n, \mu_{n+1}) + d_\infty(\mu_{n+1}, \mu_{n+2})]\} \\
 &= \max\{d_\infty(\mu_n, \mu_{n+1}), d_\infty(\mu_{n+1}, \mu_{n+2})\}.
 \end{aligned}$$

Using this one, in view of the inequalities (3.1) and (3.2), we obtain

$$d_\infty(\mu_{n+1}, \mu_{n+2}) < \beta_n k(d_\infty(\mu_n, \mu_{n+1})) \max\{d_\infty(\mu_n, \mu_{n+1}), d_\infty(\mu_{n+1}, \mu_{n+2})\}. \tag{3.3}$$

Then, by inequality (3.3), we must have

$$\max\{d_\infty(\mu_n, \mu_{n+1}), d_\infty(\mu_{n+1}, \mu_{n+2})\} = d_\infty(\mu_n, \mu_{n+1}). \tag{3.4}$$

Indeed, if (3.4) is not true, we would have

$$d_\infty(\mu_{n+1}, \mu_{n+2}) < \beta_n k(d_\infty(\mu_n, \mu_{n+1})) d_\infty(\mu_{n+1}, \mu_{n+2}),$$

by inequality (3.3). This leads to a contradiction, since $0 < \beta_n k(d_\infty(\mu_n, \mu_{n+1})) < 1 - \varepsilon_n < 1$ for all $n \in \mathbb{N}$. This shows that (3.4) holds. Subsequently, since $\beta_n k(d_\infty(\mu_n, \mu_{n+1})) \in (0, 1)$, we have

$$\begin{aligned}
 d_\infty(\mu_{n+1}, \mu_{n+2}) &< \beta_n k(d_\infty(\mu_n, \mu_{n+1})) d_\infty(\mu_n, \mu_{n+1}) \\
 &< d_\infty(\mu_n, \mu_{n+1}).
 \end{aligned} \tag{3.5}$$

This means the sequence $\{d_\infty(\mu_n, \mu_{n+1})\}$ is strictly decreasing in $[0, \infty)$. Thus, by applying Theorem 2.4, we know that

$$0 \leq \sup_{n \in \mathbb{N}} \varphi(d_\infty(\mu_n, \mu_{n+1})) < 1.$$

This implies

$$\sup_{n \in \mathbb{N}} k(d_\infty(\mu_n, \mu_{n+1})) = \frac{1}{2} \left[1 + \sup_{n \in \mathbb{N}} \varphi(d_\infty(\mu_n, \mu_{n+1})) \right] < 1. \tag{3.6}$$

On the other hand, we observe that

$$\begin{aligned}
 \beta_n k(d_\infty(\mu_n, \mu_{n+1})) &< 1 - \varepsilon_n \\
 &< 1 - \left(\frac{1 - k(d_\infty(\mu_n, \mu_{n+1}))}{2} \right) \\
 &= \frac{1 + k(d_\infty(\mu_n, \mu_{n+1}))}{2}.
 \end{aligned}$$

Using this together with (3.6), we can conclude that

$$\sup_{n \in \mathbb{N}} \beta_n k(d_\infty(\mu_n, \mu_{n+1})) < 1,$$

as required.

Step 2. We show that $\{\mu_n\}$ is a Cauchy sequence in $\mathcal{CB}(X)$.

According to step 1, let us put $c := \sup_{n \in \mathbb{N}} \beta_n k(d_\infty(\mu_n, \mu_{n+1})) \in (0, 1)$. Then, by (3.5), we have

$$\begin{aligned} d_\infty(\mu_{n+1}, \mu_{n+2}) &< \beta_n k(d_\infty(\mu_n, \mu_{n+1})) d_\infty(\mu_n, \mu_{n+1}) \\ &\leq c d_\infty(\mu_n, \mu_{n+1}), \end{aligned}$$

for each $n \in \mathbb{N}$. By using this relation, we deduce that

$$d_\infty(\mu_{n+1}, \mu_{n+2}) < c d_\infty(\mu_n, \mu_{n+1}) < \dots < c^{n+1} d_\infty(\mu_0, \mu_1),$$

for each $n \in \mathbb{N}$.

So, for arbitrary positive integers m and k , we see that

$$\begin{aligned} d_\infty(\mu_k, \mu_{k+m}) &\leq \sum_{i=k}^{k+m-1} d_\infty(\mu_i, \mu_{i+1}) \\ &< \sum_{i=k}^{k+m-1} c^i d_\infty(\mu_0, \mu_1) \\ &\leq \frac{c^k}{1-c} d_\infty(\mu_0, \mu_1). \end{aligned}$$

Since $c \in (0, 1)$, we can conclude that $\{\mu_n\}$ is a Cauchy sequence in $\mathcal{CB}(X)$, as required.

Step 3. We show that there is $\mu_* \in \mathcal{CB}(X)$ such that $\mu_* \subset F_i(\mu_*)$, for all $i \in \mathbb{N}$.

Since $\{\mu_n\}$ is a Cauchy sequence in a complete metric space $(\mathcal{CB}(X), d_\infty)$, there is $\mu_* \in \mathcal{CB}(X)$ such that $\mu_n \rightarrow \mu_*$ as $n \rightarrow \infty$. We now show that $\mu_* \subseteq F_i(\mu_*)$ for all $i \in \mathbb{N}$.

Let $i \in \mathbb{N}$ be arbitrary. Firstly, by Lemma 2.2(iii) and (iv), let us notice that

$$\begin{aligned} \rho_\infty(\mu_*, F_i(\mu_*)) &\leq d_\infty(\mu_*, \mu_j) + \rho_\infty(\mu_j, F_i(\mu_*)) \\ &\leq d_\infty(\mu_*, \mu_j) + d_\infty(F_j(\mu_{j-1}), F_i(\mu_*)), \end{aligned} \tag{3.7}$$

since $\mu_j \subset F_j(\mu_{j-1})$ for arbitrary natural numbers j such that $i \neq j$.

Subsequently, by using (3.7) and Lemma 2.2(iii), we derive

$$\begin{aligned} d_\infty(F_j(\mu_{j-1}), F_i(\mu_*)) &\leq \varphi(d_\infty(\mu_{j-1}, \mu_*)) \max\{d_\infty(\mu_{j-1}, \mu_*), \rho_\infty(\mu_{j-1}, F_j(\mu_{j-1})), \\ &\quad \rho_\infty(\mu_*, F_i(\mu_*)), 1/2[\rho_\infty(\mu_*, F_j(\mu_{j-1})) + \rho_\infty(\mu_{j-1}, F_i(\mu_*))]\} \\ &\leq \varphi(d_\infty(\mu_{j-1}, \mu_*)) \max\{d_\infty(\mu_{j-1}, \mu_*), \rho_\infty(\mu_{j-1}, F_j(\mu_{j-1})), \\ &\quad d_\infty(\mu_*, \mu_j) + d_\infty(F_j(\mu_{j-1}), F_i(\mu_*)), 1/2[d_\infty(\mu_*, \mu_j) \\ &\quad + \rho_\infty(\mu_j, F_j(\mu_{j-1})) + d_\infty(\mu_{j-1}, \mu_*) + \rho_\infty(\mu_*, F_i(\mu_*))]\} \end{aligned}$$

$$\begin{aligned} &\leq \varphi(d_\infty(\mu_{j-1}, \mu_*)) \max\{d_\infty(\mu_{j-1}, \mu_*), \rho_\infty(\mu_{j-1}, F_j(\mu_{j-1})), \\ &\quad d_\infty(\mu_j, \mu_*) + d_\infty(F_j(\mu_{j-1}), F_i(\mu_*)), 1/2[d_\infty(\mu_j, \mu_*) \\ &\quad + d_\infty(\mu_{j-1}, \mu_*) + \rho_\infty(\mu_*, F_i(\mu_*))]\}. \end{aligned} \tag{3.8}$$

We now consider the following two possible cases:

Case 1. If

$$\begin{aligned} &d_\infty(\mu_j, \mu_*) + d_\infty(F_j(\mu_{j-1}), F_i(\mu_*)) \\ &= \max\{d_\infty(\mu_{j-1}, \mu_*), \rho_\infty(\mu_{j-1}, F_j(\mu_{j-1})), d_\infty(\mu_j, \mu_*) \\ &\quad + d_\infty(F_j(\mu_{j-1}), F_i(\mu_*)), 1/2[d_\infty(\mu_j, \mu_*) + d_\infty(\mu_{j-1}, \mu_*) + \rho_\infty(\mu_*, F_i(\mu_*))]\}, \end{aligned}$$

then, by inequality (3.8), we have

$$d_\infty(F_j(\mu_{j-1}), F_i(\mu_*)) \leq \varphi(d_\infty(\mu_{j-1}, \mu_*)) [d_\infty(\mu_j, \mu_*) + d_\infty(F_j(\mu_{j-1}), F_i(\mu_*))].$$

This is equivalent to

$$d_\infty(F_j(\mu_{j-1}), F_i(\mu_*)) \leq \frac{\varphi(d_\infty(\mu_{j-1}, \mu_*))d_\infty(\mu_j, \mu_*)}{1 - \varphi(d_\infty(\mu_{j-1}, \mu_*))}.$$

Using, the previous argument together with inequality (3.7), we have

$$\begin{aligned} \rho_\infty(\mu_*, F_i(\mu_*)) &\leq d_\infty(\mu_*, \mu_j) + \frac{\varphi(d_\infty(\mu_{j-1}, \mu_*))d_\infty(\mu_j, \mu_*)}{1 - \varphi(d_\infty(\mu_{j-1}, \mu_*))} \\ &= \frac{d_\infty(\mu_*, \mu_j)}{1 - \varphi(d_\infty(\mu_{j-1}, \mu_*))}. \end{aligned} \tag{3.9}$$

Note that, since $d_\infty(\mu_j, \mu_*) \rightarrow 0$ as $j \rightarrow \infty$, without loss of generality (passing to a subsequence if necessary), we may assume that $\{d_\infty(\mu_j, \mu_*)\}_{j=1}^\infty$ is a nonincreasing sequence. Subsequently, by Theorem 2.4, we have

$$0 \leq \sup_{j \in \mathbb{N}} \varphi(d_\infty(\mu_j, \mu_*)) < 1.$$

Subsequently, by inequality (3.9), we obtain

$$\begin{aligned} \rho_\infty(\mu_*, F_i(\mu_*)) &\leq \frac{d_\infty(\mu_j, \mu_*)}{1 - \varphi(d_\infty(\mu_{j-1}, \mu_*))} \\ &\leq \frac{d_\infty(\mu_j, \mu_*)}{1 - \sup_{j \in \mathbb{N}} \varphi(d_\infty(\mu_{j-1}, \mu_*))}. \end{aligned} \tag{3.10}$$

Letting $j \rightarrow \infty$ in inequality (3.10), we have

$$\rho_\infty(\mu_*, F_i(\mu_*)) \leq 0.$$

This implies, by Lemma 2.2(i), that $\mu_* \subseteq F_i(\mu_*)$.

Case 2. Assume

$$\begin{aligned}
 & d_\infty(\mu_j, \mu_*) + d_\infty(F_j(\mu_{j-1}), F_i(\mu_*)) \\
 & \neq \max\{d_\infty(\mu_{j-1}, \mu_*), \rho_\infty(\mu_{j-1}, F_j(\mu_{j-1})), d_\infty(\mu_j, \mu_*) \\
 & \quad + d_\infty(F_j(\mu_{j-1}), F_i(\mu_*)), 1/2[d_\infty(\mu_j, \mu_*) + d_\infty(\mu_{j-1}, \mu_*) \\
 & \quad + \rho_\infty(\mu_*, F_i(\mu_*))]\}.
 \end{aligned}$$

In this situation, by inequality (3.8), we have

$$\begin{aligned}
 & d_\infty(F_j(\mu_{j-1}), F_i(\mu_*)) \\
 & \leq \varphi(d_\infty(\mu_{j-1}, \mu_*)) \max\{d_\infty(\mu_{j-1}, \mu_*), \rho_\infty(\mu_{j-1}, F_j(\mu_{j-1})), \\
 & \quad 1/2[d_\infty(\mu_j, \mu_*) + d_\infty(\mu_{j-1}, \mu_*) + \rho_\infty(\mu_*, F_i(\mu_*))]\} \\
 & \leq \varphi(d_\infty(\mu_{j-1}, \mu_*)) \max\{d_\infty(\mu_{j-1}, \mu_*), d_\infty(\mu_{j-1}, \mu_j), 1/2[d_\infty(\mu_j, \mu_*) \\
 & \quad + d_\infty(\mu_{j-1}, \mu_*) + \rho_\infty(\mu_*, F_i(\mu_*))]\}.
 \end{aligned}$$

So, it follows by inequality (3.7) that

$$\begin{aligned}
 \rho_\infty(\mu_*, F_i(\mu_*)) & \leq d_\infty(\mu_*, \mu_j) + \varphi(d_\infty(\mu_{j-1}, \mu_*)) \max\{d_\infty(\mu_{j-1}, \mu_*), d_\infty(\mu_{j-1}, \mu_j), \\
 & \quad 1/2[d_\infty(\mu_j, \mu_*) + d_\infty(\mu_{j-1}, \mu_*) + \rho_\infty(\mu_*, F_i(\mu_*))]\} \\
 & \leq d_\infty(\mu_*, \mu_j) + \sup_{j \in \mathbb{N}} \varphi(d_\infty(\mu_{j-1}, \mu_*)) \max\{d_\infty(\mu_{j-1}, \mu_*), d_\infty(\mu_{j-1}, \mu_j), \\
 & \quad 1/2[d_\infty(\mu_j, \mu_*) + d_\infty(\mu_{j-1}, \mu_*) + \rho_\infty(\mu_*, F_i(\mu_*))]\} \\
 & < d_\infty(\mu_*, \mu_j) + \max\{d_\infty(\mu_{j-1}, \mu_*), d_\infty(\mu_{j-1}, \mu_j), \\
 & \quad 1/2[d_\infty(\mu_j, \mu_*) + d_\infty(\mu_{j-1}, \mu_*) + \rho_\infty(\mu_*, F_i(\mu_*))]\}. \tag{3.11}
 \end{aligned}$$

Letting $j \rightarrow \infty$ in inequality (3.11), we obtain

$$\rho_\infty(\mu_*, F_i(\mu_*)) \leq \frac{\rho_\infty(\mu_*, F_i(\mu_*))}{2},$$

which implies that $\rho_\infty(\mu_*, F_i(\mu_*)) = 0$. Again, by Lemma 2.2(i), it follows that $\mu_* \subseteq F_i(\mu_*)$. Hence, by Cases 1 and 2, the proof is completed. □

Remark 3.2 Theorem 3.1 recovers Theorem 1.1 as a special case. Meanwhile, it improves Theorem 1.2 since we are considering a larger class of metric space settings.

Using Theorem 3.1, we also obtain the following result.

Corollary 3.3 *Let (X, d) be a complete metric space and let $\{F_i\}_{i=1}^\infty$ be a sequence of fuzzy self-mappings of $CB(X)$. If there exist nonnegative constants $a, b, c, a + 2b + 2c < 1$, such*

that for each $\mu_1, \mu_2 \in \mathcal{CB}(X)$, and for arbitrary positive integers i and $j, i \neq j$,

$$d_\infty(F_i(\mu_1), F_j(\mu_2)) \leq ad_\infty(\mu_1, \mu_2) + b\{\rho_\infty(\mu_1, F_i(\mu_1)) + \rho_\infty(\mu_2, F_j(\mu_2))\} + c\{\rho_\infty(\mu_1, F_i(\mu_2)) + \rho_\infty(\mu_2, F_i(\mu_1))\}, \tag{3.12}$$

then there exists a $\mu_* \in \mathcal{CB}(X)$ such that $\mu_* \subseteq F_i(\mu_*)$ for all $i \in \mathbb{N}$.

Proof Let us define an \mathcal{R} function $\varphi : [0, \infty) \rightarrow [0, 1)$ by

$$\varphi(t) = a + 2b + 2c \quad \text{for all } t \in [0, \infty).$$

Then one can derive that the relation (3.12) is transformed to (3.1), and so the required result follows immediately from Theorem 3.1. □

Next, motivated by the idea of Berinde [18], we now present another fuzzy fixed point theorem.

Theorem 3.4 *Let (X, d) be a complete metric space and let $\{F_i\}_{i=1}^\infty$ be a sequence of fuzzy self-mappings of $\mathcal{CB}(X)$. If there exist an \mathcal{R} -function $\varphi : [0, \infty) \rightarrow [0, 1)$ and $L \geq 0$ such that for each $\mu_1, \mu_2 \in \mathcal{CB}(X)$, and for arbitrary positive integers i and $j, i \neq j$,*

$$d_\infty(F_i(\mu_1), F_j(\mu_2)) \leq \varphi(d_\infty(\mu_1, \mu_2)) \frac{[\rho_\infty(\mu_1, F_i(\mu_1)) + \rho_\infty(\mu_2, F_j(\mu_2))]}{2} + L\rho_\infty(\mu_2, F_i(\mu_1)), \tag{3.13}$$

then there exists a $\mu_* \in \mathcal{CB}(X)$ such that $\mu_* \subseteq F_i(\mu_*)$ for all $i \in \mathbb{N}$.

Proof Let us construct and consider again a function k and the sequences which we have defined in (3.2). Recall that, by our constructive method, we also know that

$$\rho_\infty(\mu_{n-1}, F_n(\mu_{n-1})) \leq d_\infty(\mu_{n-1}, \mu_n),$$

for each $n \in \mathbb{N}$. Using this together with (3.13), we derive

$$\begin{aligned} d_\infty(\mu_{n+1}, \mu_{n+2}) &\leq \beta_n d_\infty(F_{n+1}(\mu_n), F_{n+2}(\mu_{n+1})) \\ &\leq \beta_n \varphi(d_\infty(\mu_n, \mu_{n+1})) \frac{[\rho_\infty(\mu_n, F_{n+1}(\mu_n)) + \rho_\infty(\mu_{n+1}, F_{n+2}(\mu_{n+1}))]}{2} \\ &\quad + \beta_n L\rho_\infty(\mu_{n+1}, F_{n+1}(\mu_n)) \\ &\leq \beta_n \varphi(d_\infty(\mu_n, \mu_{n+1})) \frac{[d_\infty(\mu_n, \mu_{n+1}) + d_\infty(\mu_{n+1}, \mu_{n+2})]}{2} \\ &< \beta_n k(d_\infty(\mu_n, \mu_{n+1})) \frac{[d_\infty(\mu_n, \mu_{n+1}) + d_\infty(\mu_{n+1}, \mu_{n+2})]}{2}, \end{aligned}$$

for each $n \in \mathbb{N}$. This is equivalent to

$$d_\infty(\mu_{n+1}, \mu_{n+2}) < \frac{\beta_n k(d_\infty(\mu_n, \mu_{n+1}))}{2 - \beta_n k(d_\infty(\mu_n, \mu_{n+1}))} d_\infty(\mu_n, \mu_{n+1}), \tag{3.14}$$

for each $n \in \mathbb{N}$. Observe that, since $\beta_n k(d_\infty(\mu_n, \mu_{n+1})) \in (0, 1)$, we also have

$$\frac{\beta_n k(d_\infty(\mu_n, \mu_{n+1}))}{2 - \beta_n k(d_\infty(\mu_n, \mu_{n+1}))} \in (0, 1),$$

for all $n \in \mathbb{N}$. Thus, by (3.14), we see that $\{d_\infty(\mu_n, \mu_{n+1})\}$ is a strictly decreasing sequence in $[0, \infty)$. Since φ is an \mathcal{R} -function, by applying Theorem 2.4, we have

$$0 \leq \sup_{n \in \mathbb{N}} \varphi(d_\infty(\mu_n, \mu_{n+1})) < 1.$$

Next, as we have done in order to prove Theorem 3.1, we write

$$\lambda := \sup_{n \in \mathbb{N}} \beta_n k(d_\infty(\mu_n, \mu_{n+1})) < 1.$$

Let us take $c := \frac{\lambda}{2-\lambda}$. It follows that $c \in (0, 1)$ and, by (3.14), we also have

$$\begin{aligned} d_\infty(\mu_{n+1}, \mu_{n+2}) &< \frac{\beta_n k(d_\infty(\mu_n, \mu_{n+1}))}{2 - \beta_n k(d_\infty(\mu_n, \mu_{n+1}))} d_\infty(\mu_n, \mu_{n+1}) \\ &\leq c d_\infty(\mu_n, \mu_{n+1}). \end{aligned} \tag{3.15}$$

Now, again by reasoning along the lines of proving Theorem 3.1, we can show that there is a $\mu_* \in \mathcal{CB}(X)$ which is the limit point of the considered sequence $\{\mu_n\}$, under the supremum metric d_∞ , and it satisfies

$$0 \leq \sup_{j \in \mathbb{N}} \varphi(d_\infty(\mu_j, \mu_*)) < 1. \tag{3.16}$$

Finally, we will show that $\mu_* \subseteq F_i(\mu_*)$ for all $i \in \mathbb{N}$. Let $i \in \mathbb{N}$ be arbitrary. Note again that

$$\begin{aligned} \rho_\infty(\mu_*, F_i(\mu_*)) &\leq d_\infty(\mu_*, \mu_j) + \rho_\infty(\mu_j, F_i(\mu_*)) \\ &\leq d_\infty(\mu_*, \mu_j) + d_\infty(F_j(\mu_{j-1}), F_i(\mu_*)), \end{aligned} \tag{3.17}$$

for each $j \in \mathbb{N}$ such that $j \neq i$. Consider

$$\begin{aligned} &d_\infty(F_j(\mu_{j-1}), F_i(\mu_*)) \\ &\leq \varphi(d_\infty(\mu_{j-1}, \mu_*)) \frac{[\rho_\infty(\mu_{j-1}, F_j(\mu_{j-1})) + \rho_\infty(\mu_*, F_i(\mu_*))]}{2} + L\rho_\infty(\mu_*, F_j(\mu_{j-1})) \\ &\leq \varphi(d_\infty(\mu_{j-1}, \mu_*)) \frac{d_\infty(\mu_{j-1}, \mu_j)}{2} + \varphi(d_\infty(\mu_{j-1}, \mu_*)) \frac{\rho_\infty(\mu_*, F_i(\mu_*))}{2} \\ &\quad + L(d_\infty(\mu_*, \mu_j) + \rho_\infty(\mu_j, F_j(\mu_{j-1}))) \\ &\leq \varphi(d_\infty(\mu_{j-1}, \mu_*)) \frac{d_\infty(\mu_{j-1}, \mu_j)}{2} + \varphi(d_\infty(\mu_{j-1}, \mu_*)) \frac{\rho_\infty(\mu_*, F_i(\mu_*))}{2} \\ &\quad + Ld_\infty(\mu_*, \mu_j) + L\rho_\infty(\mu_j, F_j(\mu_{j-1})), \end{aligned}$$

where $j \in \mathbb{N}$ with $j \neq i$. Subsequently, from (3.17) together with (3.16), we obtain

$$\begin{aligned}
 \rho_\infty(\mu_*, F_i(\mu_*)) &\leq d_\infty(\mu_*, \mu_j) + \varphi(d_\infty(\mu_{j-1}, \mu_*)) \frac{d_\infty(\mu_{j-1}, \mu_j)}{2} \\
 &\quad + \varphi(d_\infty(\mu_{j-1}, \mu_*)) \frac{\rho_\infty(\mu_*, F_i(\mu_*))}{2} \\
 &\quad + Ld_\infty(\mu_*, \mu_j) + L\rho_\infty(\mu_j, F_j(\mu_{j-1})) \\
 &\leq (1 + L)d_\infty(\mu_*, \mu_j) + \sup_{j \in \mathbb{N}} \varphi(d_\infty(\mu_j, \mu_*)) \frac{d_\infty(\mu_{j-1}, \mu_j)}{2} \\
 &\quad + \sup_{j \in \mathbb{N}} \varphi(d_\infty(\mu_{j-1}, \mu_*)) \frac{\rho_\infty(\mu_*, F_i(\mu_*))}{2} \\
 &< (1 + L)d_\infty(\mu_*, \mu_j) + \frac{d_\infty(\mu_{j-1}, \mu_j)}{2} + \frac{\rho_\infty(\mu_*, F_i(\mu_*))}{2}. \tag{3.18}
 \end{aligned}$$

By letting $j \rightarrow \infty$ on the right of inequality (3.18), we have

$$\rho_\infty(\mu_*, F_i(\mu_*)) \leq \frac{\rho_\infty(\mu_*, F_i(\mu_*))}{2},$$

which implies that $\rho_\infty(\mu_*, F_i(\mu_*)) = 0$. Then, by Lemma 2.2(i), we have $\mu_* \subseteq F_i(\mu_*)$. Since $i \in \mathbb{N}$ is arbitrary, we complete the proof. □

The following result can be deduced from Theorem 3.4.

Corollary 3.5 *Let (X, d) be a complete metric space and let $\{F_i\}_{i=1}^\infty$ be a sequence of fuzzy self-mappings of $\mathcal{CB}(X)$. If there exist an \mathcal{R} -function $\varphi : [0, \infty) \rightarrow [0, 1)$ and $L \geq 0$ such that for each $\mu_1, \mu_2 \in \mathcal{CB}(X)$, and for arbitrary positive integers i and j , $i \neq j$,*

$$\begin{aligned}
 d_\infty(F_i(\mu_1), F_j(\mu_2)) &\leq \varphi(d_\infty(\mu_1, \mu_2)) [\rho_\infty(\mu_1, F_i(\mu_1)) \rho_\infty(\mu_2, F_j(\mu_2))]^{1/2} \\
 &\quad + L\rho_\infty(\mu_2, F_i(\mu_1)), \tag{3.19}
 \end{aligned}$$

then there exists a $\mu_* \in \mathcal{CB}(X)$ such that $\mu_* \subseteq F_i(\mu_*)$ for all $i \in \mathbb{N}$.

Proof Since for each $a, b \in \mathbb{R}^+$, we have $\sqrt{ab} \leq \frac{a+b}{2}$, the required result follows immediately from our Theorem 3.4. □

Remark 3.6 Corollary 3.5 recovers a result which has presented in [12], when $L = 0$.

4 Conclusions

In this paper we have presented fuzzy fixed point theorems on the space of fuzzy sets under a kind of supremum metric setting. We would like to point out that this kind of space is very general and interesting. Moreover, as one can observe, a fixed point in this situation is, in fact, a fixed (fuzzy) set. This means that our presented results are very general and we recover many existing results on fixed point theory as regards both single-valued and set-valued mappings.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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