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New result on fixed point theorems for φ-contractions in Menger spaces

Huichun Hua[1], Minjiang Chen[2] and Shenghua Wang[1*]

*Correspondence: sheng-huawang@hotmail.com
[1] Department of Mathematics and Physics, North China Electric Power University, Baoding, 071003, China
Full list of author information is available at the end of the article

Abstract

Very recently, Fang (Fuzzy Sets Syst. 267:86-99, 2015) gave some fixed point theorems for probabilistic φ-contractions in Menger spaces. Fang's results improve the one of Jachymski (Nonlinear Anal. 73:2199-2203, 2010) by relaxing the restriction on the gauge function φ. In this paper, inspired by the results of Fang, we prove a new fixed point theorem for a probabilistic φ-contraction in Menger spaces in which a weaker condition on the function φ is required. Our result improves the corresponding one of Fang and some others. Finally, an example is given to illustrate our result.

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1 Introduction

Let (X, F, Δ) be a probabilistic metric space and $T : X \to X$ be a mapping. If there exists a gauge function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$F_{Tx,Ty}\big(\varphi(t)\big) \ge F_{x,y}(t) \quad \text{for all } x, y \in X \text{ and } t > 0,$$

then the mapping T is called a probabilistic φ-contraction. The probabilistic φ-contraction is a generalization of probabilistic k-contraction given by Sehgal and Bharucha-Reid [1]. In literature, many authors investigated fixed point theorems for probabilistic φ-contractions in Menger spaces; see [2–7]. On the fixed point theorems for other types of contractions in Menger or fuzzy metric spaces, please see [8–12]. Recently, Jachymski [13] proved a new fixed point theorem for a probabilistic φ-contraction in which the condition on the function φ is weakened. More precisely, the author gave the following result.

Theorem 1.1 ([13]) *Let (X, F, Δ) be a complete Menger probabilistic metric space with a continuous t-norm Δ of H-type, and let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a function satisfying conditions*:

$$0 < \varphi(t) < t \quad \textit{and} \quad \lim_{n\to\infty} \varphi^n(t) = 0 \quad \textit{for all } t > 0.$$

If $T : X \to X$ is a probabilistic φ-contraction, then T has a unique fixed point $x^* \in X$, and $\{T^n x_0\}$ converges to x^* for each $x_0 \in X$.

Although Theorem 1.1 has been a very perfect result in which the condition on the gauge function φ is very simple, Fang [14] improves Theorem 1.1 by giving a new condition on φ recently. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a function satisfying the following condition:

$$\text{for each } t > 0 \text{ there exists } r \geq t \text{ such that } \lim_{n\to\infty} \varphi^n(t) = 0. \tag{1.1}$$

Let $\mathbf{\Phi_w}$ denote the set of all functions $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the condition (1.1) and let $\mathbf{\Phi}$ denote the set of all functions $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the condition that $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t > 0$. In [14], Fang gave an example of $\varphi \in \mathbf{\Phi_w}$ but $\varphi \notin \mathbf{\Phi}$.

By using the condition (1.1), Fang gave the following result.

Theorem 1.2 ([14]) *Let (X, F, Δ) be a complete Menger space with a t-norm Δ of H-type. If $T : X \to X$ is a probabilistic φ-contraction, where $\varphi \in \mathbf{\Phi_w}$, then T has a unique fixed point $x^* \in X$, and $\{T^n x_0\}$ converges to x^* for each $x_0 \in X$.*

Since the condition (1.1) is weaker than the one in Theorem 1.1, Theorem 1.2 improves Theorem 1.1. In [14], Fang asked the following question:

Can the condition (1.1) *in Theorem* 1.2 *be replaced by a more weak condition*?

In this paper, we give a positive answer to the question of Fang by proving a new fixed point theorem for a probabilistic φ-contraction in Menger spaces. In our result, the function φ is required to satisfy a more weak condition than (1.1) and the t-norm is not required to be of H-type. Our result improves the corresponding one of Fang [14] and some others. Finally, an example is given to illustrate our result.

2 Preliminaries

In the rest of this paper, let $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$ and $\mathbb{N}$ denote the set of all natural numbers.

A mapping $F : \mathbb{R} \to [0,1]$ is called a distribution function if it is non-decreasing and left-continuous with $\inf_{t\in\mathbb{R}} F(t) = 0$. If in addition $F(0) = 0$, then F is called a distance distribution function. A distance distribution function F satisfying $\lim_{t\to\infty} F(t) = 1$ is called a Menger distance distribution function.

The set of all Menger distance distribution functions is denoted by $\mathcal{D}^+$. It is known that $\mathcal{D}^+$ is partially ordered by the usual pointwise ordering of functions, that is, $F \leq G$ if and only if $F(t) \leq G(t)$ for all $t \geq 0$. The maximal element in $\mathcal{D}^+$ on this order is the distance distribution function ϵ_0 defined by

$$\epsilon_0(t) = \begin{cases} 0, & t = 0, \\ 1, & t > 0. \end{cases}$$

Definition 2.1 ([15]) A binary operation $\Delta : [0,1] \times [0,1] \to [0,1]$ is a t-norm if Δ satisfies the following conditions:

(1) Δ is associative and commutative;
(2) $\Delta(a, 1) = a$ for all $a \in [0,1]$;
(3) $\Delta(a,b) \leq \Delta(c,d)$ whenever $a \leq c$ and $b \leq d$ for all $a,b,c,d \in [0,1]$.

Two typical examples of the continuous t-norm are $\Delta_P(a,b) = ab$ and $\Delta_M(a,b) = \min\{a,b\}$ for all $a,b \in [0,1]$.

Definition 2.2 ([16]) A t-norm Δ is said to be of Hadžić-type (for short H-type) if the family of functions $\{\Delta^m(t)\}_{m=1}^{\infty}$ is equicontinuous at $t = 1$, where

$$\Delta^1(t) = \Delta(t,t), \qquad \Delta^{m+1}(t) = \Delta\big(t, \Delta^m(t)\big), \quad m = 1,2,\dots, t \in [0,1].$$

It is easy to see that Δ_M is a t-norm of H-type but Δ_P is not of H-type. Here we give a new t-norm of H-type by Δ_M and Δ_P.

Example 2.1 Let $\Delta(x,1) = \Delta(1,x) = x$ for all $x \in [0,1]$, $\Delta(x,y) = \Delta_P(x,y)$ for all $x,y \in [0,1]$ with $\max\{x,y\} \in [0,\frac{1}{2}]$ and $\Delta(x,y) = \Delta_M(x,y)$ for all $x,y \in [0,1]$ with $\max\{x,y\} \in (\frac{1}{2},1]$. It is easy to check that Δ is a t-norm. Now we show that it is of H-type. For any given $\epsilon \in (0,\frac{1}{2})$, set $\delta = \epsilon$. Then $1-\delta = 1-\epsilon > \frac{1}{2}$. Thus, for all $t \in (1-\delta,1)$, one has $\Delta^n(t) = t > 1-\delta = 1-\epsilon$ for all $n \in \mathbb{N}$. For $\epsilon \in [\frac{1}{2},1)$, taking $\delta \in (0,\frac{1}{2})$ arbitrarily, then we have $1-\delta > \frac{1}{2} \geq 1-\epsilon$. Thus for all $t \in (1-\delta,1)$, $\Delta^n(t) = t > 1-\delta > \frac{1}{2} \geq 1-\epsilon$ for all $n \in \mathbb{N}$. Therefore, Δ is a t-norm of H-type.

Example 2.2 Let $\delta \in (0,1]$ and let Δ be a t-norm. Define Δ_δ by $\Delta_\delta(x,y) = \Delta(x,y)$, if $\max\{x,y\} \leq 1-\delta$, and $\Delta_\delta(x,y) = \min\{x,y\}$, if $\max\{x,y\} > 1-\delta$. then Δ_δ is a t-norm of H-type; see [17]. However, if $\Delta_\delta(x,1) = \Delta_\delta(1,x) = x$ for all $x \in [0,1]$, $\Delta_\delta(x,y) = \delta$ for all $x,y \in [\delta,1)$ and $\Delta_\delta(x,y) = 0$ for all $x,y \in [0,1]$ with $\min\{x,y\} \in [0,\delta)$, then Δ_δ is a t-norm but not of H-type.

For other t-norms of H-type, the reader may refer to [16].

Definition 2.3 ([18]) A triple (X,F,Δ) is called a Menger probabilistic metric space (for short, Menger space) if X is a nonempty set, Δ is a t-norm, and F is a mapping from $X \times X \to \mathcal{D}^+$ satisfying the following conditions (for $x,y \in X$, denote $F(x,y)$ by $F_{x,y}$):

(PM-1) $F_{x,y}(t) = \epsilon_0(t)$ for all $t \in \mathbb{R}$ if and only if $x = y$;
(PM-2) $F_{x,y}(t) = F_{y,x}(t)$ for all $t \in \mathbb{R}$;
(PM-3) $F_{x,y}(t+s) \geq \Delta(F_{x,z}(t), F_{z,y}(s))$ for all $x,y,z \in X$ and $t,s > 0$.

Definition 2.4 ([15]) Let (X,F,Δ) be a Menger space and $\{x_n\}$ be a sequence in X. The sequence $\{x_n\}$ is said to be convergent to $x \in X$ if $\lim_{n\to\infty} F_{x_n,x}(t) = 1$ for all $t > 0$; the sequence $\{x_n\}$ is said to be a Cauchy sequence if for any given $t > 0$ and $\epsilon \in (0,1)$, there exists $N_{\epsilon,t} \in \mathbb{N}$ such that $F_{x_n,x_m}(t) > 1-\epsilon$ whenever $m,n > N_{t,\epsilon}$; the Menger space (X,F,Δ) is said to be complete, if each Cauchy sequence in X is convergent to some point in X.

3 Main results

In this section, let $\mathbf{\Phi}_{\mathbf{w}^*}$ denote the set of all functions $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following condition:

$$\begin{aligned} &\text{for each } t_1,t_2 > 0 \text{ there exists } r \geq \max\{t_1,t_2\} \text{ and } N \in \mathbb{N} \\ &\quad \text{such that } \varphi^n(r) < \min\{t_1,t_2\} \text{ for all } n > N. \end{aligned} \tag{3.1}$$

Obviously, the condition (3.1) implies that

$$\begin{aligned} &\text{for each } t > 0 \text{ there exists } r \geq t \text{ and } N \in \mathbb{N} \\ &\quad \text{such that } \varphi^n(r) < t \text{ for all } n > N. \end{aligned} \tag{3.2}$$

It is easy to see that for each $\varphi \in \mathbf{\Phi_w}$, $\varphi \in \mathbf{\Phi_{w^*}}$. In fact, if $\varphi \in \mathbf{\Phi_w}$, then for each $t_1, t_2 > 0$, there exist $r_1 \geq t_1$ and $r_2 \geq t_2$ such that $\lim_{n\to\infty} \varphi^n(r_1) = \lim_{n\to\infty} \varphi^n(r_2) = 0$. Assume that $t_1 \leq t_2$. Then there exists $N \in \mathbb{N}$ such that $\varphi^n(r_2) < t_1$ for all $n > N$. Thus $\varphi \in \mathbf{\Phi_{w^*}}$.

However, if $\varphi \in \mathbf{\Phi_{w^*}}$, then it is unnecessary that $\varphi \in \mathbf{\Phi_w}$.

Example 3.1 Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\varphi(t) = t$ for all $t \in [0,1]$, $\varphi(t) = t - 1$ for all $t \in (1,\infty)$. Then $\varphi \in \mathbf{\Phi_{w^*}}$. In fact, for each $t_1, t_2 \in (0,\infty)$, there exists $N \in \mathbb{N}$ such that $r = 1 + N + \epsilon > \max\{t_1, t_2\}$, where $\epsilon \in (0, \min\{t_1, t_2, 1\})$. Then we have $\varphi^n(r) = \epsilon < \min\{t_1, t_2\}$ for all $n > N+1$. So $\varphi \in \mathbf{\Phi_{w^*}}$. However, since $\lim_{n\to\infty} \varphi^n(r) \neq 0$ for all $r > 0$, $\varphi \notin \mathbf{\Phi_w}$.

From Example 3.1 we see that $\mathbf{\Phi_{w^*}}$ is a proper subclass of $\mathbf{\Phi_w}$. On $\mathbf{\Phi_{w^*}}$, $\mathbf{\Phi_w}$, and $\mathbf{\Phi}$, we have $\mathbf{\Phi} \subset \mathbf{\Phi_w} \subset \mathbf{\Phi_{w^*}}$.

Lemma 3.1 *Let $\varphi \in \mathbf{\Phi_{w^*}}$. Then for each $t > 0$, there exists $r \geq t$ such that $\varphi(r) < t$.*

Proof Suppose that there is $t_0 > 0$ such that $\varphi(r) \geq t_0$ for all $r \geq t_0$. By induction, we obtain $\varphi^n(r) \geq t_0$ for all $n \in \mathbb{N}$. From (3.2) it follows that there exist $r \geq t_0$ and $N \in \mathbb{N}$ such that $\varphi^n(r) < t_0$ for all $n > N$, which contradicts $\varphi^n(r) \geq t_0$ for all $r \geq t_0$ and $n \in \mathbb{N}$. Thus for each $t > 0$, there exists $r \geq t$ such that $\varphi(r) \leq t$. This completes the proof. □

Lemma 3.2 *Let (X, F, Δ) be a Menger space and $x, y \in X$. If there exists a function $\varphi \in \mathbf{\Phi_{w^*}}$ such that*

$$F_{x,y}\bigl(\varphi(t)\bigr) \geq F_{x,y}(t), \quad \forall t > 0, \tag{3.3}$$

then $x = y$.

Proof First by a similar proof with Lemma 2.2 of [14] we can show that for all $n \in \mathbb{N}$ and $t > 0$, one has $\varphi^n(t) > 0$. By induction, from (3.3) it follows that

$$F_{x,y}\bigl(\varphi^n(t)\bigr) \geq F_{x,y}(t) \quad \text{for all } n \in \mathbb{N} \text{ and } t > 0. \tag{3.4}$$

Next we show that $F_{x,y}(t) = 1$ for all $t > 0$. In fact, if there exists $t_0 > 0$ such that $F_{x,y}(t_0) < 1$, then since $\lim_{t\to\infty} F_{x,y}(t) = 1$ there is $t_1 > t_0$ such that

$$F_{x,y}(t) > F_{x,y}(t_0) \quad \text{for all } t \geq t_1. \tag{3.5}$$

Since $\varphi \in \mathbf{\Phi_{w^*}}$, there exist $t_2 \geq \max\{t_1, t_0\}$ and $N \in \mathbb{N}$ such that $\varphi^n(t_2) < \min\{t_0, t_1\}$ for all $n > N$. By the monotonicity of $F_{x,y}(\cdot)$, from (3.4) and (3.5) it follows that, for each $n > N$,

$$F_{x,y}(t_0) \geq F_{x,y}\bigl(\varphi^n(t_2)\bigr) \geq F_{x,y}(t_2) \geq F_{x,y}(t_1) > F_{x,y}(t_0).$$

It is a contradiction. Therefore, $F_{x,y}(t) = 1$ for all $t > 0$, *i.e.*, $x = y$. This completes the proof. □

Lemma 3.3 *Let (X, F, Δ) be a Menger space where Δ is continuous at $(1,1)$ and let $\{x_n\}$ be a sequence in X. Suppose that there exists a function $\varphi \in \mathbf{\Phi_{w^*}}$ satisfying the following conditions:*

(1) $\varphi(t) > 0$ *for all* $t > 0$;
(2) $F_{x_n,x_m}(\varphi(t)) \geq F_{x_{n-1},x_{m-1}}(t)$ *for all* $n, m \in \mathbb{N}$ *and* $t > 0$.
Then $\lim_{n\to\infty} F_{x_n,x_{n+k}}(t) = 1$ *for all* $k \in \mathbb{N}$ *and* $t > 0$.

Proof It is easy to see that the condition (1) implies that $\varphi^n(t) > 0$ for all $t > 0$ and the condition (2) implies that

$$F_{x_n,x_{n+1}}\bigl(\varphi^n(t)\bigr) \geq F_{x_0,x_1}(t), \quad \forall n \in \mathbb{N} \text{ and } \forall t > 0. \tag{3.6}$$

We first prove that

$$\lim_{n\to\infty} F_{x_n,x_{n+1}}(t) = 1, \quad \forall t > 0. \tag{3.7}$$

Since $\lim_{t\to\infty} F_{x_0,x_1}(t) = 1$, for any $\epsilon \in (0,1)$, there exists $t_0 > 0$ such that $F_{x_0,x_1}(t_0) > 1-\epsilon$. For each $t > 0$, since $\varphi \in \mathbf{\Phi}_{\mathbf{w}^*}$, there exist $t_1 \geq \max\{t, t_0\}$ and $N \in \mathbb{N}$ such that $\varphi^n(t_1) < \min\{t, t_0\}$ for all $n \geq N$. By the monotonicity of $F_{x,y}(\cdot)$, from (3.6) we have

$$\begin{aligned} F_{x_n,x_{n+1}}(t) &\geq F_{x_n,x_{n+1}}\bigl(\varphi^n(t_1)\bigr) \\ &\geq F_{x_0,x_1}(t_1) \geq F_{x_0,x_n}(t_0) \\ &> 1-\epsilon \quad \text{for all } n \geq N, \end{aligned}$$

which implies that (3.7) holds. Assume that $\lim_{n\to\infty} F_{x_n,x_{n+k}}(t) = 1$ for each $k \in \mathbb{N}$ and $t > 0$. Since Δ is continuous at $(1,1)$, we have

$$F_{x_n,x_{n+k+1}}(t) \geq \Delta\bigl(F_{x_n,x_{n+k}}(t/2), F_{x_{n+k},x_{n+k+1}}(t/2)\bigr) \to \Delta(1,1) = 1 \quad \text{as } n \to \infty.$$

By induction we conclude that

$$\lim_{n\to\infty} F_{x_n,x_{n+k}}(t) = 1, \quad \forall k \in \mathbb{N} \text{ and } \forall t > 0.$$

This completes the proof. □

Lemma 3.4 *Let* (X, F, Δ) *be a Menger space where* Δ *is of H-type and continuous at* $(1,1)$ *and let* $\{x_n\}$ *be a sequence in* X. *Suppose that there exists a function* $\varphi \in \mathbf{\Phi}_{\mathbf{w}^*}$ *satisfying the conditions* (1) *and* (2) *in Lemma* 3.3. *Then* $\{x_n\}$ *is a Cauchy sequence.*

Proof Let $t > 0$. By Lemma 3.1 there is $r \geq t$ such that $\varphi(r) < t$. We show by induction that

$$F_{x_n,x_{n+k}}(t) \geq \Delta^k\bigl(F_{x_n,x_{n+1}}\bigl(t-\varphi(r)\bigr)\bigr), \quad \forall k \in \mathbb{N}. \tag{3.8}$$

Obviously, (3.8) holds for $k = 1$. Assume that (3.8) holds for some $k \in \mathbb{N}$. By (2) in Lemma 3.3 we have

$$\begin{aligned} F_{x_n,x_{n+k+1}}(t) &\geq \Delta\bigl(F_{x_n,x_{n+1}}\bigl(t-\varphi(r)\bigr), F_{x_{n+1},x_{n+k+1}}\bigl(\varphi(r)\bigr)\bigr) \\ &\geq \Delta\bigl(F_{x_n,x_{n+1}}\bigl(t-\varphi(r)\bigr), F_{x_n,x_{n+k}}(r)\bigr) \end{aligned}$$

$$\begin{aligned}
&\geq \Delta\big(F_{x_n,x_{n+1}}\big(t-\varphi(r)\big), F_{x_n,x_{n+k}}(t)\big) \\
&\geq \Delta\big(F_{x_n,x_{n+1}}\big(t-\varphi(r)\big), \Delta^k\big(F_{x_n,x_{n+1}}\big(t-\varphi(r)\big)\big)\big) \\
&= \Delta^{k+1}\big(F_{x_n,x_{n+1}}\big(t-\varphi(r)\big)\big).
\end{aligned}$$

It follows that (3.8) holds for $k+1$. So (3.8) holds for all $k \in \mathbb{N}$.

Let $t > 0$. Define $a_n = \inf_{k\geq 1} F_{x_n,x_{n+k}}(t)$. Since $\varphi \in \mathbf{\Phi}_{\mathbf{w}^*}$, by Lemma 3.1 there exists $t_0 \geq t$ such that $\varphi(t_0) < t$. So by the condition (2) we have

$$\begin{aligned}
a_n &= \inf_{k\geq 1} F_{x_n,x_{n+k}}(t) \\
&\geq \inf_{k\geq 1} F_{x_n,x_{n+k}}\big(\varphi(t_0)\big) \\
&\geq \inf_{k\geq 1} F_{x_{n-1},x_{n-1+k}}(t_0) \\
&\geq \inf_{k\geq 1} F_{x_{n-1},x_{n-1+k}}(t) \\
&= a_{n-1} \quad \text{for all } n \in \mathbb{N}.
\end{aligned}$$

So $\{a_n\}$ is non-decreasing. Since $\{a_n\}$ is bounded, there exists $a \in [0,1]$ such that $a_n \to a$ as $n \to \infty$. Assume that $a < 1$. Then there exists $\eta \in (0,1)$ such that $a + \eta < 1$. For any given $\epsilon \in (0,1/2)$, by the definition of a_n there exists $k = k(\epsilon, n) \in \mathbb{N}$ such that

$$a_n \geq F_{x_n,x_{n+k}}(t) - \epsilon/2. \tag{3.9}$$

By Lemma 3.3 one has $\lim_{n\to\infty} F_{x_n,x_{n+1}}(t-\varphi(r)) = 1$. Therefore there exist $\delta \in (0,1)$ and $N \in \mathbb{N}$ such that $F_{x_n,x_{n+1}}(t-\varphi(r)) \in (1-\delta, 1)$ for all $n > N$. Since Δ is of H-type, $\Delta^k(F_{x_n,x_{n+1}}(t-\varphi(r))) > 1-\epsilon/2$ for all $n > N$ and all $k \in \mathbb{N}$. Further combing (3.8) and (3.9) we get

$$1 > a + \eta > a_n \geq 1 - \epsilon$$

for all $n > N$, which implies that

$$1 > a + \delta > a \geq 1.$$

It is a contradiction. So $a = 1$. Since $a_n \to 1$ as $n \to \infty$, there exists $N' \in \mathbb{N}$ such that $a_n > 1 - \epsilon$ for all $n > N$. Then by the definition of $\{a_n\}$, we have

$$F_{x_n,x_{n+k}}(t) > 1 - \epsilon$$

for all $n \in \mathbb{N}$ with $n > N'$ and all $k \in \mathbb{N}$. Thus $\{x_n\}$ is a Cauchy sequence. This completes the proof. □

Theorem 3.1 *Let (X, F, Δ) be a complete Menger space where Δ is of H-type and continuous at $(1,1)$. Let $T : X \to X$ be a probabilistic φ-contraction, where $\varphi \in \mathbf{\Phi}_{\mathbf{w}^*}$ satisfies $\varphi(t) > 0$ for all $t > 0$. Then T has a unique fixed point $x^* \in X$, and $\{T^n x_0\}$ converges to x^* for each $x_0 \in X$.*

Proof Take $x_0 \in X$ arbitrarily and define the sequence $\{x_n\}$ by $x_n = Tx_{n-1}$ for each $n \in \mathbb{N}$. Since T is a probabilistic φ-contraction, we have

$$F_{x_n,x_m}\big(\varphi(t)\big) = F_{Tx_{n-1},Tx_{m-1}}\big(\varphi(t)\big) \geq F_{x_{n-1},x_{m-1}}(t), \quad \forall m,n \in \mathbb{N} \text{ and } \forall t > 0.$$

So, from Lemma 3.4 it follows that $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$.

Next we show that x^* is a fixed point of T. For any $t > 0$, Lemma 3.1 shows that there exists $r \geq t$ such that $\varphi(r) < t$. By the monotonicity of Δ we get

$$\begin{aligned} F_{x^*,Tx^*}(t) &\geq \Delta\big(F_{x^*,x_{n+1}}\big(t-\varphi(r)\big), F_{x_{n+1},Tx^*}\big(\varphi(r)\big)\big) \\ &= \Delta\big(F_{x^*,x_{n+1}}\big(t-\varphi(r)\big), F_{Tx_n,Tx^*}\big(\varphi(r)\big)\big) \\ &\geq \Delta\big(F_{x^*,x_{n+1}}\big(t-\varphi(r)\big), F_{x_n,x^*}(r)\big) \\ &\geq \Delta(c_n, c_n), \end{aligned} \tag{3.10}$$

where $c_n = \min\{F_{x^*,x_{n+1}}(t-\varphi(r)), F_{x_n,x^*}(r)\}$. Since $c_n \to 1$ as $n \to \infty$ and Δ is continuous at $(1,1)$, from (3.10) we have

$$F_{x^*,Tx^*}(t) \geq \Delta(c_n, c_n) \to \Delta(1,1) = 1,$$

which implies that $x^* = Tx^*$.

Finally, we prove that x^* is the unique fixed point of T. Suppose that T has another fixed point $x' \in X$. Then we have

$$F_{x^*,x'}\big(\varphi(t)\big) = F_{Tx^*,Tx'}\big(\varphi(t)\big) \geq F_{x^*,x'}(t), \quad \forall t > 0.$$

From Lemma 3.2 it follows that $x^* = x'$. Thus x^* is the unique fixed point of T. This completes the proof. □

Corollary 3.1 *Let (X, F, Δ) be a complete Menger space where Δ is of H-type and continuous at $(1,1)$. Let $T_0, T_1 : X \to X$ be two mappings such that*

$$F_{T_0x,T_0y}\big(\varphi(t)\big) \geq F_{x,y}(t) \quad \textit{and} \quad F_{T_1x,T_1y}(t) \geq F_{x,y}(t) \quad \textit{for all } x,y \in X \textit{ and } t > 0, \tag{3.11}$$

where $\varphi \in \mathbf{\Phi}_{\mathbf{w}^*}$ satisfies $\varphi(t) > 0$ for all $t > 0$. If T_0 commutes with T_1, then T_0 and T_1 have a unique common fixed point in X.

Proof Let $T = T_0T_1$. Then (3.11) implies that T is a probabilistic φ-contraction. From Theorem 3.1 it follows that T has a unique fixed point $x^* \in X$. Since T_0 commutes with T_1, we have $T_0T_1x^* = T_1T_0x^*$. Further we have $T(T_0x^*) = (T_0T_1)(T_0x^*) = T_0(T_0T_1x^*) = T_0(Tx^*) = T_0x^*$, which implies that T_0x^* is a fixed point of T. Since T has a unique fixed point x^*, one has $T_0x^* = x^*$. Similarly, we have $T_1x^* = x^*$. Thus x^* is the common fixed point of T_0 and T_1. Assume that $x' \in X$ is another common fixed point of T_0 and T_1. Since T_0 commutes with T_1, we have $T(T_0x') = (T_0T_1)(T_0x') = T_0(T_0T_1x') = T_0(T_1T_0x') = T_0x'$, which implies that T_0x' is the fixed point of T. Since x^* is a unique fixed point of T, one has $x' = T_0x' = x^*$. Thus x^* is the unique common fixed point of T_0 and T_1. This completes the proof. □

Finally, we give an example to illustrate Theorem 3.1.

Example 3.2 Let $X = \{3^{n+2} : n \in \mathbb{N}\} \cup \{0,3\}$ and define the mapping $F : X \times X \to \mathcal{D}^+$ by $F_{x,y}(0) = 0$ for all $x, y \in X$, $F_{x,x}(t) = 1$ for all $x \in X$ and $t > 0$,

$$F_{0,3}(t) = F_{3,0}(t) = \begin{cases} \frac{3}{5}, & 0 < t \leq 3, \\ 1, & t > 3 \end{cases} \quad \text{and} \quad F_{x,y}(t) = F_{y,x}(t) = \begin{cases} \frac{1}{2}, & 0 < t \leq |x-y|, \\ 1, & t > |x-y| \end{cases}$$

for all $x, y \in X$ with $x \neq y$ and $\{x,y\} \neq \{0,3\}$. It is easy to see that (X, F, Δ_M) is a complete Menger space.

Let $T : X \to X$ be a mapping defined by $T0 = T3 = T27 = 0$ and $T3^{n+3} = 3^{n+2}$ for each $n \in \mathbb{N}$. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a function defined by

$$\varphi(t) = \begin{cases} t, & \text{if } 0 \leq t \leq 1, \\ t-1, & \text{if } t > 1. \end{cases}$$

Then $\varphi \in \mathbf{\Phi}_{\mathbf{w}^*}$, but $\varphi \notin \mathbf{\Phi}_{\mathbf{w}}$; see Example 3.1.

Next we show that T is a probabilistic φ-contraction, *i.e.*, T satisfies the following condition:

$$F_{Tx,Ty}\bigl(\varphi(t)\bigr) \geq F_{x,y}(t) \quad \text{for all } x, y \in X \text{ and } t > 0. \tag{3.12}$$

First, it is easy to see that for $x, y \in \{0,3,27\}$, (3.12) holds for all $t > 0$ since $T0 = T3 = T27 = 0$. Next we show that (3.12) holds for all $x, y \in X$ with $x \neq y$ and $\{x,y\} \nsubseteq \{0,3,27\}$ and $t > 0$. Obviously, if $|Tx - Ty| < \varphi(t)$, then $F_{Tx,Ty}(\varphi(t)) = 1 \geq F_{x,y}(t)$. So (3.12) holds. Now we consider all $x, y \in X$ with $x \neq y$ and $\{x,y\} \nsubseteq \{0,3,27\}$ and $t > 0$ with $|Tx - Ty| \geq \varphi(t)$ by the following cases:

(a) For $(x,y) \in \{(0,3^{n+3}), (3,3^{n+3}), (27,3^{n+3}) : n \in \mathbb{N}\}$, it is easy to conclude that $\varphi(t) \leq |Tx - Ty|$ implies that $t \leq |x-y|$ for all $t > 0$. Thus if $\varphi(t) \leq |Tx - Ty|$, then

$$F_{Tx,Ty}\bigl(\varphi(t)\bigr) = \frac{1}{2} = F_{x,y}(t) \quad \text{for all } t > 0.$$

Therefore (3.12) holds.

(b) For $(x,y) \in \{(3^{n+3}, 3^{m+3}) : m, n \in \mathbb{N} \text{ with } m > n\}$, we have $\varphi(t) \leq |Tx - Ty| = 3^{m+2} - 3^{n+2} < 3(3^{m+2} - 3^{n+2}) = |y - x|$ for $t \in (0,1]$. For $t > 1$, from $\varphi(t) = t - 1 \leq |Tx - Ty| = 3^{m+2} - 3^{n+2}$, we have $t \leq 3^{m+2} - 3^{n+2} + 1 < 3^{m+3} - 3^{n+3} = |x - y|$ since $3^{m+3} - 3^{n+3} - 3^{m+2} + 3^{n+2} = 2(3^{m+2} - 3^{n+2}) > 1$. So $\varphi(t) \leq |Tx - Ty|$ implies that $t \leq |x-y|$ for all $t > 0$. Thus if $\varphi(t) \leq |Tx - Ty|$, then

$$F_{Tx,Ty}\bigl(\varphi(t)\bigr) = \frac{1}{2} = F_{x,y}(t) \quad \text{for all } t > 0.$$

Therefore (3.12) holds.

By the discussion above, (3.12) holds for all $x, y \in X$ and $t > 0$. Therefore, T is a probabilistic φ-contraction. All the conditions of Theorem 3.1 are satisfied. By Theorem 3.1, T has a unique fixed point $x^* \in X$. Obviously, $x^* = 0$ is the unique fixed point of T. However, since $\varphi \notin \mathbf{\Phi}_{\mathbf{w}}$, Theorem 1.2, *i.e.*, Theorem 3.1 of [14] cannot be applied to this example.

4 Conclusion

In this paper, we prove a new fixed point theorems for a probabilistic φ-contraction in Menger spaces. In the theorem, a more weak condition on the gauge function φ is required. Thus our result improves Theorem 1.2 of Fang [14] and some others, such as Jachymski [13], Ćirić [2], and Xiao *et al.* [19]. By using Theorem 3.1, it is easy to prove some fixed point theorems for φ-contraction in fuzzy metric spaces like Theorems 4.1-4.4 in [14]. For shortening the length of this paper, we omit the proofs of these theorems.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics and Physics, North China Electric Power University, Baoding, 071003, China. [2]Department of Mathematics and Sciences, Shijiazhuang University of Economics, Shijiazhuang, 050031, China.

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