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Modified semi-implicit midpoint rule for nonexpansive mappings

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Abstract
The purpose of the paper is to construct iterative methods for finding the fixed points of nonexpansive mappings. We present a modified semi-implicit midpoint rule with the viscosity technique. We prove that the suggested method converges strongly to a special fixed point of nonexpansive mappings under some different control conditions. Some applications are also included.

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Keywords: modified semi-implicit midpoint rule; nonexpansive mapping; strong convergence

1 Introduction

The implicit midpoint rule is one of the powerful numerical methods for solving ordinary differential equations and differential algebraic equations. For related works, please refer to [1–9].

For the ordinary differential equation

$$x' = f(t), \qquad x(0) = x_0, \tag{1.1}$$

the implicit midpoint rule generates a sequence $\{x_n\}$ by the recursion procedure

$$x_{n+1} = x_n + hf\left(\frac{x_n + x_{n+1}}{2}\right), \quad n \geq 0, \tag{1.2}$$

where $h > 0$ is a stepsize. It is known that if $f : \mathbb{R}^N \to \mathbb{R}^N$ is Lipschitz continuous and sufficiently smooth, then the sequence $\{x_n\}$ generated by (1.2) converges to the exact solution of (1.1) as $h \to 0$ uniformly over $t \in [0, \bar{t}]$ for any fixed $\bar{t} > 0$.

If we write the function f in the form $f(t) = g(t) - t$, then differential equation (1.1) becomes $x' = g(t) - t$. Then the equilibrium problem associated with the differential equation is the fixed point problem $t = g(t)$.

Based on the above fact, Alghamdi *et al.* [10] presented the following semi-implicit midpoint rule for nonexpansive mappings:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T\left(\frac{x_n + x_{n+1}}{2}\right), \quad n \geq 0, \tag{1.3}$$

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where $\alpha_n \in (0,1)$ and $T : H \to H$ is a nonexpansive mapping. They proved the weak convergence of (1.3) under some additional conditions on $\{\alpha_n\}$.

Furthermore, in [11], Xu *et al.* used contractions to regularize the semi-implicit midpoint rule (1.3) and presented the following viscosity implicit midpoint rule for nonexpansive mappings:

$$x_{n+1} = \alpha_n Q(x_n) + (1-\alpha_n)T\left(\frac{x_n + x_{n+1}}{2}\right), \quad n \geq 0, \tag{1.4}$$

where $\alpha_n \in (0,1)$ and Q is a contraction.

Xu *et al.* [11] showed the following strong convergence theorem.

Theorem 1.1 *Let H be a Hilbert space, C be a nonempty, closed, and convex subset of H, and $T : C \to C$ be a nonexpansive mapping such that* $\mathrm{Fix}(T) \neq \emptyset$. *Let $Q : C \to C$ be a contraction with coefficient $\alpha \in [0,1)$. Assume that the sequence $\{\alpha_n\}$ satisfies the following three restrictions*:

(C1): $\lim_{n\to\infty} \alpha_n = 0$;

(C2): $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(C3): *either* $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *or* $\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = 1$.

Then the sequence $\{x_n\}$ generated by (1.4) *converges in norm to a fixed point q of T, which is also the unique solution of the variational inequality*

$$\langle (I-Q)q, x-q \rangle \geq 0, \quad \forall x \in \mathrm{Fix}(T). \tag{1.5}$$

In other words, q is the unique fixed point of the contraction $P_{\mathrm{Fix}(T)}Q$, that is, $P_{\mathrm{Fix}(T)}Q(q) = q$.

Remark 1.2 The usefulness of (1.4) is that it can be used to find a periodic solution of the time-dependent nonlinear evolution equation (see [11])

$$\frac{du}{dt} + A(t)u = g(t,u), \quad t \geq 0,$$

where $A(t)$ is a family of closed linear operators in a Hilbert space H and g maps $\mathbb{R}^1 \times H$ into H.

Remark 1.3 Note that the proof of Theorem 1.1 in [11] is technical. However, Step 6 in the proof of Theorem 1.1 is also complicated.

Fixed point method has attracted so much attention. Now we briefly recall some related historic approaches.

Browder [12] introduced an implicit scheme as follows. Fix $u \in C$ and, for each $t \in (0,1)$, let x_t be the unique fixed point in C of the contraction T_t which maps C into C: $T_t x = tu + (1-t)Tx$, $x \in C$. Browder proved that $s\text{-}\lim_{t\downarrow 0} x_t = P_{\mathrm{Fix}(T)}u$. That is, the strong limit of $\{x_t\}$ as $t \to 0^+$ is the fixed point of T which is nearest from $\mathrm{Fix}(T)$ to u.

Halpern [13], on the other hand, introduced an explicit scheme. Again fix a $u \in C$. Then with a sequence $\{t_n\}$ in $(0,1)$ and an arbitrary initial guess $x_0 \in C$, we can define a sequence $\{x_n\}$ through the recursive formula

$$x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n, \quad n \geq 0. \tag{1.6}$$

Lions [14] proved the strong convergence of (1.6) under conditions (C1), (C2) and

$$\text{(C4)}: \lim_{n\to\infty} \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n^2} = 0.$$

It is now known that this sequence $\{x_n\}$ converges in norm to the same limit $P_{\mathrm{Fix}(T)}u$ as Browder's implicit scheme if the sequence $\{\alpha_n\}$ satisfies assumptions (C1), (C2), and (C3) above.

Moudafi [15] presented an explicit viscosity method for nonexpansive mappings which generates a sequence $\{x_n\}$ through the iteration process

$$x_{n+1} = \alpha_n Q(x_n) + (1 - \alpha_n)Tx_n, \quad n \geq 0. \tag{1.7}$$

Moudafi proved the strong convergence of (1.7) under conditions (C1), (C2), and

$$\text{(C5)}: \lim_{n\to\infty} \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n \alpha_{n-1}} = 0.$$

Refinements in Hilbert spaces and extensions to Banach spaces were obtained by Xu [16]. This technique uses (strict) contractions to regularize a nonexpansive mapping for the purpose of selecting a particular fixed point of the nonexpansive mapping, for instance, the fixed point of minimal norm, or of a solution to another variational inequality.

Motivated and inspired by the above work, in this paper we aim to construct a unified iterative algorithm for finding the fixed points of nonexpansive mappings. We present a modified semi-implicit midpoint rule with the viscosity technique for nonexpansive mappings. We prove that the suggested algorithm converges strongly to a special fixed point of nonexpansive mappings under some different conditions. Some applications are also included.

2 Tools

2.1 Some notations

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. Let C be a nonempty closed convex subset of H.

Recall that a mapping $T : C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$. We use $\mathrm{Fix}(T)$ to denote the set of fixed points of T.

A mapping $Q : C \to C$ is said to be contractive if there exists a constant $\alpha \in (0, 1)$ such that

$$\|Q(x) - Q(y)\| \leq \alpha \|x - y\|$$

for all $x, y \in C$. In this case, Q is called α-contraction.

Let C be a nonempty closed convex subset of H. For every point $x \in H$, there exists a unique nearest point in C denoted by $P_C x$ such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C.$$

The mapping P_C is called the metric projection of H onto C. It is well known that P_C is a nonexpansive mapping and is characterized by the following property:

$$\langle x - P_C x, y - P_C x\rangle \leq 0, \quad \forall x \in H, y \in C. \tag{2.1}$$

2.2 Existing algorithm and convergence result

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $Q : C \to C$ be an α-contraction and $T : C \to C$ be a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$.

Algorithm 2.1 For given $y_0 \in C$ arbitrarily, let the sequence $\{y_n\}$ be defined iteratively by the manner

$$y_n = \alpha_n Q(y_n) + (1 - \alpha_n) T y_n, \quad n \geq 0, \tag{2.2}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$.

Theorem 2.2 ([16]) *The sequence* $\{y_n\}$ *generated by* (2.2) *converges strongly to* $q = P_{\mathrm{Fix}(T)} Q(q)$ *provided* $\lim_{n\to\infty} \alpha_n = 0$.

3 Some lemmas

The following demiclosedness principles for nonexpansive mappings are well known.

Lemma 3.1 ([17]) *Let* C *be a nonempty closed convex subset of a Hilbert space* H, *and let* $T : C \to C$ *be a nonexpansive mapping with* $\mathrm{Fix}(T) \neq \emptyset$. *Assume that* $\{y_n\}$ *is a sequence in* C *such that* $y_n \rightharpoonup x^\dagger$ *and* $(I - T)y_n \to 0$. *Then* $x^\dagger \in \mathrm{Fix}(T)$.

Lemma 3.2 ([18]) *Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in a Banach space* E *and* $\{\beta_n\}$ *be a sequence in* $[0, 1]$ *with* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. *Suppose that* $x_{n+1} = (1 - \beta_n)x_n + \beta_n z_n$ *for all* $n \geq 0$ *and* $\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$. *Then* $\lim_{n\to\infty} \|z_n - x_n\| = 0$.

Lemma 3.3 ([19]) *Let* $\{a_n\}_{n\in\mathbb{N}}$ *be a sequence of nonnegative real numbers satisfying the following relation*:

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n \sigma_n + \delta_n, \quad n \geq 0,$$

where

(i) $\{\alpha_n\}_{n\in\mathbb{N}} \subset [0, 1]$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $\limsup_{n\to\infty} \sigma_n \leq 0$;
(iii) $\sum_{n=1}^{\infty} \delta_n < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

4 Main results

In this section, we firstly present the following unified algorithm.

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$. Let $Q : C \to C$ be an α-contraction.

Algorithm 4.1 For given $x_0 \in C$ arbitrarily, let the sequence $\{x_n\}$ be generated by the manner

$$x_{n+1} = \alpha_n Q(x_n) + \beta_n x_n + \gamma_n T\left(\frac{x_n + x_{n+1}}{2}\right), \quad n \geq 0, \tag{4.1}$$

where $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset [0,1)$, and $\{\gamma_n\} \subset (0,1)$ are three sequences satisfying $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \geq 0$.

Remark 4.2 Equation (4.1) is well defined. As a matter of fact, for fixed $u \in C$, we can define a mapping

$$x \mapsto T_u x := \alpha Q(u) + \beta u + \gamma T\left(\frac{u+x}{2}\right).$$

Then we have

$$\begin{aligned}\|T_u x - T_u y\| &= \gamma \left\| T\left(\frac{u+x}{2}\right) - T\left(\frac{u+y}{2}\right)\right\| \\ &\leq \frac{\gamma}{2}\|x-y\|.\end{aligned}$$

This means T_u is a contraction with coefficient $\frac{\gamma}{2} \in (0, \frac{1}{2})$. Hence, Algorithm 4.1 is well defined.

Now, we show the boundedness of the sequence $\{x_n\}$.

Conclusion 4.3 *The sequence* $\{x_n\}$ *generated by* (4.1) *is bounded.*

Proof Pick any $x^\dagger \in \operatorname{Fix}(T)$. From (4.1), we have

$$\begin{aligned}\|x_{n+1} - x^\dagger\| &= \Big\| \alpha_n\big(Q(x_n) - Q(x^\dagger)\big) + \alpha_n\big(Q(x^\dagger) - x^\dagger\big) + \beta_n\big(x_n - x^\dagger\big) \\ &\quad + \gamma_n\left(T\left(\frac{x_n + x_{n+1}}{2}\right) - x^\dagger\right)\Big\| \\ &\leq \alpha_n \|Q(x_n) - Q(x^\dagger)\| + \alpha_n\|Q(x^\dagger) - x^\dagger\| + \beta_n\|x_n - x^\dagger\| \\ &\quad + \gamma_n\left\|T\left(\frac{x_n + x_{n+1}}{2}\right) - x^\dagger\right\| \\ &\leq \alpha_n \alpha \|x_n - x^\dagger\| + \alpha_n \|Q(x^\dagger) - x^\dagger\| + \beta_n \|x_n - x^\dagger\| \\ &\quad + \frac{\gamma_n}{2}\|x_n - x^\dagger\| + \frac{\gamma_n}{2}\|x_{n+1} - x^\dagger\|.\end{aligned}$$

It follows that

$$\begin{aligned}\|x_{n+1} - x^\dagger\| &\leq \frac{1 + \beta_n + (2\alpha - 1)\alpha_n}{1 + \beta_n + \alpha_n}\|x_n - x^\dagger\| + \frac{2\alpha_n}{1 + \beta_n + \alpha_n}\|Q(x^\dagger) - x^\dagger\| \\ &= \left[1 - \frac{2(1-\alpha)\alpha_n}{1 + \beta_n + \alpha_n}\right]\|x_n - x^\dagger\| + \frac{2(1-\alpha)\alpha_n}{1 + \beta_n + \alpha_n}\frac{1}{1-\alpha}\|Q(x^\dagger) - x^\dagger\| \\ &\leq \max\left\{\|x_n - x^\dagger\|, \frac{1}{1-\alpha}\|Q(x^\dagger) - x^\dagger\|\right\}.\end{aligned}$$

By induction, we can deduce

$$\|x_n - x^\dagger\| \le \max\left\{\|x_0 - x^\dagger\|, \frac{1}{1-\alpha}\|Q(x^\dagger) - x^\dagger\|\right\}.$$

Hence, $\{x_n\}$ is bounded. This completes the proof. □

Now we give the following result.

Theorem 4.4 *The sequence* $\{x_n\}$ *generated by* (4.1) *converges strongly to* $q = P_{\mathrm{Fix}(T)}Q(q)$ *provided* $\{\alpha_n\}$ *satisfies* (C1)-(C3) (*as stated in Theorem* 1.1) *and* $\{\beta_n\}$ *satisfies*

$$\text{(C6): } \lim_{n\to\infty}\frac{\beta_n}{\alpha_n} = 0.$$

Proof Set $y_{n+1} = \alpha_n Q(y_n) + (1-\alpha_n)T(\frac{y_n+y_{n+1}}{2})$ for all n. Then we have

$$\begin{aligned}\|x_{n+1} - y_{n+1}\| &= \left\|\alpha_n\big(Q(x_n) - Q(y_n)\big) + \beta_n\left(x_n - T\left(\frac{y_n + y_{n+1}}{2}\right)\right)\right.\\ &\quad \left. + \gamma_n\left(T\left(\frac{x_n + x_{n+1}}{2}\right) - T\left(\frac{y_n + y_{n+1}}{2}\right)\right)\right\|\\ &\le \alpha\alpha_n\|x_n - y_n\| + \beta_n\left\|x_n - T\left(\frac{y_n + y_{n+1}}{2}\right)\right\| + \frac{\gamma_n}{2}\|x_n - y_n\|\\ &\quad + \frac{\gamma_n}{2}\|x_{n+1} - y_{n+1}\|. \end{aligned} \tag{4.2}$$

It is obvious that $\{x_n\}$ and $\{y_n\}$ are bounded by Conclusion 4.3. Hence, we deduce from (4.2) that

$$\begin{aligned}\|x_{n+1} - y_{n+1}\| &\le \left[1 - \frac{2(1-\alpha)\alpha_n}{2-\gamma_n}\right]\|x_n - y_n\| + \beta_n M_1\\ &= (1-\sigma_n)\|x_n - y_n\| + \sigma_n\frac{\beta_n}{\sigma_n}M_1, \end{aligned} \tag{4.3}$$

where $\sigma_n = \frac{2(1-\alpha)\alpha_n}{2-\gamma_n}$ and M_1 is a constant such that $\sup_n\{2\|x_n - T(\frac{y_n+y_{n+1}}{2})\|\} \le M_1$. Note that $\limsup_{n\to\infty}\frac{\beta_n}{\sigma_n} \le 0$. Apply Lemma 3.3 to (4.3) to conclude that $\|x_{n+1} - y_{n+1}\| \to 0$. Consequently, $x_n \to q = P_{\mathrm{Fix}(T)}Q(q)$ according to Theorem 1.1. This completes the proof. □

Remark 4.5 The proof of Theorem 4.4 is very simple.

Remark 4.6 In (4.1), if we choose $\beta_n \equiv 0$ for all n, then (4.1) is reduced to (1.4). Thus, our Algorithm 4.1 includes Algorithm 1.4 as a special case, and Theorem 1.1 is also a special case of our Theorem 4.4.

Next, we can define the following algorithm.

Algorithm 4.7 For given $y_0 \in C$ arbitrarily, let the sequence $\{y_n\}$ be defined iteratively by the manner

$$y_n = \alpha_n Q(y_n) + \beta_n y_n + \gamma_n Ty_n, \quad n \ge 0, \tag{4.4}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are the same sequences as stated in Algorithm 4.1.

Proposition 4.8 *The sequence* $\{y_n\}$ *generated by* (4.4) *converges strongly to* $q = P_{\mathrm{Fix}(T)}Q(q)$ *provided* $\lim_{n\to\infty}\alpha_n = 0$.

In fact, we can rewrite (4.4) as $y_n = \frac{\alpha_n}{1-\beta_n}Q(y_n) + (1 - \frac{\alpha_n}{1-\beta_n})Ty_n$ for all n. Thus, Proposition 4.8 can be deduced from Theorem 2.2.

Next we use Proposition 4.8 to show the convergence analysis of Algorithm 4.1 under other control conditions.

Let the sequences $\{x_n\}$ and $\{y_n\}$ be generated by (4.1) and (4.4), respectively. Note that the sequences $\{x_n\}$ and $\{y_n\}$ are all bounded. First, we have the following estimation:

$$\begin{aligned}\|x_{n+1} - y_n\| &= \left\|\alpha_n\big(Q(x_n) - Q(y_n)\big) + \beta_n(x_n - y_n) + \gamma_n\left(T\left(\frac{x_n + x_{n+1}}{2}\right) - Ty_n\right)\right\| \\ &\le \alpha_n\alpha\|x_n - y_n\| + \beta_n\|x_n - y_n\| + \gamma_n\frac{\|x_n - y_n\|}{2} + \gamma_n\frac{\|x_{n+1} - y_n\|}{2}.\end{aligned}$$

It follows that

$$\begin{aligned}\|x_{n+1} - y_n\| &\le \left[1 - \frac{2(1-\alpha)\alpha_n}{1 + \alpha_n + \beta_n}\right]\|x_n - y_n\| \\ &\le \left[1 - \frac{2(1-\alpha)\alpha_n}{1 + \alpha_n + \beta_n}\right]\|x_n - y_{n-1}\| + \|y_n - y_{n-1}\|.\end{aligned}$$

It is easily seen that if $\sum_{n=0}^{\infty}\alpha_n = \infty$ and $\lim_{n\to\infty}\frac{\|y_n - y_{n-1}\|}{\alpha_n} = 0$, then we get $\lim_{n\to\infty}\|x_{n+1} - y_n\| = 0$ by Lemma 3.3. Consequently, $x_n \to q = P_{\mathrm{Fix}(T)}Q(q)$ provided $\lim_{n\to\infty}\alpha_n = 0$.

Next, we estimate $\|y_n - y_{n-1}\|$. From (4.4), we have

$$\begin{aligned}\|y_n - y_{n-1}\| &= \big\|\alpha_n\big(Q(y_n) - Q(y_{n-1})\big) + (\alpha_n - \alpha_{n-1})Q(y_{n-1}) + \beta_n(y_n - y_{n-1}) \\ &\quad + (\beta_n - \beta_{n-1})y_{n-1} + \gamma_n(Ty_n - Ty_{n-1}) + (\gamma_n - \gamma_{n-1})Ty_{n-1}\big\| \\ &\le (\alpha\alpha_n + \beta_n + \gamma_n)\|y_n - y_{n-1}\| + |\alpha_n - \alpha_{n-1}|\big\|Q(y_{n-1})\big\| \\ &\quad + |\beta_n - \beta_{n-1}|\|y_{n-1}\| + |\gamma_n - \gamma_{n-1}|\|Ty_{n-1}\|.\end{aligned}$$

Hence,

$$\begin{aligned}\frac{\|y_n - y_{n-1}\|}{\alpha_n} &\le \frac{|\alpha_n - \alpha_{n-1}|}{(1-\alpha)\alpha_n^2}\big(\big\|Q(y_{n-1})\big\| + \|Ty_{n-1}\|\big) \\ &\quad + \frac{|\beta_n - \beta_{n-1}|}{(1-\alpha)\alpha_n^2}\big(\|y_{n-1}\| + \|Ty_{n-1}\|\big).\end{aligned}$$

If $\lim_{n\to\infty}\frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n^2} = \lim_{n\to\infty}\frac{|\beta_n - \beta_{n-1}|}{\alpha_n^2} = 0$, we derive that $\lim_{n\to\infty}\frac{\|y_n - y_{n-1}\|}{\alpha_n} = 0$. So, we obtain immediately the following theorem.

Theorem 4.9 *Assume that* $\{\alpha_n\}$ *satisfies* (C1), (C2), *and* (C4) *and* $\{\beta_n\}$ *satisfies*

$$\text{(C7):}\ \lim_{n\to\infty}\frac{|\beta_n - \beta_{n-1}|}{\alpha_n^2} = 0.$$

Then the sequence $\{x_n\}$ *generated by* (4.1) *converges strongly to* $q = P_{\mathrm{Fix}(T)}Q(q)$.

Remark 4.10 Note that conditions (C1), (C2), and (C4) were presented by Lions in [14]. At the same time, (C7) is different from (C6). In fact, we can choose $\beta_n = \beta \in (0,1)$ in (C7).

Next, we will give another control condition instead of (C4) and (C7).

Theorem 4.11 *Assume that* $\{\alpha_n\}$ *satisfies* (C1) *and* (C2) *and* $\{\beta_n\}$ *satisfies*

$$\text{(C8)}: 0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$$

and

$$\text{(C9)}: \lim_{n\to\infty} (\beta_{n+1} - \beta_n) = 0.$$

Then the sequence $\{x_n\}$ *generated by* (4.1) *converges strongly to* $q = P_{\mathrm{Fix}(T)}Q(q)$.

Proof From Conclusion 4.3, we can choose a constant M such that

$$\sup_n \left\{ \frac{3}{1-\beta_n} \left(\|Q(x_n)\| + \left\| T\left(\frac{x_n + x_{n+1}}{2}\right) \right\| + \left\| x_n - T\left(\frac{x_n + x_{n+1}}{2}\right) \right\| \right) \right\} \leq M.$$

Set $y_n = \frac{x_{n+1} - \beta_n x_n}{1-\beta_n}$ for all $n \geq 0$. Thus, we have

$$\begin{aligned}
y_{n+1} - y_n &= \frac{x_{n+2} - \beta_{n+1}x_{n+1}}{1-\beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1-\beta_n} \\
&= \frac{\alpha_{n+1}Q(x_{n+1}) + (1-\alpha_{n+1}-\beta_{n+1})T(\frac{x_{n+1}+x_{n+2}}{2})}{1-\beta_{n+1}} \\
&\quad - \frac{\alpha_n Q(x_n) + (1-\alpha_n-\beta_n)T(\frac{x_n+x_{n+1}}{2})}{1-\beta_n} \\
&= \frac{\alpha_{n+1}}{1-\beta_{n+1}}\big(Q(x_{n+1}) - Q(x_n)\big) \\
&\quad + \frac{1-\alpha_{n+1}-\beta_{n+1}}{1-\beta_{n+1}}\left[T\left(\frac{x_{n+1}+x_{n+2}}{2}\right) - T\left(\frac{x_n+x_{n+1}}{2}\right) \right] \\
&\quad + \left(\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n} \right)\left(Q(x_n) - T\left(\frac{x_n+x_{n+1}}{2}\right) \right).
\end{aligned}$$

It follows that

$$\begin{aligned}
\|y_{n+1} - y_n\| &\leq \left(\frac{\alpha_{n+1}}{1-\beta_{n+1}} + \frac{\alpha_n}{1-\beta_n} \right) \left\| Q(x_n) - T\left(\frac{x_n+x_{n+1}}{2}\right) \right\| \\
&\quad + \frac{1-\alpha_{n+1}-\beta_{n+1}}{1-\beta_{n+1}} \left[\frac{\|x_{n+1}-x_n\|}{2} + \frac{\|x_{n+2}-x_{n+1}\|}{2} \right] \\
&\quad + \frac{\alpha\alpha_{n+1}}{1-\beta_{n+1}} \|x_{n+1} - x_n\|.
\end{aligned} \tag{4.5}$$

From (4.1), we have

$$\begin{aligned}
\|x_{n+2} - x_{n+1}\| = \bigg\| &\alpha_{n+1}\big(Q(x_{n+1}) - Q(x_n)\big) + (\alpha_{n+1} - \alpha_n)Q(x_n) \\
&+ \beta_{n+1}(x_{n+1} - x_n) + (\beta_{n+1} - \beta_n)\left(x_n - T\left(\frac{x_n + x_{n+1}}{2}\right) \right)
\end{aligned}$$

$$
\begin{aligned}
&+ (1-\alpha_{n+1}-\beta_{n+1})\left(T\left(\frac{x_{n+1}+x_{n+2}}{2}\right)-T\left(\frac{x_n+x_{n+1}}{2}\right)\right)\\
&+ (\alpha_n-\alpha_{n+1})T\left(\frac{x_n+x_{n+1}}{2}\right)\Bigg\|\\
\le{}& \alpha\alpha_{n+1}\|x_{n+1}-x_n\| + (\alpha_{n+1}+\alpha_n)\big\|Q(x_n)\big\| + \beta_{n+1}\|x_{n+1}-x_n\|\\
&+ (1-\alpha_{n+1}-\beta_{n+1})\left(\frac{\|x_{n+1}-x_n\|}{2}+\frac{\|x_{n+2}-x_{n+1}\|}{2}\right)\\
&+ (\alpha_n+\alpha_{n+1})\left\|T\left(\frac{x_n+x_{n+1}}{2}\right)\right\| + |\beta_{n+1}-\beta_n|\left\|x_n-T\left(\frac{x_n+x_{n+1}}{2}\right)\right\|.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\|x_{n+2}-x_{n+1}\| \le{}& \left[1-\frac{2(1-\alpha)\alpha_{n+1}}{1+\alpha_{n+1}+\beta_{n+1}}\right]\|x_{n+1}-x_n\|\\
&+ \frac{2(\alpha_{n+1}+\alpha_n)}{1+\alpha_{n+1}+\beta_{n+1}}\left(\big\|Q(x_n)\big\| + \left\|T\left(\frac{x_n+x_{n+1}}{2}\right)\right\|\right)\\
&+ \frac{2|\beta_{n+1}-\beta_n|}{1+\alpha_{n+1}+\beta_{n+1}}\left\|x_n-T\left(\frac{x_n+x_{n+1}}{2}\right)\right\|\\
\le{}& \|x_{n+1}-x_n\| + M\big(\alpha_n+\alpha_{n+1}+|\beta_{n+1}-\beta_n|\big).
\end{aligned}
\tag{4.6}
$$

Substitute (4.6) into (4.5) to get

$$
\|y_{n+1}-y_n\| \le \left[1-\frac{(1-\alpha)\alpha_{n+1}}{1-\beta_{n+1}}\right]\|x_{n+1}-x_n\| + 2M\big(\alpha_{n+1}+\alpha_n+|\beta_{n+1}-\beta_n|\big).
$$

Hence,

$$
\limsup_{n\to\infty}\big(\|y_{n+1}-y_n\|-\|x_{n+1}-x_n\|\big)\le 0.
$$

This together with Lemma 3.2 implies that

$$
\lim_{n\to\infty}\|y_n-x_n\| = 0.
$$

Note that

$$
y_n-x_n=\frac{x_{n+1}-x_n}{1-\beta_n}.
$$

So,

$$
\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0. \tag{4.7}
$$

Again, from (4.1), we have

$$
\begin{aligned}
\|x_n-Tx_n\| &\le \|x_n-x_{n+1}\| + \|x_{n+1}-Tx_n\|\\
&\le \|x_n-x_{n+1}\| + \alpha_n\big\|Q(x_n)-Tx_n\big\| + \beta_n\|x_n-Tx_n\|\\
&\quad + (1-\alpha_n-\beta_n)\frac{1}{2}\|x_n-x_{n+1}\|.
\end{aligned}
$$

It follows that

$$\|x_n - Tx_n\| \le \frac{\alpha_n}{1-\beta_n}\|Q(x_n) - Tx_n\| + \frac{3-\alpha_n-\beta_n}{2(1-\beta_n)}\|x_{n+1} - x_n\|.$$

This together with (C1) and (4.7) implies that

$$\lim_{n\to\infty}\|x_n - Tx_n\| = 0. \tag{4.8}$$

Next, we prove that

$$\limsup_{n\to\infty}\langle q - Q(q), q - x_n\rangle \le 0, \tag{4.9}$$

where $q \in \mathrm{Fix}(T)$ is the unique fixed point of the contraction $P_{\mathrm{Fix}(T)}Q$, that is, $q = P_{\mathrm{Fix}(T)}Q(q)$.

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ converges weakly to a point $\breve{x}$ and

$$\begin{aligned}
&\limsup_{n\to\infty}\langle P_{\mathrm{Fix}(T)}Q(q) - Q(q), P_{\mathrm{Fix}(T)}Q(q) - x_n\rangle \\
&\quad = \lim_{i\to\infty}\langle P_{\mathrm{Fix}(T)}Q(q) - Q(q), P_{\mathrm{Fix}(T)}Q(q) - x_{n_i}\rangle.
\end{aligned} \tag{4.10}$$

By Lemma 3.1 and (4.8), we deduce $\breve{x} \in \mathrm{Fix}(T)$. This together with (2.1) implies that

$$\begin{aligned}
&\limsup_{n\to\infty}\langle P_{\mathrm{Fix}(T)}Q(q) - Q(q), P_{\mathrm{Fix}(T)}Q(q) - x_n\rangle \\
&\quad = \lim_{i\to\infty}\langle P_{\mathrm{Fix}(T)}Q(q) - Q(q), P_{\mathrm{Fix}(T)}Q(q) - x_{n_i}\rangle \\
&\quad = \langle P_{\mathrm{Fix}(T)}Q(q) - Q(q), P_{\mathrm{Fix}(T)}Q(q) - \breve{x}\rangle \\
&\quad \le 0.
\end{aligned}$$

Finally, we prove that $x_n \to q$. From (4.1), we have

$$\begin{aligned}
\|x_{n+1} - q\|^2 &= \alpha_n\langle Q(x_n) - Q(q), x_{n+1} - q\rangle + \alpha_n\langle Q(q) - q, x_{n+1} - q\rangle \\
&\quad + (1-\alpha_n-\beta_n)\left\langle T\left(\frac{x_n + x_{n+1}}{2}\right) - q, x_{n+1} - q\right\rangle \\
&\quad + \beta_n\langle x_n - q, x_{n+1} - q\rangle \\
&\le \alpha_n\alpha\|x_n - q\|\|x_{n+1} - q\| + \alpha_n\langle Q(q) - q, x_{n+1} - q\rangle \\
&\quad + (1-\alpha_n-\beta_n)\frac{1}{2}\big(\|x_n - q\| + \|x_{n+1} - q\|\big)\|x_{n+1} - q\| \\
&\quad + \beta_n\|x_n - q\|\|x_{n+1} - q\| \\
&\le \frac{1+\beta_n+(2\alpha-1)\alpha_n}{4}\|x_n - q\|^2 + \frac{3-\beta_n-(3-2\alpha)\alpha_n}{4}\|x_{n+1} - q\|^2 \\
&\quad + \alpha_n\langle Q(q) - q, x_{n+1} - q\rangle.
\end{aligned}$$

It follows that

$$\|x_{n+1}-q\|^2 \le \left[1-\frac{4(1-\alpha)\alpha_n}{1+\beta_n+(3-2\alpha)\alpha_n}\right]\|x_n-q\|^2 + \frac{4\alpha_n}{1+\beta_n+(3-2\alpha)\alpha_n}\langle Q(q)-q, x_{n+1}-q\rangle. \tag{4.11}$$

Apply Lemma 3.3 and (4.9) to (4.11) to deduce that $x_n \to q$. This completes the proof. □

Remark 4.12 Note that condition (C8) has been used in a large number of references. Theorems 4.4, 4.9, and 4.11 demonstrate the strong convergence of Algorithm 4.1 under different control conditions on parameters $\{\alpha_n\}$ and $\{\beta_n\}$. Our algorithm and results provide a unified framework for the class problem of algorithmic approach to the fixed point of nonlinear operators.

5 Applications

5.1 Application to variational inequalities

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : H \to H$ be a single-valued monotone operator such that $C \subset \operatorname{dom}(A)$. Now we consider the following variational inequality:

$$\langle Ax^*, x-x^*\rangle \ge 0, \quad x \in C. \tag{5.1}$$

It is known that (5.1) is equivalent to the fixed point problem, for any $\lambda > 0$,

$$P_C(I-\lambda A)x^* = x^*. \tag{5.2}$$

If A is Lipschitzian and α-inverse-strongly monotone, then $P_C(I-\lambda A)$ is nonexpansive provided $0 < \lambda < 2\alpha$. Thus, we can get the following theorem.

Theorem 5.1 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : H \to H$ be a Lipschitzian and α-inverse-strongly monotone operator. Let $Q : C \to C$ be a contraction. Assume* (5.1) *is solvable. Let $\{x_n\}$ be a sequence generated by the manner*

$$x_{n+1} = \alpha_n Q(x_n) + \beta_n x_n + \gamma_n P_C(I-\lambda A)\left(\frac{x_n+x_{n+1}}{2}\right), \quad n \ge 0, \tag{5.3}$$

where $\lambda \in (0, 2\alpha)$ and $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy one of the following conditions: (C1), (C2), (C4), *and* (C7) *or* (C1), (C2), (C3), *and* (C6) *or* (C1), (C2), (C8), *and* (C9). *Then the sequence $\{x_n\}$ converges strongly to a solution x^* of* (5.1) *which is also a solution to the variational inequality*

$$\langle (I-Q)x^*, x-x^*\rangle \ge 0, \quad x \in A^{-1}(0).$$

5.2 Application to hierarchical minimization

Consider the following hierarchical minimization problem:

$$\min_{x\in S_0} \psi_1(x), \tag{5.4}$$

where $S_0 := \arg\min_{x\in H} \psi_0(x)$ and ψ_0, ψ_1 are two lower semi-continuous convex functions from H to $\mathbb{R}$. Assume that $S_0 \neq \emptyset$. Set $S = \arg\min_{x\in S_0} \psi_1(x)$ and assume $S \neq \emptyset$.

Assume that ψ_0 and ψ_1 are differentiable and their gradients satisfy the Lipschitz continuity conditions

$$\left\|\nabla\psi_0(x) - \nabla\psi_0(y)\right\| \leq L_0\|x-y\|, \qquad \left\|\nabla\psi_1(x) - \nabla\psi_1(y)\right\| \leq L_1\|x-y\| \tag{5.5}$$

for all $x, y \in H$. Note that the Lipschitz continuity (5.5) implies that $\nabla\psi_i$ is $\frac{1}{L_i}$-inverse-strongly monotone. Consequently, $(I - \gamma_i\nabla\psi_i)$ is nonexpansive provided $0 < \gamma_i < 2/L_i$ and $S_0 = \operatorname{Fix}((I - \gamma_1\nabla\psi_0))$.

The optimality condition for $x^* \in S_0$ to be a solution of the hierarchical minimization (5.4) is the variational inequality

$$x^* \in S_0, \quad \left\langle\nabla\psi_1\left(x^*\right), x - x^*\right\rangle \geq 0, \quad x \in S_0. \tag{5.6}$$

Hence, we have the following theorem.

Theorem 5.2 *Assume that the hierarchical minimization problem* (5.4) *is solvable. Assume* (5.5) *and* $0 < \gamma_i < 2/L_i$. *Let* $Q : C \to C$ *be a contraction. Define a sequence* $\{x_n\}$ *by the manner*

$$x_{n+1} = \alpha_n Q(x_n) + \beta_n x_n + \gamma_n P_{S_0}(I - \lambda\nabla\psi_1)\left(\frac{x_n + x_{n+1}}{2}\right), \quad n \geq 0, \tag{5.7}$$

where $\{\alpha_n\}$ *and* $\{\beta_n\}$ *satisfy one of the following conditions*: (C1), (C2), (C4), *and* (C7) *or* (C1), (C2), (C3), *and* (C6) *or* (C1), (C2), (C8), *and* (C9). *Then the sequence* $\{x_n\}$ *converges strongly to a solution* x^* *of* (5.6) *which is also a solution to the variational inequality*

$$\left\langle(I - Q)x^*, x - x^*\right\rangle \geq 0, \quad x \in S.$$

5.3 Periodic solution of a nonlinear evolution equation

Consider the time-dependent nonlinear equation of evolution in H given by

$$\frac{du}{dt} + A(t)u = f(t,u), \quad t \geq 0, \tag{5.8}$$

where $A(t)$ is a family of closed linear operators in a Hilbert space H and f maps $\mathbb{R}^1 \times H$ into H.

We assume that $A(t)$ and $f(t,u)$ are periodic in t with a common period $\xi > 0$.

An interesting result on the existence of periodic solutions of equation (5.8) is due to Browder [20].

Theorem 5.3 *Suppose that* $A(t)$ *and* $f(t,u)$ *are periodic in* T *of period* $\xi > 0$ *and satisfy the following assumptions*:

(i) *For each* t *and each pair* $u, v \in H$,

$$\operatorname{Re}\left\langle f(t,u) - f(t,v), u - v\right\rangle \leq 0.$$

(ii) *For each t and each $u \in D(A(t))$*, $\operatorname{Re}\langle A(t)u, u\rangle \geq 0$.
(iii) *There exists a mild solution u of equation* (5.1) *on $\mathbb{R}^+$ for each initial value $v \in H$.*
(iv) *There exists some $R > 0$ such that*

$$\operatorname{Re}\langle f(t,u), u\rangle < 0$$

for $\|u\| = R$ and all $t \in [0,\xi]$.

Then there exists an element v of H with $\|v\| < R$ such that the mild solution of equation (5.8) *with the initial condition $u(0) = v$ is of period ξ.*

Define a mapping $T : H \to H$ by

$$Tv = u(\xi), \quad v \in H, \tag{5.9}$$

where u solves (5.8) with $u(0) = v$.

Then each fixed point of T corresponds to a periodic solution of equation (5.8) with period ξ. Since

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\{\|u(t)\|^2\} &= -\operatorname{Re}\langle A(t)u(t), u(t)\rangle + \operatorname{Re}\langle f(t,u(t)), u(t)\rangle \\ &\leq \operatorname{Re}\langle f(t,u(t)), u(t)\rangle,\end{aligned}$$

we see that for any value of t in $[0,\xi]$ for which $\|u(t)\| = R$, we have $\frac{d}{dt}\{\|u(t)\|^2\} < 0$. Hence, $\|u(\xi)\| \leq R$, and T maps the closed ball $B := \{v \in H : \|v\| \leq R\}$ into itself.

At the same time, we note that T is nonexpansive. As a matter of fact, if v and v_1 are two elements of B, $u(t)$ and $u_1(t)$ are the corresponding mild solutions, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\{\|u(t) - u_1(t)\|^2\} &= -\operatorname{Re}\langle A(t)(u(t) - u_1(t)), u(t) - u_1(t)\rangle \\ &\quad + \operatorname{Re}\langle f(t,u(t)) - f(t,u_1(t)), u(t) - u_1(t)\rangle \\ &\leq 0.\end{aligned}$$

Hence, $\|u(\xi) - u_1(\xi)\| \leq \|u(0) - u_1(0)\|$, *i.e.*, $\|Tv - Tv_1\| \leq \|v - v_1\|$.

Consequently, T has a fixed point which we denote by v, and the corresponding solution u of (5.8) with the initial condition $u(0) = v$ is a desired periodic solution of (5.8) with period ξ. In other words, to find a periodic solution u of (5.8) is equivalent to finding a fixed point of T.

Thus, our method is applicable to (5.8). It turns out that under one of the following conditions: (C1), (C2), (C4), and (C7) or (C1), (C2), (C3), and (C6) or (C1), (C2), (C8), and (C9), the sequence $\{v_n\}$ generated by the manner

$$v_{n+1} = \alpha_n Q(v_n) + \beta_n v_n + \gamma_n T\left(\frac{v_n + v_{n+1}}{2}\right), \quad n \geq 0$$

converges strongly to a fixed point v of T, and the mild solution of (5.8) with the initial value $u(0) = \xi$ is a periodic solution of (5.8).

5.4 Fredholm integral equation

Consider a Fredholm integral equation of the form

$$x(t) = g(t) + \int_0^t F\bigl(t,s,x(s)\bigr)\,ds, \quad t \in [0,1], \tag{5.10}$$

where g is a continuous function on $[0,1]$ and $F : [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous. Note that if F satisfies the Lipschitz continuity condition

$$\bigl|F(t,s,x) - F(t,s,y)\bigr| \le |x - y|, \quad t,s \in [0,1], x,y \in \mathbb{R},$$

then equation (5.10) has at least one solution in $L^2[0,1]$ (see [14]).

Define a mapping $T : L^2[0,1] \to L^2[0,1]$ by

$$(Tx)(t) = g(t) + \int_0^t F\bigl(t,s,x(s)\bigr)\,ds, \quad t \in [0,1]. \tag{5.11}$$

It is easily seen that T is nonexpansive. In fact, we have, for $x,y \in L^2[0,1]$,

$$\begin{aligned}\|Tx - Ty\|^2 &= \int_0^1 \bigl|Tx(t) - Ty(t)\bigr|^2\,dt \\ &= \int_0^1 \left|\int_0^1 \bigl(F\bigl(t,s,x(s)\bigr) - F\bigl(t,s,y(s)\bigr)\bigr)\,ds\right|^2 dt \\ &\le \int_0^1 \left|\int_0^2 \bigl|x(s) - y(s)\bigr|\,ds\right|^2 dt \\ &\le \int_0^1 \bigl|x(s) - y(s)\bigr|^2\,ds \\ &= \|x - y\|^2.\end{aligned}$$

This means that to find the solution of integral equation (5.10) is reduced to finding a fixed point of the nonexpansive mapping T in the Hilbert space $L^2[0,1]$.

Initiating with any function $y_0 \in L^2[0,1]$, define a sequence of functions $\{y_n\}$ in $L^2[0,1]$ by

$$y_{n+1} = \alpha_n Q(y_n) + \beta_n y_n + \gamma_n T\left(\frac{y_n + y_{n+1}}{2}\right), \quad n \ge 0.$$

Then the sequence $\{y_n\}$ converges strongly in $L^2[0,1]$ to the solution of integral equation (5.10) under one of the following conditions: (C1), (C2), (C4), and (C7) or (C1), (C2), (C3), and (C6) or (C1), (C2), (C8), and (C9).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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