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# Further generalized contraction mapping principle and best proximity theorem in metric spaces

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**Abstract**
The aim of this paper is to prove a more generalized contraction mapping principle. By using this more generalized contraction mapping principle, a further generalized best proximity theorem was established. Some concrete results have been derived by using the above two theorems. The results of this paper improve many important results published recently in the literature.

**Keywords:** contraction mapping principle; complete metric spaces; fixed point; best proximity theorem

## 1 Introduction

The Banach contraction mapping principle is a classical and powerful tool in nonlinear analysis. Weak contractions are generalizations of Banach contraction mapping which have been studied by several authors, and in particular some types of weak contractions in complete metric spaces were introduced in [1]. Let $T$ be a self-map of a metric space $(X,d)$ and $\phi:[0,+\infty)\to[0,+\infty)$ be a function. We say that $T$ is a $\phi$-contraction if

$$d(Tx,Ty)\le\phi\bigl(d(x,y)\bigr),\quad \forall x,y\in X.$$

In 1968, Browder [2] proved that if $\phi$ is nondecreasing and right continuous, and $(X,d)$ is complete, then $T$ has a unique fixed point $x^*$ and $\lim_{n\to+\infty}T^nx_0=x^*$ for any given $x_0\in X$. Subsequently, his result was extended in 1969 by Boyd and Wong [3] by weakening the hypothesis on $\phi$ where it suffices to assume that $\phi$ is right upper semi-continuous (not necessarily monotonic). For a comprehensive study of relations between several contractive conditions, see [4, 5].

In 1973, Geraghty [6] introduced the Geraghty-contraction and obtained the fixed point theorem.

**Definition 1.1** Let $(X,d)$ be a metric space. A mapping $T:X\to X$ is said to be a Geraghty-contraction if there exists $\beta\in\Gamma$ such that for any $x,y\in X$,

$$d(Tx,Ty)\le\beta\bigl(d(x,y)\bigr)d(x,y),$$

where the class $\Gamma$ denotes those functions $\beta : [0,+\infty) \to [0,+\infty)$ satisfying the following condition: $\beta(t_n) \to 1 \Rightarrow t_n \to 0$.

**Theorem 1.2** ([6]) *Let $(X,d)$ be a complete metric space and $T : X \to X$ be a Geraghty-contraction. Then $T$ has a unique fixed point $x^*$ and for any given $x_0 \in X$, the iterative sequence $T^n x_0$ converges to $x^*$.*

In 2012, Samet *et al.* [7] defined the notion of $\alpha$-admissible mappings as follows.

**Definition 1.3** ([7]) Let $\alpha : X \times X \to [0,+\infty)$ be a function. We say that a self-mapping $T : X \to X$ is $\alpha$-admissible if

$$x, y \in X, \quad \alpha(x,y) \geq 1 \quad \Rightarrow \quad \alpha(Tx,Ty) \geq 1.$$

By using this concept, they proved some fixed point results.

**Theorem 1.4** ([7]) *Let $(X,d)$ be a complete metric space and $T : X \to X$ be an $\alpha$-admissible mapping. Assume that the following conditions hold*:

(i) *for all $x, y \in X$, we have*

$$\alpha(x,y)d(Tx,Ty) \leq \psi(x,y),$$

*where $\psi : [0,+\infty) \to [0,+\infty)$ is a nondecreasing function such that*

$$\sum_{n=1}^{+\infty} \psi^n(t) < +\infty, \quad \forall t > 0;$$

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*

(iii) *either $T$ is continuous or for any sequence $\{x_n\}$ in $X$ with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \geq 0$ and $x_n \to x$ as $n \to +\infty$, then $\alpha(x_n, x) \geq 1$.*

*Then $T$ has a fixed point.*

In particular, existence of a fixed point for weak contractions and generalized contractions was extended to partially ordered metric spaces in [1, 8–19]. Among them, some involve altering distance functions. Such functions were introduced by Khan *et al.* in [20], where they presented some fixed point theorems with the help of such functions. We recall the definition of altering distance function.

**Definition 1.5** An altering distance function is a function $\psi : [0,\infty) \to [0,\infty)$ which satisfies

(a) $\psi$ is continuous and nondecreasing;

(b) $\psi = 0$ if and only if $t = 0$.

Recently, Harjani and Sadarangani proved some fixed point theorems for weak contraction and generalized contractions in partially ordered metric spaces by using the altering distance function in [11, 19] respectively. Their results improve the theorems of [8].

**Theorem 1.6** ([11]) *Let $(X,\leq)$ be a partially ordered set and suppose that there exists a metric $d \in X$ such that $(X,d)$ is a complete metric space. Let $f : X \to X$ be a continuous and* *nondecreasing mapping such that*

$$d\big(f(x),f(y)\big) \leq d(x,y) - \psi\big(d(x,y)\big) \quad \text{for } x \geq y,$$

*where* $\psi : [0,\infty) \to [0,\infty)$ *is a continuous and nondecreasing function such that* $\psi$ *is positive in* $(0,\infty)$, $\psi(0) = 0$ *and* $\lim_{t\to\infty} \psi(t) = \infty$. *If there exists* $x_0 \in X$ *with* $x_0 \leq f(x_0)$, *then* $f$ *has a fixed point.*

**Theorem 1.7** ([19]) *Let* $(X,\leq)$ *be a partially ordered set and suppose that there exists a metric* $d \in X$ *such that* $(X,d)$ *is a complete metric space. Let* $f : X \to X$ *be a continuous and nondecreasing mapping such that*

$$\psi d\big(f(x),f(y)\big) \leq \psi\big(d(x,y)\big) - \phi\big(d(x,y)\big) \quad \text{for } x \geq y,$$

*where* $\psi$ *and* $\phi$ *are altering distance functions. If there exists* $x_0 \in X$ *with* $x_0 \leq f(x_0)$, *then* $f$ *has a fixed point.*

Subsequently, Amini-Harandi and Emami proved another fixed point theorem for contraction type maps in partially ordered metric spaces in [10]. The following class of functions is used in [10].

**Theorem 1.8** ([10]) *Let* $(X,\leq)$ *be a partially ordered set and suppose that there exists a metric* $d$ *such that* $(X,d)$ *is a complete metric space. Let* $f : X \to X$ *be an increasing mapping such that there exists an element* $x_0 \in X$ *with* $x_0 \leq f(x_0)$. *Suppose that there exists* $\beta \in \Gamma$ *such that*

$$d\big(f(x),f(y)\big) \leq \beta\big(d(x,y)\big)d(x,y) \quad \text{for each } x,y \in X \text{ with } x \geq y.$$

*Assume that either* $f$ *is continuous or* $X$ *is such that if an increasing sequence* $x_n \to x \in X$, *then* $x_n \leq x$, $\forall n$. *Besides, if for each* $x,y \in X$ *there exists* $z \in X$ *which is comparable to* $x$ *and* $y$, *then* $f$ *has a unique fixed point.*

In 2012, Yan *et al.* proved the following result.

**Theorem 1.9** ([1]) *Let* $X$ *be a partially ordered set and suppose that there exists a metric* $d$ *in* $x$ *such that* $(X,d)$ *is a complete metric space. Let* $T : X \to X$ *be a continuous and nondecreasing mapping such that*

$$\psi\big(d(Tx,Ty)\big) \leq \phi\big(d(x,y)\big), \quad \forall x \geq y,$$

*where* $\psi$ *is an altering distance function and* $\phi : [0,\infty) \to [0,\infty)$ *is a continuous function with the condition* $\psi(t) > \phi(t)$ *for all* $t > 0$. *If there exists* $x_0 \in X$ *such that* $x_0 \leq Tx_0$, *then* $T$ *has a fixed point.*

Several problems can be changed as equations of the form $Tx = x$, where $T$ is a given self-mapping defined on a subset of a metric space, a normed linear space, a topological vector space or some suitable space. However, if $T$ is a non-self mapping from $A$ to $B$, then the aforementioned equation does not necessarily admit a solution. In this case, it is worth consideration to find an approximate solution $x$ in $A$ such that the error $d(x, Tx)$ is minimum, where $d$ is the distance function. In view of the fact that $d(x, Tx)$ is at least $d(A,B)$, a best proximity point theorem (for short BPPT) guarantees the global minimization of $d(x, Tx)$ by the requirement that an approximate solution $x$ satisfies the condition $d(x, Tx) = d(A,B)$. Such optimal approximate solutions are called best proximity points of the mapping $T$. Interestingly, best proximity point theorems also serve as a natural generalization of fixed point theorems since a best proximity point becomes a fixed point if the mapping under consideration is a self mapping. Research on the best proximity point is an important topic in the nonlinear functional analysis and applications (see [21–34]).

Let $A$, $B$ be two nonempty subsets of a complete metric space and consider a mapping $T : A \to B$. The best proximity point problem is whether we can find an element $x_0 \in A$ such that $d(x_0, Tx_0) = \min\{d(x, Tx) : x \in A\}$. Since $d(x, Tx) \geq d(A,B)$ for any $x \in A$, in fact, the optimal solution to this problem is the one for which the value $d(A,B)$ is attained.

Let $A$, $B$ be two nonempty subsets of a metric space $(X,d)$. We denote by $A_0$ and $B_0$ the following sets:

$$A_0 = \{x \in A : d(x,y) = d(A,B) \text{ for some } y \in B\},$$
$$B_0 = \{y \in B : d(x,y) = d(A,B) \text{ for some } x \in A\},$$

where $d(A,B) = \inf\{d(x,y) : x \in A \text{ and } y \in B\}$.

It is interesting to note that $A_0$ and $B_0$ are contained in the boundaries of $A$ and $B$ respectively provided $A$ and $B$ are closed subsets of a normed linear space such that $d(A,B) > 0$ [28, 29].

**Definition 1.10** ([33]) Let $(A,B)$ be a pair of nonempty subsets of a metric space $(X,d)$ with $A_0 \neq \emptyset$. Then the pair $(A,B)$ is said to have the *P-property* if and only if for any $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$,

$$\begin{cases} d(x_1,y_1) = d(A,B), \\ d(x_2,y_2) = d(A,B) \end{cases} \quad \Rightarrow \quad d(x_1,x_2) = d(y_1,y_2).$$

In [14], the authors proved that any pair $(A,B)$ of nonempty closed convex subsets of a real Hilbert space $H$ satisfies the $P$-property.

In [28], $P$-property was weakened to weak $P$-property and an example satisfying $P$-property but not weak $P$-property can be found there.

**Definition 1.11** ([28]) Let $(A,B)$ be a pair of nonempty subsets of a metric space $(X,d)$ with $A_0 \neq \emptyset$. Then the pair $(A,B)$ is said to have the *weak P-property* if and only if for any $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$,

$$\begin{cases} d(x_1,y_1) = d(A,B), \\ d(x_2,y_2) = d(A,B) \end{cases} \quad \Rightarrow \quad d(x_1,x_2) \leq d(y_1,y_2).$$

**Example** ([28]) Consider $(R^2, d)$, where $d$ is the Euclidean distance and the subsets $A = \{(0,0)\}$ and $B = \{y = 1 + \sqrt{1 - x^2}\}$.

Obviously, $A_0 = \{(0,0)\}$, $B_0 = \{(-1,1),(1,1)\}$ and $d(A,B) = \sqrt{2}$. Furthermore,

$$d\big((0,0),(-1,1)\big) = d\big((0,0),(1,1)\big) = \sqrt{2};$$

however,

$$0 = d\big((0,0),(0,0)\big) < d\big((-1,1),(1,1)\big) = 2.$$

We can see that the pair $(A,B)$ satisfies the weak $P$-property but not the $P$-property.

**Definition 1.12** ([34]) Let $(A,B)$ be a pair of nonempty closed subsets of a complete metric space $(X,d)$. A mapping $f : A \to B$ is said to be *weakly contractive* provided that

$$d\big(f(x),f(y)\big) \leq \bar{\alpha}(x,y)d(x,y)$$

for all $x,y \in A$, where the function $\bar{\alpha} : X \times X \to [0,1)$ holds, for every $0 < a < b$, that

$$\theta(a,b) = \sup\big\{\bar{\alpha}(x,y) : a \leq d(x,y) \leq b\big\} < 1.$$

**Theorem 1.13** ([34]) *Let $(A,B)$ be a pair of nonempty closed subsets of a complete metric space $(X,d)$ such that $A_0 \neq \emptyset$. Let $T : A \to B$ be a weakly contractive mapping defined as in Definition* 1.12. *Suppose that $T(A_0) \subseteq B_0$ and the pair $(A,B)$ has the weak $P$-property. Then $T$ has a unique best proximity point $x^* \in A_0$ and the iteration sequence $\{x_{2k}\}_{n=0}^{\infty}$ defined by*

$$x_{2k+1} = Tx_{2k}, \qquad d(x_{2k+2},x_{2k+1}) = d(A,B), \quad k = 0,1,2,\ldots$$

*converges, for every $x_0 \in A_0$, to $x^*$.*

The aim of this paper is to prove a further generalized contraction mapping principle. By using this further generalized contraction mapping principle, the authors prove a further generalized best proximity theorem. Some concrete results are derived by using the above two theorems. The results of this paper modify and improve many other important recent results.

## 2 Further generalized contraction mapping principle

In what follows, we prove the following theorem which generalizes many related results in the literature.

**Theorem 2.1** *Let $(X,d)$ be a complete metric space. Let $T : X \to X$ be a mapping such that*

$$\psi\big(d(Tx,Ty)\big) \leq \phi\big(d(x,y)\big), \quad \forall x,y \in X, \tag{2.1}$$

*where $\psi,\phi : [0,+\infty) \to [0,+\infty)$ are two functions with the conditions*:

(1) $\psi(a) \leq \phi(b) \quad \Rightarrow \quad a \leq b$;

(2) $\begin{cases} \psi(a_n) \leq \phi(b_n), \\ a_n \to \varepsilon, \qquad b_n \to \varepsilon \end{cases} \quad \Rightarrow \quad \varepsilon = 0.$

*Then $T$ has a unique fixed point and for any given $x_0 \in X$, the iterative sequence $T^n x_0$ converges to this fixed point.*

*Proof* For any given $x_0 \in X$, we define an iterative sequence as follows:

$$x_1 = Tx_0, \qquad x_2 = Tx_1, \qquad \ldots, \qquad x_{n+1} = Tx_n, \qquad \ldots. \tag{2.2}$$

Then, for each integer $n \geq 1$, from (2.1) and (2.2) we get

$$\psi\big(d(x_{n+1}, x_n)\big) = \psi\big(d(Tx_n, Tx_{n-1})\big) \leq \phi\big(d(x_n, x_{n-1})\big). \tag{2.3}$$

Using condition (1) we have

$$d(x_{n+1}, x_n) \leq d(x_n, x_{n-1})$$

for all $n \geq 1$. Hence the sequence $d(x_{n+1}, x_n)$ is nonincreasing and, consequently, there exists $r \geq 0$ such that

$$d(x_{n+1}, x_n) \to r$$

as $n \to \infty$. By using condition (2) we know $r = 0$.

In what follows, we show that $\{x_n\}$ is a Cauchy sequence. Suppose that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find subsequences $\{x_{n_k}\}$, $\{x_{m_k}\}$ with $n_k > m_k > k$ such that

$$d(x_{n_k}, x_{m_k}) \geq \varepsilon \tag{2.4}$$

for all $k \geq 1$. Further, corresponding to $m_k$ we can choose $n_k$ in such a way that it is the smallest integer with $n_k > m_k$ satisfying (2.4). Then

$$d(x_{n_k-1}, x_{m_k}) < \varepsilon. \tag{2.5}$$

From (2.4) and (2.5), we have

$$\varepsilon \leq d(x_{n_k}, x_{m_k}) \leq d(x_{n_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{m_k}) < d(x_{n_k}, x_{n_k-1}) + \varepsilon.$$

Letting $k \to \infty$, we get

$$\lim_{k\to\infty} d(x_{n_k}, x_{m_k}) = \varepsilon. \tag{2.6}$$

By using the triangular inequality, we have

$$d(x_{n_k}, x_{m_k}) \leq d(x_{n_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{m_k-1}) + d(x_{m_k-1}, x_{m_k}),$$
$$d(x_{n_k-1}, x_{m_k-1}) \leq d(x_{n_k-1}, x_{n_k}) + d(x_{n_k}, x_{m_k}) + d(x_{m_k}, x_{m_k-1}).$$

Letting $k \to \infty$ in the above two inequalities and applying (2.6), we have

$$\lim_{k\to\infty} d(x_{n_k-1}, x_{m_k-1}) = \varepsilon.$$

Since

$$\psi\bigl(d(x_{n_k}, x_{m_k})\bigr) \le \phi\bigl(d(x_{n_k-1}, x_{m_k-1})\bigr).$$

By using condition (2) we know $\varepsilon = 0$, which is a contradiction. This shows that $\{x_n\}$ is a Cauchy sequence and, since $X$ is a complete metric space, there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$. From (2.1) and (2.2) we have

$$\psi\bigl(d(x_n, Tz)\bigr) \le \phi\bigl(d(x_{n-1}, z)\bigr).$$

By using condition (1) we get

$$d(x_n, Tz) \le d(x_{n-1}, z),$$

so that $d(x_n, Tz) \to 0$, as $n \to +\infty$. Therefore

$$d(z, Tz) \le d(x_n, z) + d(x_n, Tz) \to 0$$

as $n \to +\infty$. This implies $z = Tz$ and proves that $z$ is a fixed point. Next we prove the uniqueness of the fixed point. Assume that there exist two fixed points $z$ and $w$. Then from (2.1) we have that

$$\psi\bigl(d(z, w)\bigr) = \psi\bigl(d(Tz, Tw)\bigr) \le \phi\bigl(d(z, w)\bigr),$$

by using condition (2) we know $d(z, w) = 0$ and hence $z = w$. This completes the proof. □

**Example 2.2** The following functions satisfy conditions (1) and (2) of Theorem 2.1:

$$\text{(a)} \quad \begin{cases} \psi_1(t) = t, \\ \phi_1(t) = \alpha t, \end{cases}$$

where $0 < \alpha < 1$ is a constant;

$$\text{(b)} \quad \begin{cases} \psi_2(t) = t^2, \\ \phi_2(t) = \ln(t^2 + 1); \end{cases}$$

$$\text{(c)} \quad \begin{cases} \psi_3(t) = t, \\ \phi_3(t) = \begin{cases} t^2, & 0 \le t \le \frac{1}{2}, \\ t - \frac{3}{8}, & \frac{1}{2} < t < +\infty; \end{cases} \end{cases}$$

$$\text{(d)} \quad \begin{cases} \psi_4(t) = \begin{cases} t, & 0 \le t \le 1, \\ t - \frac{1}{2}, & 1 < t < +\infty, \end{cases} \\ \phi_4(t) = \begin{cases} \frac{t}{2}, & 0 \le t \le 1, \\ t - \frac{4}{5}, & 1 < t < +\infty; \end{cases} \end{cases}$$

$$\text{(e)}\quad \begin{cases} \psi_5(t) = \begin{cases} t, & 0 \le t < 1, \\ \alpha t^2, & 1 \le t < +\infty, \end{cases} \\ \phi_5(t) = \begin{cases} t^2, & 0 \le t < 1, \\ \beta t, & 1 \le t < +\infty, \end{cases} \end{cases}$$

where $0 < \beta < \alpha$ are constants.

If we choose $\psi_5(t)$, $\phi_5(t)$ in Theorem 2.1, then we can get the following result.

**Theorem 2.3** *Let $(X, d)$ be a complete metric space. Let $T : X \to X$ be a mapping such that*

$$\begin{aligned} &0 \le d(Tx, Ty) < 1 \quad \Rightarrow \quad d(Tx, Ty) \le \bigl(d(x, y)\bigr)^2, \\ &d(Tx, Ty) \ge 1 \quad \Rightarrow \quad \alpha\bigl(d(Tx, Ty)\bigr)^2 \le \beta d(x, y) \end{aligned}$$

*for any $x, y \in X$. Then $T$ has a unique fixed point and for any given $x_0 \in X$, the iterative sequence $T^n x_0$ converges to this fixed point.*

If we choose $\psi_4(t)$, $\phi_4(t)$ in Theorem 2.1, then we can get the following result.

**Theorem 2.4** *Let $(X, d)$ be a complete metric space. Let $T : X \to X$ be a mapping such that*

$$\begin{aligned} &0 \le d(Tx, Ty) \le 1 \quad \Rightarrow \quad d(Tx, Ty) \le \frac{1}{2} d(x, y), \\ &1 < d(Tx, Ty) \quad \Rightarrow \quad d(Tx, Ty) \le d(x, y) - \frac{3}{10} \end{aligned}$$

*for any $x, y \in X$. Then $T$ has a unique fixed point and for any given $x_0 \in X$, the iterative sequence $T^n x_0$ converges to this fixed point.*

If we choose $\psi_3(t)$, $\phi_3(t)$ in Theorem 2.1, then we can get the following result.

**Theorem 2.5** *Let $(X, d)$ be a complete metric space. Let $T : X \to X$ be a mapping such that*

$$\begin{aligned} &0 \le d(x, y) \le \frac{1}{2} \quad \Rightarrow \quad d(Tx, Ty) \le \bigl(d(x, y)\bigr)^2, \\ &\frac{1}{2} < d(x, y) \quad \Rightarrow \quad d(Tx, Ty) \le d(x, y) - \frac{3}{8} \end{aligned}$$

*for any $x, y \in X$. Then $T$ has a unique fixed point and for any given $x_0 \in X$, the iterative sequence $T^n x_0$ converges to this fixed point.*

It is easy to prove the following conclusion and corollary.

**Corollary 2.6** *Let $\psi, \phi : [0, +\infty) \to [0, +\infty)$ be two functions with the conditions:*

(i) $\psi(0) = \phi(0)$;
(ii) $\psi(t) > \phi(t)$, $\forall t > 0$;
(iii) *$\psi$ is lower semi-continuous, $\phi$ is upper semi-continuous.*

*Then $\psi(t)$, $\phi(t)$ satisfy conditions* (1) *and* (2).

**Corollary 2.7** *Let $(X,d)$ be a complete metric space. Let $T : X \to X$ be a mapping such that*

$$\psi\bigl(d(Tx,Ty)\bigr) \le \phi\bigl(d(x,y)\bigr), \quad \forall x,y \in X,$$

*where $\psi,\phi : [0,+\infty) \to [0,+\infty)$ are two functions with the conditions* (i), (ii) *and* (iii). *Then $T$ has a unique fixed point, and for any given $x_0 \in X$, the iterative sequence $T^n x_0$ converges to this fixed point.*

## 3 Further generalized best proximity point theorems

Before giving our main results, we first introduce the notion of $(\varphi,\psi)$-$P$-property.

**Definition 3.1** Let $(A,B)$ be a pair of nonempty subsets of a metric space $(X,d)$ with $A_0 \neq \emptyset$. Then the pair $(A,B)$ is said to have the $(\psi,\varphi)$-$P$-property if and only if for any $x_1,x_2 \in A_0$ and $y_1,y_2 \in B_0$,

$$\begin{cases} d(x_1,y_1) = d(A,B) \\ d(x_2,y_2) = d(A,B) \end{cases} \quad \Rightarrow \quad \psi\bigl(d(x_1,x_2)\bigr) \le \varphi\bigl(d(y_1,y_2)\bigr),$$

where $\psi,\varphi : [0,+\infty) \to [0,+\infty)$ are two functions.

**Theorem 3.2** *Let $(A,B)$ be a pair of nonempty closed subsets of a complete metric space $(X,d)$ such that $A_0 \neq \emptyset$. Let $\psi,\varphi,\phi : [0,+\infty) \to [0,+\infty)$ be three functions with the conditions*:

(1) $\psi(a) \le \phi(b) \quad \Rightarrow \quad a \le b$;

(2) $\begin{cases} \psi(a_n) \le \phi(b_n), \\ a_n \to \varepsilon, \quad b_n \to \varepsilon \end{cases} \quad \Rightarrow \quad \varepsilon = 0$;

(3) $\psi(t_n) \to 0 \quad \Rightarrow \quad t_n \to 0$;

(4) $t_n \to 0 \quad \Rightarrow \quad \varphi(t_n) \to 0$;

(5) $\varphi(a) \le \phi(b) \quad \Rightarrow \quad a \le b$.

*Let $T : A \to B$ be a mapping such that*

$$\varphi\bigl(d(Tx,Ty)\bigr) \le \phi\bigl(d(x,y)\bigr), \quad \forall x,y \in A. \tag{3.1}$$

*Suppose that the pair $(A,B)$ has the $(\psi,\varphi)$-$P$-property and $T(A_0) \subseteq B_0$. Then there exists a unique $x^*$ in $A$ such that $d(x^*,Tx^*) = d(A,B)$.*

*Proof* We first prove that $B_0$ is closed. Let $\{y_n\} \subseteq B_0$ be a sequence such that $y_n \to q \in B$. It follows from the $(\psi,\varphi)$-$P$-property that

$$\varphi\bigl(d(y_n,y_m)\bigr) \to 0 \quad \Rightarrow \quad \psi\bigl(d(x_n,x_m)\bigr) \to 0$$

as $n,m \to \infty$, where $x_n,x_m \in A_0$ and $d(x_n,y_n) = d(A,B)$, $d(x_m,y_m) = d(A,B)$. This together with conditions (3) and (4) implies that $\{x_n\}$ is a Cauchy sequence so that $\{x_n\}$ converges strongly to a point $p \in A$. By the continuity of metric $d$ we have $d(p,q) = d(A,B)$, that is, $q \in B_0$, and hence $B_0$ is closed.

Let $\overline{A}_0$ be the closure of $A_0$. We claim that $T(\overline{A}_0) \subseteq B_0$. In fact, if $x \in \overline{A}_0 \setminus A_0$, then there exists a sequence $\{x_n\} \subseteq A_0$ such that $x_n \to x$. From (3.1) and condition (5) we have

$$d(Tx,Ty) \le d(x,y), \quad \forall x,y \in A.$$

This together with the closedness of $B_0$ implies that $Tx = \lim_{n\to\infty} Tx_n \in B_0$. That is, $T(\overline{A}_0) \subseteq B_0$.

Define an operator $P_{A_0} : T(\overline{A}_0) \to A_0$ by $P_{A_0}y = \{x \in A_0 : d(x,y) = d(A,B)\}$. Since the pair $(A,B)$ has the $(\psi,\varphi)$-$P$-property and $T$ satisfies condition (3.1), we have

$$\psi\big(d(P_{A_0}Tx_1, P_{A_0}Tx_2)\big) \le \varphi\big(d(Tx_1,Tx_2)\big) \le \phi\big(d(x_1,x_2)\big)$$

for any $x_1, x_2 \in \overline{A}_0$. This shows that $P_{A_0}T : \overline{A}_0 \to \overline{A}_0$ is a mapping from a complete metric subspace $\overline{A}_0$ into itself, and it satisfies the conditions of Theorem 2.1. By using Theorem 2.1, we can get that $P_{A_0}T$ has a unique fixed point $x^*$. That is, $P_{A_0}Tx^* = x^* \in A_0$. It implies that

$$d\big(x^*, Tx^*\big) = d(A,B).$$

Therefore, $x^*$ is the unique element in $A_0$ such that $d(x^*,Tx^*) = d(A,B)$. It is easy to see that $x^*$ is also the unique one in $A$ such that $d(x^*,Tx^*) = d(A,B)$. □

**Theorem 3.3** *Let $(A,B)$ be a pair of nonempty closed subsets of a complete metric space $(X,d)$ such that $A_0 \neq \emptyset$. Let $\psi, \phi : [0,+\infty) \to [0,+\infty)$ be two functions with the conditions*:

(1) $\psi(a) \le \phi(b) \quad \Rightarrow \quad a \le b$;

(2) $\begin{cases} \psi(a_n) \le \phi(b_n), \\ a_n \to \varepsilon, \qquad b_n \to \varepsilon \end{cases} \quad \Rightarrow \quad \varepsilon = 0;$

(3) $\psi(t_n) \to 0 \quad \Leftrightarrow \quad t_n \to 0$

*and $\psi(t)$ is nondecreasing. Let $T : A \to B$ be a mapping such that*

$$\psi\big(d(Tx,Ty)\big) \le \phi\big(d(x,y)\big), \quad \forall x,y \in A. \tag{3.2}$$

*Suppose that the pair $(A,B)$ has the weak $P$-property and $T(A_0) \subseteq B_0$. Then there exists a unique $x^*$ in $A$ such that $d(x^*,Tx^*) = d(A,B)$.*

*Proof* Let $\varphi(t) = \psi(t)$ for all $t \in [0,+\infty)$. Then the pair $(A,B)$ having the weak $P$-property implies that the pair $(A,B)$ has the $(\psi,\varphi)$-$P$-property. Condition (3) of Theorem 3.3 implies conditions (3), (4) of Theorem 3.2 and (3.2) implies (3.1). By using Theorem 3.2 we get the conclusion of Theorem 3.3. □

If we choose $\psi_3(t)$, $\phi_3(t)$ in Theorem 3.3, then we can get the following result.

**Theorem 3.4** *Let $(A,B)$ be a pair of nonempty closed subsets of a complete metric space $(X,d)$ such that $A_0 \neq \emptyset$. Let $T: A \to B$ be a mapping such that*

$$\begin{aligned} &0 \le d(x,y) \le \frac{1}{2} \quad \Rightarrow \quad d(Tx,Ty) \le \bigl(d(x,y)\bigr)^2, \\ &\frac{1}{2} < d(x,y) \quad \Rightarrow \quad d(Tx,Ty) \le d(x,y) - \frac{3}{8} \end{aligned}$$

*for any $x,y \in A$. Suppose that the pair $(A,B)$ has the weak P-property and $T(A_0) \subseteq B_0$. Then there exists a unique $x^*$ in $A$ such that $d(x^*,Tx^*) = d(A,B)$.*

If we choose $\psi_4(t)$, $\phi_4(t)$ in Theorem 3.3, then we can get the following result.

**Theorem 3.5** *Let $(A,B)$ be a pair of nonempty closed subsets of a complete metric space $(X,d)$ such that $A_0 \neq \emptyset$. Let $T: A \to B$ be a mapping such that*

$$\begin{aligned} &0 \le d(Tx,Ty) \le 1 \quad \Rightarrow \quad d(Tx,Ty) \le \frac{1}{2} d(x,y), \\ &1 < d(Tx,Ty) \quad \Rightarrow \quad d(Tx,Ty) \le d(x,y) - \frac{3}{10} \end{aligned}$$

*for any $x,y \in X$. Suppose that the pair $(A,B)$ has the weak P-property and $T(A_0) \subseteq B_0$. Then there exists a unique $x^*$ in $A$ such that $d(x^*,Tx^*) = d(A,B)$.*

**Example 3.6** Let $X = R^2$ and

$$\begin{aligned} &A = \bigl\{(0,y) \in R^2 : 0 \le y \le 1 \text{ or } y = -n^2, n = 2,3,\ldots\bigr\}, \\ &B = \left\{(1,y) \in R^2 : 0 \le y \le 1 \text{ or } -\infty < y \le -4 + \frac{3}{10}\right\}. \end{aligned}$$

It is easy to see that

$$\begin{aligned} &A_0 = \bigl\{(0,y) \in R^2 : 0 \le y \le 1\bigr\}, \\ &B_0 = \bigl\{(1,y) \in R^2 : 0 \le y \le 1\bigr\}. \end{aligned}$$

Let $T: A \to B$ be defined by

$$T(0,y) = \begin{cases} (1, \frac{y}{2}) & \text{if } 0 \le y \le 1, \\ (1, -n^2 + \frac{3n}{10}) & \text{if } y = -n^2, n = 2,3,\ldots. \end{cases}$$

It is obvious that $A$, $B$ are closed sets of $R^2$, $A_0 \neq \emptyset$, the pair $(A,B)$ has the weak $P$-property and $T(A_0) \subset B_0$. On the other hand, from the definition of $T$, it is not hard to see that

$$\begin{aligned} &0 \le \bigl|T(0,x) - T(0,y)\bigr| \le 1 \quad \Rightarrow \quad \bigl|T(0,x) - T(0,y)\bigr| = \frac{1}{2}\bigl|(0,x) - (0,y)\bigr|, \\ &1 < \bigl|T(0,x) - T(0,y)\bigr| \quad \Rightarrow \quad \bigl|T(0,x) - T(0,y)\bigr| \le \bigl|(0,x) - (0,y)\bigr| - \frac{3}{10} \end{aligned}$$

for all $(0,x),(0,y) \in A$. Therefore, by using Theorem 3.5, there exists a unique $x^* = (0,0)$ in $A$ such that $d(x^*, Tx^*) = d(A,B) = 1$, where $Tx^* = (1,0)$. Note that the above mapping $T$ is a further generalized contraction, but not a contraction. In fact, for any $(0,-n^2),(0,-m^2) \in A$, $m > n$, we have

$$\left|T\left(0,-n^2\right) - T\left(0,-m^2\right)\right| = m^2 - \frac{3m}{10} - n^2 + \frac{3n}{10},$$

hence

$$\frac{|T(0,-n^2) - T(0,-m^2)|}{|(0,-n^2) - (0,m^2)|} = 1 + \frac{3}{10}\frac{m-n}{(m+n)(m-n)} \to 1$$

as $m,n \to +\infty$.

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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