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On an extension of contractivity conditions via auxiliary functions

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Abstract

In this manuscript, we study sufficient conditions on the functions that appears in very complex contractivity conditions introduced in a recent manuscript by Liu *et al.* in order to guarantee the existence and uniqueness of common fixed points of four self-mappings.

Keywords: fixed point theorem; fixed point; common fixed point; contractivity condition

1 Introduction

Originally, a *Banach contractive mapping* [1] is a mapping $T : X \to X$ from a metric space (X, d) into itself for which there exists $\lambda \in [0, 1)$ such that

$$d(Tx, Ty) \le \lambda d(x, y) \quad \text{for all } x, y \in X.$$

In [2], Rhoades did a complete study about different notions of contractive mapping that were appearing after Banach's pioneering notion, including, among others, the following ones:

- Kannan [3], 1969,

$$d(Tx, Ty) \le \lambda \big(d(x, Tx) + d(y, Ty) \big), \quad \text{where } \lambda \in [0, 1/2).$$

- Reich [4], 1971,

$$d(Tx, Ty) \le ad(x, Tx) + bd(y, Ty) + cd(x, y), \quad \text{where } a + b + c < 1.$$

- Bianchini [5], 1972,

$$d(Tx, Ty) \le \lambda \max\big\{ d(x, Tx), d(y, Ty) \big\}, \quad \text{where } \lambda \in [0, 1).$$

- Chatterjea [6], 1972,

$$d(Tx, Ty) \le \lambda \big(d(x, Ty) + d(y, Tx) \big), \quad \text{where } \lambda \in [0, 1/2).$$

- Sehgal [7], 1972,

$$d(Tx, Ty) < \max\{d(x, Tx), d(y, Ty), d(x, y)\}.$$

- Hardy and Rogers [8], 1973,

$$d(Tx, Ty) \le a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty) + a_4 d(x, Ty) + a_5 d(y, Tx),$$

  where $a_1 + a_2 + a_3 + a_1 + a_5 < 1$.

- Ćirić [9], 1974,

$$d(Tx, Ty) \le \lambda \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\},$$

  where $\lambda \in [0, 1)$.

Later, some auxiliary functions were appearing in the contractivity condition. To cite some of them, the following ones are well-known contractivity conditions.

- Boyd and Wong [10], 1969,

$$d(Tx, Ty) \le \varphi\big(d(x, y)\big).$$

- Rhoades [11], 2001,

$$d(Tx, Ty) \le d(x, y) - \phi\big(d(x, y)\big).$$

- Dutta and Choudhury [12], 2008,

$$\psi\big(d(Tx, Ty)\big) \le \psi\big(d(x, y)\big) - \phi\big(d(x, y)\big).$$

- Aydi *et al.* [13], 2013,

$$\psi\big(d(Tx, Ty)\big) \le \psi\big(M(x, y)\big) - \phi\big(M(x, y)\big) + LN(x, y),$$

  where $L \ge 0$,

$$M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right\} \quad \text{and}$$

$$N(x, y) = \max\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}.$$

- Berzig *et al.* [14], 2014,

$$\psi\big(d(Tx, Ty)\big) \le \alpha(x, y)\psi\big(M(x, y)\big) - \beta(x, y)\varphi\big(M(x, y)\big),$$

  where $M(x, y)$ is given by one of the following cases:

  - $M_1(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \dfrac{d(x, Ty) + d(y, Tx)}{2}\right\}$ (type I);

- $M_2(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}$ (type II);

- $M_3(x, y) = \max\left\{d(x, y), \dfrac{d(x, Tx) + d(y, Ty)}{2}, \dfrac{d(x, Ty) + d(y, Tx)}{2}\right\}$ (type III);

- $M_4(x, y) = \max\left\{d(x, y), \dfrac{d(x, Tx) + d(y, Ty)}{2}\right\}$ (type IV);

- $M_5(x, y) = d(x, y)$ (type V).

After the appearance of Rhoades' theorem (see [11]), many results have been introduced involving contractivity conditions in which some families of functions play a key role, even in partially ordered metric spaces (see, for instance, [12, 14–33]). As we can observe, each time, more and more terms have been included in the right-hand side of the contractivity inequality. As a consequence, new results are getting better.

In a very recent manuscript, Liu *et al.* [34] introduced common fixed point theorems for self-mappings $A, B, T, S : X \to X$ involving some very complex contractivity conditions, as the following one.

Theorem 1 (Liu *et al.* [34], Theorem 2.1) *Let A, B, S, and T be self-mappings in a metric space (X, d) such that*

$\{A, T\}$ *and* $\{B, S\}$ *are weakly compatible*;

$T(X) \subseteq B(X)$ *and* $S(X) \subseteq A(X)$;

one of $A(X), B(X), S(X),$ *and* $T(X)$ *is complete*;

$$d(Tx, Sy) \le \psi\big(M_1(x, y)\big), \quad \forall x, y \in X, \tag{1}$$

where ψ_1 is in Φ_3 and M_1 is defined, for all $x, y \in X$, by

$$
\begin{aligned}
M_1(x, y) = \ \max\bigg\{ & d(Ax, By), d(Ax, Tx), d(By, Sy), \frac{d(Ax, Sy) + d(Tx, By)}{2}, \\
& \frac{d(Ax, Sy)d(Tx, By)}{1 + d(Ax, By)}, \frac{d(Ax, Tx)d(By, Sy)}{1 + d(Ax, By)}, \\
& \frac{1 + d(Ax, Sy) + d(Tx, By)}{1 + d(Ax, Tx) + d(By, Sy)} d(Ax, Tx) \bigg\}.
\end{aligned}
\tag{2}
$$

Then A, B, S, and T have a unique common fixed point in X.

Other statements were proved in the mentioned paper including similar contractivity conditions involving functions $M_2, M_3 : X^2 \to [0, \infty)$ as follows:

$$
\begin{aligned}
M_2(x, y) = \ \max\bigg\{ & d(Ax, By), d(Ax, Tx), d(By, Sy), \frac{d(Ax, Sy) + d(Tx, By)}{2}, \\
& \frac{1 + d(Ax, Tx)}{1 + d(Ax, By)} d(By, Sy), \frac{1 + d(By, Sy)}{1 + d(Ax, By)} d(Ax, Tx), \\
& \frac{1 + d(Ax, Sy) + d(Tx, By)}{1 + d(Ax, Tx) + d(By, Sy)} d(By, Sy) \bigg\} \quad \text{and}
\end{aligned}
$$

$$M_3(x,y) = \max\left\{d(Ax,By), d(Ax,Tx), d(By,Sy), \frac{d(Ax,Sy) + d(Tx,By)}{2},\right.$$
$$\left.\frac{d(Ax,Sy)d(Tx,By)}{1 + d(Ax,By)}\right\}.$$

Obviously, in the future, new contractivity conditions will appear involving more and more terms in the right-hand side of the contractivity inequality. It is not our purpose to complicate such situation. On the contrary, *the main aim of the present manuscript is to study what kind of functions might we include in an efficient contractivity condition depending on the terms we wish to consider* (like $d(Ax,By)$, $d(Ax,Tx)$, $d(By,Sy)$, etc.). To do that, we present some families of very general functions that we can use in a contractivity condition so that Theorem 1 remains true. In other words, for instance, we illustrate what kind of functions we could consider involving the terms $d(Ax,Sy)$ and $d(Tx,By)$. Thus, we show how we can replace $(d(Ax,Sy) + d(Tx,By))/2$ in (2) by another term $\phi_1(d(Ax,Sy), d(Tx,By))$ considering a function ϕ_1, which generalizes the particular case $\phi(t,s) = (t + s)/2$. It is our purpose that, in the future, when some authors wish to include new terms in the contractivity conditions, they first ponder on what the sufficient conditions are on the functions involved in the arguments of the contractivity condition in order for their main statements to remain true.

2 Preliminaries

We will follow notations given in [34]. Throughout this paper, $\mathbb{N}$ denotes the set of all positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and X stands for a nonempty set. Given $n \in \mathbb{N}$, X^n will denote the Cartesian space $X \times X \times \overset{(n)}{\cdots} \times X$ of n identical copies of X.

Definition 2 A *fixed point* of a self-mapping $T : X \to X$ is a point $x \in X$ such that $Tx = x$. A *coincidence point of two or more operators* $T_1, T_2, \ldots, T_n : X \to X$ is a point $x \in X$ such that $T_1x = T_2x = \cdots = T_nx$. A *common fixed point of two or more operators* $T_1, T_2, \ldots, T_n : X \to X$ is a point $x \in X$ such that $T_1x = T_2x = \cdots = T_nx = x$.

Definition 3 Given a nonempty set X, a pair $\{T, g\}$ of self-mappings $T, g : X \to X$ are said to be *weakly compatible* if $Tgx = gTx$ for all $x \in X$ such that $Tx = gx$.

Let Φ_1, Φ_2, and Φ_3 the following families of control functions:

$$\Phi_1 = \left\{\phi : [0, \infty) \to [0, \infty) : \phi \text{ is continuous, nondecreasing and } \phi^{-1}(\{0\}) = \{0\}\right\},$$

$$\Phi_2 = \left\{\phi : [0, \infty) \to [0, \infty) : \phi \text{ is lower semi-continuous and } \phi^{-1}(\{0\}) = \{0\}\right\}, \quad \text{and}$$

$$\Phi_3 = \left\{\phi : [0, \infty) \to [0, \infty) : \phi \text{ is upper semi-continuous, and} \right.$$

$$\left. \lim_{n \to \infty} a_n = 0 \text{ for each sequence } \{a_n\}_{n \in \mathbb{N}} \subset [0, \infty) \text{ with } a_{n+1} \le \phi(a_n), \forall n \in \mathbb{N}\right\}.$$

The functions in Φ_1 are known as *altering distance functions* (see [35]).

Lemma 4 ([34], Lemma 1.1) *If $\phi \in \Phi_3$, then $\phi(0) = 0$ and $\phi(t) < t$ for all $t > 0$.*

Henceforth, (X, d) stands for a metric space.

Proposition 5 *Let (X, d) be a metric space and let $\{x_n\} \subseteq X$ be a sequence such that* $\lim_{n \to \infty} d(x_n, x_{n+1}) = 0$. *If $\{x_n\}$ is not a Cauchy sequence, then $\{x_{2n}\}$ is not a Cauchy sequence. In particular, there exist $\varepsilon_0 > 0$ and two subsequences $\{x_{2n(k)}\}$ and $\{x_{2m(k)}\}$ of $\{x_{2n}\}$ such that*

$$2k < 2n(k) < 2m(k), \quad d(x_{2n(k)}, x_{2m(k)-2}) \leq \varepsilon_0 < d(x_{2n(k)}, x_{2m(k)}) \quad \textit{for all } k \in \mathbb{N},$$

$$\lim_{n \to \infty} d(x_{2n(k)}, x_{2m(k)}) = \lim_{n \to \infty} d(x_{2n(k)}, x_{2m(k)-1}) = \lim_{n \to \infty} d(x_{2n(k)}, x_{2m(k)+1})$$

$$= \lim_{n \to \infty} d(x_{2n(k)-1}, x_{2m(k)-1}) = \lim_{n \to \infty} d(x_{2n(k)-1}, x_{2m(k)+1}) = \varepsilon_0.$$

Proof By contradiction, assume that $\{x_{2n}\}$ is a Cauchy sequence. Let $\varepsilon > 0$ be arbitrary. Since $\{d(x_n, x_{n+1})\} \to 0$, there exists $n_1 \in \mathbb{N}$ such that

$$d(x_n, x_{n+1}) \leq \frac{\varepsilon}{3} \quad \text{for all } n \geq n_1.$$

As $\{x_{2n}\}$ is a Cauchy sequence, there exists $n_2 \in \mathbb{N}$ such that

$$d(x_{2n}, x_{2m}) \leq \frac{\varepsilon}{3} \quad \text{for all } m, n \in \mathbb{N} \text{ such that } 2m \geq 2n \geq n_2.$$

Let $n_0 = \max\{n_1, n_2\}$ and let $n, m \in \mathbb{N}$ be such that $m > n > n_0$. Let define

$$p = \begin{cases} n/2, & \text{if } n \text{ is even,} \\ (n+1)/2, & \text{if } n \text{ is odd;} \end{cases} \qquad q = \begin{cases} m/2, & \text{if } m \text{ is even,} \\ (m+1)/2, & \text{if } m \text{ is odd.} \end{cases}$$

Then $2p = n$ if n is even and $2p = n + 1$ if n is odd. Similarly, $2q = m$ if m is even and $2q = m + 1$ if m is odd. Since $2q \geq 2p \geq n_2$,

$$d(x_n, x_m) \leq d(x_n, x_{2p}) + d(x_{2p}, x_{2q}) + d(x_{2q}, x_m) \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

This means that $\{x_n\}$ is a Cauchy sequence, which is false. As a consequence, $\{x_{2n}\}$ cannot be a Cauchy sequence. $\qquad \square$

3 Common fixed point theorems in metric spaces

For convenience, we introduce the following families of functions. Let F_1 be the family of all functions $\phi : [0, \infty)^n \to [0, \infty)$, with $n \geq 2$, such that, for all $r, z_3, z_4, \ldots, z_n \in [0, \infty)$:

(F_{11}) ϕ is continuous on its first two arguments;
(F_{12}) $\phi(0, r, z_3, z_4, \ldots, z_n) \leq \frac{r}{2}$;
(F_{13}) $\phi(r, r, z_3, z_4, \ldots, z_n) \leq r$ for all $r > 0$;
(F_{14}) $\phi(r, 0, z_3, z_4, \ldots, z_n) \leq r$ for all $r > 0$.

Examples of functions in F_1 are the following ones:

$$\phi(r, s) = \frac{r + s}{2};$$

$$\phi(r, s) = \max\left\{r, \frac{s}{2}\right\};$$

$$\phi(r,s) = \max\left\{r, \frac{r+s}{2}\right\};$$

$$\phi(r,s,t) = \frac{1}{1+t^2}\frac{r+s}{2};$$

$$\phi(r,s,t,u) = \frac{e^{-u^2}}{1+t^4}\max\left\{r, \frac{r+s}{2}\right\}.$$

Let F_2 be the family of all functions $\phi : [0,\infty)^n \to [0,\infty)$, with $n \geq 3$, such that, for all $r,s,z_4,z_5,\ldots,z_n \in [0,\infty)$:

(F_{21}) ϕ is continuous on its first three arguments;

(F_{22}) $\phi(0,r,s,z_4,z_5,\ldots,z_n) \leq \max\{\frac{r}{2},s\}$;

(F_{23}) $\phi(r,r,r,z_4,z_5,\ldots,z_n) \leq r$ for all $r > 0$;

(F_{24}) $\phi(r,0,0,z_4,z_5,\ldots,z_n) \leq r$ for all $r > 0$.

Examples of functions in F_2 are the following ones:

$$\phi(r,s,t) = \frac{rs}{L+t} \quad \text{where } L > 0;$$

$$\phi(r,s,t) = \frac{rs}{L+t+e^t} \quad \text{where } L > 0;$$

$$\phi(r,t,s,z_1,z_2) = \max\{r,s,t\}.$$

Let F_3 be the family of all functions $\phi : [0,\infty)^n \to [0,\infty)$, with $n \geq 3$, such that, for all $r,s,z_4,z_5,\ldots,z_n \in [0,\infty)$:

(F_{31}) ϕ is continuous on its first three arguments;

(F_{32}) $\phi(r,s,s,z_4,z_5,\ldots,z_n) \leq \max\{r,s\}$;

(F_{33}) $\phi(0,0,r,z_4,z_5,\ldots,z_n) \leq r$ for all $r > 0$;

(F_{34}) $\phi(0,r,0,z_4,z_5,\ldots,z_n) \leq r$ for all $r > 0$.

For instance,

$$\phi(r,s,t) = \frac{rs}{L+t} \quad \text{where } L > 0,$$

$$\phi(r,t,s,z_1,z_2) = \max\{r,s,t\}.$$

Let F_4 be the family of all functions $\phi : [0,\infty)^n \to [0,\infty)$, with $n \geq 4$, such that, for all $r,s,t,u,z_3,z_4,\ldots,z_n \in [0,\infty)$:

(F_{41}) ϕ is continuous on its first four arguments;

(F_{42}) $\phi(0,r,s,t,z_5,z_6,\ldots,z_n) \leq \max\{\frac{r}{2},s,t\}$;

(F_{43}) $\phi(r,r,0,0,z_5,z_6,\ldots,z_n) \leq r$ for all $r > 0$;

(F_{44}) $\phi(r,0,0,r,z_5,z_6,\ldots,z_n) \leq r$ for all $r > 0$.

For instance,

$$\phi(r,s,t,u) = \frac{L+r+s}{L+t+u}t \quad \text{where } L > 0;$$

$$\phi(r,s,t,u) = \max\left\{\frac{r+s}{2},t,u\right\};$$

$$\phi(r,s,t,u,z_1,z_2) = \frac{\arctan(z_1)}{2 + z_2^2} \max\left\{\frac{r+s}{2}, t, u\right\}.$$

The main theorem of the present manuscript is the following one. Notice that we assume that ϕ_1 has two arguments, ϕ_2 and ϕ_3 have three arguments and ϕ_4 has four arguments because other arguments are not important through the proof.

Theorem 6 *Let $A, B, S, T : X \to X$ be self-mappings in a metric space (X, d) such that*

    (a) $\{A, T\}$ *and* $\{B, S\}$ *are weakly compatible;*

    (b) $T(X) \subseteq B(X)$ *and* $S(X) \subseteq A(X)$;

    (c)   *one of* $A(X), B(X), S(X),$ *and* $T(X)$ *is complete.*

Also assume that there exist $\phi_1 \in \mathsf{F}_1, \phi_2 \in \mathsf{F}_2, \phi_3 \in \mathsf{F}_3, \phi_4 \in \mathsf{F}_4,$ *and* $\psi \in \Phi_3$ *such that*

$$d(Tx, Sy) \leq \psi\big(M(x,y)\big) \quad \text{for all } x, y \in X, \tag{3}$$

where

$$\begin{aligned}
M(x,y) = \ &\max\big\{ d(Ax, By), d(Ax, Tx), d(By, Sy), \phi_1\big(d(Ax, Sy), d(Tx, By)\big), \\
&\phi_2\big(d(Ax, Sy), d(Tx, By), d(Ax, By)\big), \phi_3\big(d(Ax, Tx), d(By, Sy), d(Ax, By)\big) \\
&\phi_4\big(d(Ax, Sy), d(Tx, By), d(Ax, Tx), d(By, Sy)\big)\big\}.
\end{aligned} \tag{4}$$

Then A, B, S, and T have a unique common fixed point in X.

Proof Let $x_0 \in X$ be an arbitrary point and let $y_1 = Tx_0$. Since $Tx_0 \in T(X) \subseteq B(X)$, there exists $x_1 \in X$ such that $y_1 = Tx_0 = Bx_1$. Let $y_2 = Sx_1$. Since $Sx_1 \in S(X) \subseteq A(X)$, there exists $x_2 \in X$ such that $y_2 = Sx_1 = Ax_2$. Let $y_3 = Tx_2$. Repeating again and again this process, we can find two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$y_{2n+1} = Bx_{2n+1} = Tx_{2n} \quad \text{and} \quad y_{2n+2} = Ax_{2n+2} = Sx_{2n+1} \quad \text{for all } n \in \mathbb{N}_0. \tag{5}$$

Let define $d_n = d(y_n, y_{n+1})$ for all $n \in \mathbb{N}_0$. We claim that $\lim_{n\to\infty} d_n = 0$. Indeed, by (3) and (5), for all $n \in \mathbb{N}$,

$$d_{2n} = d(y_{2n}, y_{2n+1}) = d(y_{2n+1}, y_{2n}) = d(Tx_{2n}, Sx_{2n-1}) \leq \psi\big(M(x_{2n}, x_{2n-1})\big). \tag{6}$$

Taking into account the properties that define functions in $\{\mathsf{F}_i\}_{i=1}^4$, we deduce that

$$d(Ax_{2n}, Bx_{2n-1}) = d(y_{2n}, y_{2n-1}) = d(y_{2n-1}, y_{2n}) = d_{2n-1};$$

$$d(Ax_{2n}, Tx_{2n}) = d(y_{2n}, y_{2n+1}) = d_{2n};$$

$$d(Bx_{2n-1}, Sx_{2n-1}) = d(y_{2n-1}, y_{2n}) = d_{2n-1};$$

$$\phi_1\big(d(Ax_{2n}, Sx_{2n-1}), d(Tx_{2n}, Bx_{2n-1})\big)$$

$$\quad = \phi_1\big(d(y_{2n}, y_{2n}), d(y_{2n+1}, y_{2n-1})\big)$$

$$= \phi_1\big(0, d(y_{2n-1}, y_{2n+1})\big) \overset{(F_{12})}{\leq} \frac{d(y_{2n-1}, y_{2n+1})}{2}$$

$$\leq \frac{d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})}{2}$$

$$\leq \max\big\{d(y_{2n+1}, y_{2n}), d(y_{2n}, y_{2n-1})\big\} = \max\{d_{2n-1}, d_{2n}\};$$

$$\phi_2\big(d(Ax_{2n}, Sx_{2n-1}), d(Tx_{2n}, Bx_{2n-1}), d(Ax_{2n}, Bx_{2n-1})\big)$$

$$= \phi_2\big(d(y_{2n}, y_{2n}), d(y_{2n+1}, y_{2n-1}), d(y_{2n}, y_{2n-1})\big)$$

$$= \phi_2\big(0, d(y_{2n+1}, y_{2n-1}), d(y_{2n}, y_{2n-1})\big) \overset{(F_{22})}{\leq} \max\{d_{2n-1}, d_{2n}\};$$

$$\phi_3\big(d(Ax_{2n}, Tx_{2n}), d(Bx_{2n-1}, Sx_{2n-1}), d(Ax_{2n}, Bx_{2n-1})\big)$$

$$= \phi_3\big(d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n-1})\big)$$

$$= \phi_3(d_{2n}, d_{2n-1}, d_{2n-1}) \overset{(F_{32})}{\leq} \max\{d_{2n-1}, d_{2n}\};$$

$$\phi_4\big(d(Ax_{2n}, Sx_{2n-1}), d(Tx_{2n}, Bx_{2n-1}), d(Ax_{2n}, Tx_{2n}), d(Bx_{2n-1}, Sx_{2n-1})\big)$$

$$= \phi_4\big(d(y_{2n}, y_{2n}), d(y_{2n+1}, y_{2n-1}), d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n})\big)$$

$$= \phi_4\big(0, d(y_{2n+1}, y_{2n-1}), d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n})\big)$$

$$\overset{(F_{42})}{\leq} \max\left\{\frac{d(y_{2n+1}, y_{2n-1})}{2}, d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n})\right\}$$

$$\leq \max\left\{\frac{d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})}{2}, d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n})\right\}$$

$$= \max\{d_{2n-1}, d_{2n}\}.$$

Hence,

$$M(x_{2n}, x_{2n-1}) = \max\big\{d(Ax_{2n}, Bx_{2n-1}), d(Ax_{2n}, Tx_{2n}), d(Bx_{2n-1}, Sx_{2n-1}),$$

$$\phi_1\big(d(Ax_{2n}, Sx_{2n-1}), d(Tx_{2n}, Bx_{2n-1})\big),$$

$$\phi_2\big(d(Ax_{2n}, Sx_{2n-1}), d(Tx_{2n}, Bx_{2n-1}), d(Ax_{2n}, Bx_{2n-1})\big),$$

$$\phi_3\big(d(Ax_{2n}, Tx_{2n}), d(Bx_{2n-1}, Sx_{2n-1}), d(Ax_{2n}, Bx_{2n-1})\big),$$

$$\phi_4\big(d(Ax_{2n}, Sx_{2n-1}), d(Tx_{2n}, Bx_{2n-1}), d(Ax_{2n}, Tx_{2n}), d(Bx_{2n-1}, Sx_{2n-1})\big)\big\}$$

$$= \max\{d_{2n-1}, d_{2n}\}.$$

By (6),

$$d_{2n} \leq \psi\big(M(x_{2n}, x_{2n-1})\big) = \psi\big(\max\{d_{2n-1}, d_{2n}\}\big).$$

If $d_{2n-1} < d_{2n}$ for some n, then $d_{2n} > 0$ and we get the contradiction $d_{2n} \leq \psi(\max\{d_{2n-1}, d_{2n}\}) = \psi(d_{2n})$ (recall Lemma 4). As a consequence, we deduce that $d_{2n} \leq d_{2n-1}$ for all $n \in \mathbb{N}$. Similarly, it can be proved that $d_{2n+1} \leq \psi(\{d_{2n}, d_{2n+1}\})$, so $d_{2n+1} \leq d_{2n}$ for all $n \in \mathbb{N}$. Hence,

$$d_{n+1} \leq \psi(d_n) \quad \text{and} \quad d_{n+1} \leq d_n \quad \text{for all } n \in \mathbb{N}.$$

Since $\{d_n\}$ is a non-increasing sequence of non-negative real numbers, it is convergent. Let $L \geq 0$ be its limit. Taking into account that $\psi \in \Phi_3$ and $d_{n+1} \leq \psi(d_n)$ for all $n \in \mathbb{N}$, we deduce that $\{d_n\} \to 0$, that is,

$$\lim_{n \to \infty} d(y_n, y_{n+1}) = 0.$$

Next, let us show that $\{y_n\}$ is a Cauchy sequence in (X, d) reasoning by contradiction. Assume that $\{y_n\}$ is not a Cauchy sequence. By Proposition 5, the sequence $\{y_{2n}\}$ is not a Cauchy sequence. In particular, there exist $\varepsilon_0 > 0$ and two subsequences $\{y_{2n(k)}\}$ and $\{y_{2m(k)}\}$ of $\{y_{2n}\}$ such that

$$2k < 2n(k) < 2m(k), \quad d(y_{2n(k)}, y_{2m(k)-2}) \leq \varepsilon_0 < d(y_{2n(k)}, y_{2m(k)}) \quad \text{for all } k \in \mathbb{N},$$

$$\lim_{n \to \infty} d(y_{2n(k)}, y_{2m(k)}) = \lim_{n \to \infty} d(y_{2n(k)}, y_{2m(k)-1}) = \lim_{n \to \infty} d(y_{2n(k)}, y_{2m(k)+1})$$

$$= \lim_{n \to \infty} d(y_{2n(k)-1}, y_{2m(k)-1}) = \lim_{n \to \infty} d(y_{2n(k)-1}, y_{2m(k)+1}) = \varepsilon_0. \tag{7}$$

By (3) and (5), for all $k \in \mathbb{N}$,

$$d(y_{2m(k)+1}, y_{2n(k)}) = d(Tx_{2m(k)}, Sx_{2n(k)-1}) \leq \psi\big(M(x_{2m(k)}, x_{2n(k)-1})\big). \tag{8}$$

Notice that

$$\lim_{k \to \infty} d(Ax_{2m(k)}, Bx_{2n(k)-1}) = \lim_{k \to \infty} d(y_{2m(k)}, y_{2n(k)-1}) = \varepsilon_0;$$

$$\lim_{k \to \infty} d(Ax_{2m(k)}, Tx_{2m(k)}) = \lim_{k \to \infty} d(y_{2m(k)}, y_{2m(k)+1}) = 0;$$

$$\lim_{k \to \infty} d(Bx_{2n(k)-1}, Sx_{2n(k)-1}) = \lim_{k \to \infty} d(y_{2n(k)-1}, y_{2n(k)}) = 0;$$

$$\lim_{k \to \infty} \phi_1\big(d(Ax_{2m(k)}, Sx_{2n(k)-1}), d(Tx_{2m(k)}, Bx_{2n(k)-1})\big)$$

$$= \lim_{k \to \infty} \phi_1\big(d(y_{2m(k)}, y_{2n(k)}), d(y_{2m(k)+1}, y_{2n(k)-1})\big)$$

$$\overset{(F_{11})}{=} \phi_1\Big(\lim_{k \to \infty} d(y_{2m(k)}, y_{2n(k)}), \lim_{k \to \infty} d(y_{2m(k)+1}, y_{2n(k)-1})\Big) = \phi_1(\varepsilon_0, \varepsilon_0) \overset{(F_{13})}{\leq} \varepsilon_0;$$

$$\lim_{k \to \infty} \phi_2\big(d(Ax_{2m(k)}, Sx_{2n(k)-1}), d(Tx_{2m(k)}, Bx_{2n(k)-1}), d(Ax_{2m(k)}, Bx_{2n(k)-1})\big)$$

$$= \lim_{k \to \infty} \phi_2\big(d(y_{2m(k)}, y_{2n(k)}), d(y_{2m(k)+1}, y_{2n(k)-1}), d(y_{2m(k)}, y_{2n(k)-1})\big)$$

$$\overset{(F_{21})}{=} \phi_2\Big(\lim_{k \to \infty} d(y_{2m(k)}, y_{2n(k)}), \lim_{k \to \infty} d(y_{2m(k)+1}, y_{2n(k)-1}), \lim_{k \to \infty} d(y_{2m(k)}, y_{2n(k)-1})\Big)$$

$$= \phi_2(\varepsilon_0, \varepsilon_0, \varepsilon_0) \overset{(F_{23})}{\leq} \varepsilon_0;$$

$$\lim_{k \to \infty} \phi_3\big(d(Ax_{2m(k)}, Tx_{2m(k)}), d(Bx_{2n(k)-1}, Sx_{2n(k)-1}), d(Ax_{2m(k)}, Bx_{2n(k)-1})\big)$$

$$= \lim_{k \to \infty} \phi_3\big(d(y_{2m(k)}, y_{2m(k)+1}), d(y_{2n(k)-1}, y_{2n(k)}), d(y_{2m(k)}, y_{2n(k)-1})\big)$$

$$\overset{(F_{31})}{=} \phi_3\Big(\lim_{k \to \infty} d(y_{2m(k)}, y_{2m(k)+1}), \lim_{k \to \infty} d(y_{2n(k)-1}, y_{2n(k)}), \lim_{k \to \infty} d(y_{2m(k)}, y_{2n(k)-1})\Big)$$

$$= \phi_3(0, 0, \varepsilon_0) \overset{(F_{33})}{\leq} \varepsilon_0;$$

$$\lim_{k\to\infty} \phi_4\big(d(Ax_{2m(k)}, Sx_{2n(k)-1}), d(Tx_{2m(k)}, Bx_{2n(k)-1}), d(Ax_{2m(k)}, Tx_{2m(k)}),$$

$$d(Bx_{2n(k)-1}, Sx_{2n(k)-1})\big)$$

$$= \lim_{k\to\infty} \phi_4\big(d(y_{2m(k)}, y_{2n(k)}), d(y_{2m(k)+1}, y_{2n(k)-1}), d(y_{2m(k)}, y_{2m(k)+1}), d(y_{2n(k)-1}, y_{2n(k)})\big)$$

$$\overset{(F_{41})}{=} \phi_4\Big(\lim_{k\to\infty} d(y_{2m(k)}, y_{2n(k)}), \lim_{k\to\infty} d(y_{2m(k)+1}, y_{2n(k)-1}),$$

$$\lim_{k\to\infty} d(y_{2m(k)}, y_{2m(k)+1}), \lim_{k\to\infty} d(y_{2n(k)-1}, y_{2n(k)})\Big) = \phi_4(\varepsilon_0, \varepsilon_0, 0, 0) \overset{(F_{43})}{\le} \varepsilon_0.$$

As a result, it follows from (8) and (7),

$$\lim_{k\to\infty} M(x_{2m(k)}, x_{2n(k)-1})$$

$$= \lim_{k\to\infty} \max\big\{d(Ax_{2m(k)}, Bx_{2n(k)-1}), d(Ax_{2m(k)}, Tx_{2m(k)}), d(Bx_{2n(k)-1}, Sx_{2n(k)-1}),$$

$$\phi_1\big(d(Ax_{2m(k)}, Sx_{2n(k)-1}), d(Tx_{2m(k)}, Bx_{2n(k)-1})\big),$$

$$\phi_2\big(d(Ax_{2m(k)}, Sx_{2n(k)-1}), d(Tx_{2m(k)}, Bx_{2n(k)-1}), d(Ax_{2m(k)}, Bx_{2n(k)-1})\big),$$

$$\phi_3\big(d(Ax_{2m(k)}, Tx_{2m(k)}), d(Bx_{2n(k)-1}, Sx_{2n(k)-1}), d(Ax_{2m(k)}, Bx_{2n(k)-1})\big),$$

$$\phi_4\big(d(Ax_{2m(k)}, Sx_{2n(k)-1}), d(Tx_{2m(k)}, Bx_{2n(k)-1}),$$

$$d(Ax_{2m(k)}, Tx_{2m(k)}), d(Bx_{2n(k)-1}, Sx_{2n(k)-1})\big)\big\}$$

$$= \varepsilon_0.$$

As ψ is upper semi-continuous, we deduce that

$$\varepsilon_0 = \lim_{n\to\infty} d(y_{2m(k)+1}, y_{2n(k)}) = \limsup_{n\to\infty} d(y_{2m(k)+1}, y_{2n(k)})$$

$$\le \limsup_{n\to\infty} \psi\big(M(x_{2m(k)}, x_{2n(k)-1})\big) \le \psi(\varepsilon_0) < \varepsilon_0,$$

which is a contradiction. As a result, $\{y_n\}$ is a Cauchy sequence in (X, d). Then there exists $z \in X$ such that $\{y_n\} \to z$. In particular,

$$z = \lim_{n\to\infty} y_n = \lim_{n\to\infty} Tx_{2n} = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Sx_{2n-1}.$$

Next we distinguish some cases depending on the complete set. Notice that

$$\{y_{2n+2} = Ax_{2n+2} = Sx_{2n+1}\} \subseteq S(X) \subseteq A(X) \quad \text{and}$$

$$\{y_{2n+1} = Bx_{2n+1} = Tx_{2n}\} \subseteq T(X) \subseteq B(X).$$

Assume that $A(X)$ (or $S(X)$) is complete. Then there exists $u \in X$ such that $z = Au$. To show that $Tu = z$, suppose, by contradiction, that $d(Tu, z) > 0$. Therefore, for all $n \in \mathbb{N}$,

$$d(Tu, y_{2n+2}) = d(Tu, Sx_{2n+1}) \le \psi\big(M(u, x_{2n+1})\big). \tag{9}$$

Notice that

$$\lim_{n\to\infty} d(Au, Bx_{2n+1}) = \lim_{n\to\infty} d(z, y_{2n+1}) = 0;$$

$$\lim_{n\to\infty} d(Au, Tu) = d(z, Tu) > 0;$$

$$\lim_{n\to\infty} d(Bx_{2n+1}, Sx_{2n+1}) = d(y_{2n+1}, y_{2n+2}) = 0;$$

$$\lim_{n\to\infty} \phi_1\big(d(Au, Sx_{2n+1}), d(Tu, Bx_{2n+1})\big)$$

$$= \lim_{n\to\infty} \phi_1\big(d(z, y_{2n+2}), d(Tu, y_{2n+1})\big)$$

$$\overset{(F_{11})}{=} \phi_1\Big(\lim_{n\to\infty} d(z, y_{2n+2}), \lim_{n\to\infty} d(Tu, y_{2n+1})\Big)$$

$$= \phi_1\big(0, d(Tu, z)\big) \overset{(F_{12})}{\leq} \frac{d(Tu, z)}{2} < d(Tu, z);$$

$$\lim_{n\to\infty} \phi_2\big(d(Au, Sx_{2n+1}), d(Tu, Bx_{2n+1}), d(Au, Bx_{2n+1})\big)$$

$$= \lim_{n\to\infty} \phi_2\big(d(z, y_{2n+2}), d(Tu, y_{2n+1}), d(z, y_{2n+1})\big)$$

$$\overset{(F_{21})}{=} \phi_2\Big(\lim_{n\to\infty} d(z, y_{2n+2}), \lim_{n\to\infty} d(Tu, y_{2n+1}), \lim_{n\to\infty} d(z, y_{2n+1})\Big)$$

$$= \phi_2\big(0, d(Tu, z), 0\big) \overset{(F_{22})}{\leq} d(Tu, z);$$

$$\lim_{n\to\infty} \phi_3\big(d(Au, Tu), d(Bx_{2n+1}, Sx_{2n+1}), d(Au, Bx_{2n+1})\big)$$

$$= \lim_{n\to\infty} \phi_3\big(d(z, Tu), d(y_{2n+1}, y_{2n+2}), d(z, y_{2n+1})\big)$$

$$\overset{(F_{31})}{=} \phi_3\Big(\lim_{n\to\infty} d(z, Tu), \lim_{n\to\infty} d(y_{2n+1}, y_{2n+2}), \lim_{n\to\infty} d(z, y_{2n+1})\Big)$$

$$= \phi_3\big(d(z, Tu), 0, 0\big) \overset{(F_{32})}{\leq} d(z, Tu);$$

$$\lim_{n\to\infty} \phi_4\big(d(Au, Sx_{2n+1}), d(Tu, Bx_{2n+1}), d(Au, Tu), d(Bx_{2n+1}, Sx_{2n+1})\big)$$

$$= \lim_{n\to\infty} \phi_4\big(d(z, y_{2n+2}), d(Tu, y_{2n+1}), d(z, Tu), d(y_{2n+1}, y_{2n+2})\big)$$

$$\overset{(F_{41})}{=} \phi_4\Big(\lim_{n\to\infty} d(z, y_{2n+2}), \lim_{n\to\infty} d(Tu, y_{2n+1}), \lim_{n\to\infty} d(z, Tu), \lim_{n\to\infty} d(y_{2n+1}, y_{2n+2})\Big)$$

$$= \phi_4\big(0, d(Tu, z), d(z, Tu), 0\big) \overset{(F_{42})}{\leq} d(z, Tu).$$

Since

$$\lim_{n\to\infty} M(u, x_{2n+1})$$

$$= \lim_{n\to\infty} \max\big\{d(Au, Bx_{2n+1}), d(Au, Tu), d(Bx_{2n+1}, Sx_{2n+1}),$$

$$\phi_1\big(d(Au, Sx_{2n+1}), d(Tu, Bx_{2n+1})\big), \phi_2\big(d(Au, Sx_{2n+1}), d(Tu, Bx_{2n+1}), d(Au, Bx_{2n+1})\big),$$

$$\phi_3\big(d(Au, Tu), d(Bx_{2n+1}, Sx_{2n+1}), d(Au, Bx_{2n+1})\big),$$

$$\phi_4\big(d(Au, Sx_{2n+1}), d(Tu, Bx_{2n+1}), d(Au, Tu), d(Bx_{2n+1}, Sx_{2n+1})\big)\big\}$$

$$= d(z, Tu) > 0,$$

letting $n \to \infty$ in (9), the upper semi-continuity of ψ yields

$$d(z, Tu) = \limsup_{n \to \infty} d(Tu, y_{2n+2}) \leq \limsup_{n \to \infty} \psi\big(M(u, x_{2n+1})\big)$$

$$\leq \psi\Big(\limsup_{n \to \infty} M(u, x_{2n+1})\Big) = \psi\big(d(z, Tu)\big) < d(z, Tu),$$

which is a contradiction. As a consequence, $Tu = z = Au$. As $T(X) \subseteq B(X)$, there exists $w \in X$ such that $Tu = Bw$. Then $Au = Tu = z = Bw$. We claim that $z = Sw$. To prove it, suppose that $d(z, Sw) > 0$. By (3), for all $n \in \mathbb{N}$,

$$d(y_{2n+1}, Sw) = d(Tx_{2n}, Sw) \leq \psi\big(M(x_{2n}, w)\big). \tag{10}$$

Notice that

$$\lim_{n \to \infty} d(Ax_{2n}, Bw) = \lim_{n \to \infty} d(y_{2n}, z) = 0;$$

$$\lim_{n \to \infty} d(Ax_{2n}, Tx_{2n}) = \lim_{n \to \infty} d(y_{2n}, y_{2n+1}) = 0;$$

$$\lim_{n \to \infty} d(Bw, Sw) = d(z, Sw) > 0;$$

$$\lim_{n \to \infty} \phi_1\big(d(Ax_{2n}, Sw), d(Tx_{2n}, Bw)\big)$$

$$= \lim_{n \to \infty} \phi_1\big(d(y_{2n}, Sw), d(y_{2n+1}, z)\big)$$

$$\overset{(F_{11})}{=} \phi_1\Big(\lim_{n \to \infty} d(y_{2n}, Sw), \lim_{n \to \infty} d(y_{2n+1}, z)\Big)$$

$$= \phi_1\big(d(z, Sw), 0\big) \overset{(F_{14})}{\leq} d(z, Sw);$$

$$\lim_{n \to \infty} \phi_2\big(d(Ax_{2n}, Sw), d(Tx_{2n}, Bw), d(Ax_{2n}, Bw)\big)$$

$$= \lim_{n \to \infty} \phi_2\big(d(y_{2n}, Sw), d(y_{2n+1}, z), d(y_{2n}, z)\big)$$

$$\overset{(F_{21})}{=} \phi_2\Big(\lim_{n \to \infty} d(y_{2n}, Sw), \lim_{n \to \infty} d(y_{2n+1}, z), \lim_{n \to \infty} d(y_{2n}, z)\Big)$$

$$= \phi_2\big(d(z, Sw), 0, 0\big) \overset{(F_{24})}{\leq} d(z, Sw);$$

$$\lim_{n \to \infty} \phi_3\big(d(Ax_{2n}, Tx_{2n}), d(Bw, Sw), d(Ax_{2n}, Bw)\big)$$

$$= \lim_{n \to \infty} \phi_3\big(d(y_{2n}, y_{2n+1}), d(z, Sw), d(y_{2n}, z)\big)$$

$$\overset{(F_{31})}{=} \phi_3\Big(\lim_{n \to \infty} d(y_{2n}, y_{2n+1}), \lim_{n \to \infty} d(z, Sw), \lim_{n \to \infty} d(y_{2n}, z)\Big)$$

$$= \phi_3\big(0, d(z, Sw), 0\big) \overset{(F_{34})}{\leq} d(z, Sw);$$

$$\lim_{n \to \infty} \phi_4\big(d(Ax_{2n}, Sw), d(Tx_{2n}, Bw), d(Ax_{2n}, Tx_{2n}), d(Bw, Sw)\big)$$

$$= \lim_{n \to \infty} \phi_4\big(d(y_{2n}, Sw), d(y_{2n+1}, z), d(y_{2n}, y_{2n+1}), d(z, Sw)\big)$$

$$\overset{(F_{41})}{=} \phi_4\Big(\lim_{n \to \infty} d(y_{2n}, Sw), \lim_{n \to \infty} d(y_{2n+1}, z), \lim_{n \to \infty} d(y_{2n}, y_{2n+1}), \lim_{n \to \infty} d(z, Sw)\Big)$$

$$= \phi_4\big(d(z, Sw), 0, 0, d(z, Sw)\big) \overset{(F_{44})}{\leq} d(z, Sw).$$

Hence,

$$
\lim_{n\to\infty} M(x_{2n}, w) = \lim_{n\to\infty} \max\big\{ d(Ax_{2n}, Bw), d(Ax_{2n}, Tx_{2n}), d(Bw, Sw),
$$

$$
\phi_1\big(d(Ax_{2n}, Sw), d(Tx_{2n}, Bw)\big),
$$

$$
\phi_2\big(d(Ax_{2n}, Sw), d(Tx_{2n}, Bw), d(Ax_{2n}, Bw)\big),
$$

$$
\phi_3\big(d(Ax_{2n}, Tx_{2n}), d(Bw, Sw), d(Ax_{2n}, Bw)\big),
$$

$$
\phi_4\big(d(Ax_{2n}, Sw), d(Tx_{2n}, Bw), d(Ax_{2n}, Tx_{2n}), d(Bw, Sw)\big)\big\}
$$

$$
= d(z, Sw) > 0.
$$

Again, letting $n \to \infty$ in (10) and using the upper semi-continuity of ψ, we deduce that

$$
d(z, Sw) = \limsup_{n\to\infty} d(y_{2n+1}, Sw) \le \limsup_{n\to\infty} \psi\big(M(x_{2n}, w)\big)
$$

$$
\le \psi\Big(\lim_{n\to\infty} M(x_{2n}, w)\Big) = \psi\big(d(z, Sw)\big) < d(z, Sw),
$$

which is a contradiction. Thus, $Sw = z$, which means that $z = Tu = Au = Bw = Sw$. As $\{A, T\}$ and $\{B, S\}$ are weakly compatible,

$$
Az = ATu = TAu = Tz \quad \text{and} \quad Bz = BSw = SBw = Sz.
$$

Next, let us show that $Tz = Sz$. On the contrary, suppose that $d(Tz, Sz) > 0$. Hence, by (3),

$$
d(Tz, Sz) \le \psi\big(M(z, z)\big). \tag{11}
$$

Notice that

$$
d(Az, Bz) = d(Tz, Sz) > 0;
$$

$$
d(Az, Tz) = d(Tz, Tz) = 0;
$$

$$
d(Bz, Sz) = d(Sz, Sz) = 0;
$$

$$
\phi_1\big(d(Az, Sz), d(Tz, Bz)\big) = \phi_1\big(d(Tz, Sz), d(Tz, Sz)\big) \overset{(F_{13})}{\le} d(Tz, Sz);
$$

$$
\phi_2\big(d(Az, Sz), d(Tz, Bz), d(Az, Bz)\big)
$$

$$
= \phi_2\big(d(Tz, Sz), d(Tz, Sz), d(Tz, Sz)\big) \overset{(F_{23})}{\le} d(Tz, Sz);
$$

$$
\phi_3\big(d(Az, Tz), d(Bz, Sz), d(Az, Bz)\big)
$$

$$
= \phi_3\big(d(Tz, Tz), d(Sz, Sz), d(Tz, Sz)\big)
$$

$$
= \phi_3\big(0, 0, d(Tz, Sz)\big) \overset{(F_{33})}{\le} d(Tz, Sz);
$$

$$
\phi_4\big(d(Az, Sz), d(Tz, Bz), d(Az, Tz), d(Bz, Sz)\big)
$$

$$
= \phi_4\big(d(Tz, Sz), d(Tz, Sz), d(Tz, Tz), d(Sz, Sz)\big)
$$

$$
= \phi_4\big(d(Tz, Sz), d(Tz, Sz), 0, 0\big) \overset{(F_{43})}{\le} d(Tz, Sz).
$$

Therefore,

$$M(z,z) = \max\{d(Az,Bz), d(Az,Tz), d(Bz,Sz),$$
$$\phi_1\big(d(Az,Sz), d(Tz,Bz)\big), \phi_2\big(d(Az,Sz), d(Tz,Bz), d(Az,Bz)\big),$$
$$\phi_3\big(d(Az,Tz), d(Bz,Sz), d(Az,Bz)\big),$$
$$\phi_4\big(d(Az,Sz), d(Tz,Bz), d(Az,Tz), d(Bz,Sz)\big)\}$$
$$= d(Tz,Sz) > 0.$$

Again, it follows from (11) that

$$d(Tz,Sz) \le \psi\big(M(z,z)\big) = \psi\big(d(Tz,Sz)\big) < d(Tz,Sz),$$

which is a contradiction. As a consequence, we conclude that $Tz = Sz$, which means that

$$Az = Tz = Sz = Bz.$$

In particular, z is a coincidence point of A, B, T, and S.

Next, let us show that $Tz = z$. Reasoning by contradiction, assume that $d(Tz,z) > 0$. By (3),

$$d(Tz,z) = d(Tz,Sw) \le \psi\big(M(z,w)\big). \tag{12}$$

Notice that

$$d(Az,Bw) = d(Tz,Sw) = d(Tz,z) > 0;$$
$$d(Az,Tz) = d(Tz,Tz) = 0;$$
$$d(Bw,Sw) = d(Sw,Sw) = 0;$$
$$\phi_1\big(d(Az,Sw), d(Tz,Bw)\big) = \phi_1\big(d(Tz,Sw), d(Tz,Sw)\big) \overset{(F_{13})}{\le} d(Tz,z);$$
$$\phi_2\big(d(Az,Sw), d(Tz,Bw), d(Az,Bw)\big) = \phi_2\big(d(Tz,z), d(Tz,z), d(Tz,z)\big) \overset{(F_{23})}{\le} d(Tz,z);$$
$$\phi_3\big(d(Az,Tz), d(Bw,Sw), d(Az,Bw)\big)$$
$$= \phi_3\big(d(Tz,Tz), d(z,z), d(Tz,z)\big) = \phi_3\big(0,0,d(Tz,z)\big) \overset{(F_{33})}{\le} d(Tz,z);$$
$$\phi_4\big(d(Az,Sw), d(Tz,Bw), d(Az,Tz), d(Bw,Sw)\big)$$
$$= \phi_4\big(d(Tz,z), d(Tz,z), d(Tz,Tz), d(z,z)\big) = \phi_4\big(d(Tz,z), d(Tz,z), 0, 0\big) \overset{(F_{43})}{\le} d(Tz,z).$$

As a result,

$$M(z,w) = \max\{d(Az,Bw), d(Az,Tz), d(Bw,Sw), \phi_1\big(d(Az,Sw), d(Tz,Bw)\big),$$
$$\phi_2\big(d(Az,Sw), d(Tz,Bw), d(Az,Bw)\big), \phi_3\big(d(Az,Tz), d(Bw,Sw), d(Az,Bw)\big)$$
$$\phi_4\big(d(Az,Sw), d(Tz,Bw), d(Az,Tz), d(Bw,Sw)\big)\} = d(Tz,z) > 0.$$

Again, it follows from (12) that

$$d(Tz, z) \leq \psi\big(M(z, w)\big) < d(Tz, z), \tag{13}$$

which is a contradiction. Thus, $Tz = z$, so z is a common fixed point of A, B, T, and S.

Finally, let us show that z is the unique common fixed point of A, T, S, and B. Assume that $x, y \in X$ are two different common fixed points of A, T, S, and B. Then $d(x, y) > 0$. Notice that

$$d(Ax, By) = d(x, y) > 0;$$

$$d(Ax, Tx) = d(x, x) = 0;$$

$$d(By, Sy) = d(y, y) = 0;$$

$$\phi_1\big(d(Ax, Sy), d(Tx, By)\big) = \phi_1\big(d(x, y), d(x, y)\big) \overset{(F_{13})}{\leq} d(x, y);$$

$$\phi_2\big(d(Ax, Sy), d(Tx, By), d(Ax, By)\big) = \phi_2\big(d(x, y), d(x, y), d(x, y)\big) \overset{(F_{23})}{\leq} d(x, y);$$

$$\phi_3\big(d(Ax, Tx), d(By, Sy), d(Ax, By)\big)$$

$$= \phi_3\big(d(x, x), d(y, y), d(x, y)\big) = \phi_3\big(0, 0, d(x, y)\big) \overset{(F_{33})}{\leq} d(x, y);$$

$$\phi_4\big(d(Ax, Sy), d(Tx, By), d(Ax, Tx), d(By, Sy)\big)$$

$$= \phi_4\big(d(x, y), d(x, y), d(x, x), d(y, y)\big) = \phi_4\big(d(x, y), d(x, y), 0, 0\big) \overset{(F_{43})}{\leq} d(x, y).$$

Therefore,

$$M(x, y) = \max\big\{ d(Ax, By), d(Ax, Tx), d(By, Sy), \phi_1\big(d(Ax, Sy), d(Tx, By)\big),$$

$$\phi_2\big(d(Ax, Sy), d(Tx, By), d(Ax, By)\big), \phi_3\big(d(Ax, Tx), d(By, Sy), d(Ax, By)\big)$$

$$\phi_4\big(d(Ax, Sy), d(Tx, By), d(Ax, Tx), d(By, Sy)\big)\big\}$$

$$= d(x, y) > 0.$$

As a result,

$$d(x, y) = d(Tx, Sy) \leq \psi\big(M(x, y)\big) < d(x, y), \tag{14}$$

which is a contradiction. Then $x = y$ and A, B, S, and T have a unique common fixed point in X. □

Using whatever functions $\phi_1 \in \mathsf{F}_1$, $\phi_2 \in \mathsf{F}_2$, $\phi_3 \in \mathsf{F}_3$, and $\phi_4 \in \mathsf{F}_4$ (as we have shown in the first part of this section), we can obtain a large variety of different corollaries. For instance, the following one.

Corollary 7 *Theorem* 1 *immediately follows from Theorem* 6.

Proof It is only necessary to take, for all $r, s, t, u \in [0, \infty)$,

$$\phi_1(r, s) = \frac{r + s}{2}, \qquad \phi_2(r, s, t) = \phi_3(r, s, t,) = \frac{rs}{1 + t}, \quad \text{and} \quad \phi_4(r, s, t, u) = \frac{1 + r + s}{1 + t + u}t.$$

Hence $\phi_1 \in \mathsf{F}_1$, $\phi_2 \in \mathsf{F}_2$, $\phi_3 \in \mathsf{F}_3$, and $\phi_4 \in \mathsf{F}_4$. Furthermore, for all $x, y \in X$,

$$\phi_1\big(d(Ax, Sy), d(Tx, By)\big) = \frac{d(Ax, Sy) + d(Tx, By)}{2},$$

$$\phi_2\big(d(Ax, Sy), d(Tx, By), d(Ax, By)\big) = \frac{d(Ax, Sy)d(Tx, By)}{1 + d(Ax, By)},$$

$$\phi_3\big(d(Ax, Tx), d(By, Sy), d(Ax, By)\big) = \frac{d(Ax, Tx)d(By, Sy)}{1 + d(Ax, By)}, \quad \text{and}$$

$$\phi_4\big(d(Ax, Sy), d(Tx, By), d(Ax, Tx), d(By, Sy)\big) = \frac{1 + d(Ax, Sy) + d(Tx, By)}{1 + d(Ax, Tx) + d(By, Sy)} d(Ax, Tx).$$

As a result, the contractivity condition (1) implies (3), and Theorem 6 is applicable. □

As a particular case, we highlight the situation in which $T = B$ and $S = A$. In such a case, we obtain the following consequence.

Corollary 8 *Let $T, S : X \to X$ be self-mappings in a metric space (X, d) such that*

(a) *$\{T, S\}$ is weakly compatible;*

(b) *one of $T(X)$ or $S(X)$ is complete.*

Also assume that there exist $\phi_1 \in \mathsf{F}_1$, $\phi_2 \in \mathsf{F}_2$, $\phi_3 \in \mathsf{F}_3$, $\phi_4 \in \mathsf{F}_4$, and $\psi \in \Phi_3$ such that

$$d(Tx, Sy) \le \psi\big(M(x, y)\big) \quad \text{for all } x, y \in X,$$

where

$$M(x, y) = \max\big\{ d(Sx, Ty), d(Sx, Tx), d(Ty, Sy), \phi_1\big(d(Sx, Sy), d(Tx, Ty)\big),$$
$$\phi_2\big(d(Sx, Sy), d(Tx, Ty), d(Sx, Ty)\big), \phi_3\big(d(Sx, Tx), d(Ty, Sy), d(Sx, Ty)\big)$$
$$\phi_4\big(d(Sx, Sy), d(Tx, Ty), d(Sx, Tx), d(Ty, Sy)\big) \big\}.$$

Then T and S have a unique common fixed point in X.

The following example shows that Theorem 6 improves Theorem 1. It is based on Example 2.3 in [34].

Example 9 Let $X = [0, 1]$ be endowed with the Euclidean metric $d(x, y) = |x - y|$ for all $x, y \in X$, and let $A, B, S, T : X \to X$ be the self-mappings given, for all $x \in X$, by

$$Ax = x^2, \qquad Bx = \frac{x^2}{2}, \qquad Sx = 0, \qquad Tx = \begin{cases} 0, & \text{if } x \in [0, 1), \\ 0.9, & \text{if } x = 1. \end{cases}$$

Notice that, for all $x, y \in X$,

$$d(Tx, Sy) = \begin{cases} 0.9, & \text{if } x = 1, \\ 0, & \text{otherwise.} \end{cases}$$

If $x < 1$, the contractivity condition

$$d(Tx, Sy) \leq \psi\big(M(x, y)\big) \quad \text{for all } y \in X, \tag{15}$$

is obvious whatever $\psi \in \Phi_3$. Next, let $\psi : [0, \infty) \to [0, \infty)$ and $\phi_1 : [0, \infty)^2 \to [0, \infty)$ be the functions given by

$$\psi(r) = 0.9r \quad \text{and} \quad \phi_1(r, s) = \max\left\{r, \frac{r+s}{2}\right\} \quad \text{for all } r, s \geq 0.$$

Clearly, $\psi \in \Phi_3$ and $\phi_1 \in \mathsf{F}_1$. Notice that if $x = 1$, then $d(Ax, Sy) = d(1, 0) = 1$, so $\phi_1(d(Ax, Sy), d(Tx, By)) = 1$. Hence, for $x = 1$,

$$
\begin{aligned}
M(x, y) = \ & \max\big\{d(Ax, By), d(Ax, Tx), d(By, Sy), \phi_1\big(d(Ax, Sy), d(Tx, By)\big), \\
& \phi_2\big(d(Ax, Sy), d(Tx, By), d(Ax, By)\big), \phi_3\big(d(Ax, Tx), d(By, Sy), d(Ax, By)\big) \\
& \phi_4\big(d(Ax, Sy), d(Tx, By), d(Ax, Tx), d(By, Sy)\big)\big\} \geq 1.
\end{aligned}
$$

In such a case, for all $y \in X$,

$$d\big(T(1), Sy\big) = 0.9 = \psi(1) \leq \psi\big(M(1, y)\big),$$

which means that the contractivity condition (3) holds for all $x, y \in X$. The pairs $\{A, T\}$ and $\{B, S\}$ are weakly compatible because the unique solution of equations $Ax = Tx$ and $Bx = Sx$ is $x = 0$. As all hypotheses of Theorem 6 are satisfied, we conclude that A, B, S, and T have a unique common fixed point in X.

Notice that Theorem 1 is not applicable because if $x = 1$ and $y = 0.5$, then $d(T(1), S(0.5)) = 0.9$ and

$$
\begin{aligned}
M_1(1, 0.5) = \ & \max\bigg\{d\big(A(1), B(0.5)\big), d\big(A(1), T(1)\big), d\big(B(0.5), S(0.5)\big), \\
& \frac{d(A(1), S(0.5)) + d(T(1), B(0.5))}{2}, \frac{d(A(1), S(0.5))d(T(1), B(0.5))}{1 + d(A(1), B(0.5))}, \\
& \frac{d(A(1), T(1))d(B(0.5), S(0.5))}{1 + d(A(1), B(0.5))}, \\
& \frac{1 + d(A(1), S(0.5)) + d(T(1), B(0.5))}{1 + d(A(1), T(1)) + d(B(0.5), S(0.5))} d\big(A(1), T(1)\big)\bigg\} \\
= \ & \max\{0.875, 0.1, 0.125, 0.8875, 0.4133, 0.00667, 0.2265\} = 0.8875.
\end{aligned}
$$

As a result, $d(T(1), S(0)) = 0.9 > 0.79875 = \psi(0.8875) = \psi(M_1(1, 0.5))$, that is, inequality (1) does not hold.

To be precise, we point out that our contractivity condition holds because we have employed $\phi_1(r, s) = \max\{r, \frac{r+s}{2}\}$, which is better for our kind of contractivity conditions than $\phi_1'(r, s) = \frac{r+s}{2}$, as Liu *et al.* used in Theorem 1.

4 Common fixed point theorems under α conditions

The study of fixed point theory in ordered metric spaces was initiated by Ran and Reurings in [36] and by Nieto and Rodríguez-López in [37]. Later, improved results were obtained

by many authors (see, for instance, [17, 20, 22, 23, 25–27, 31, 33]). In this section we obtain a version of Theorem 6 involving a function α that can generalize a partial order.

In the sequel, let $\alpha : X \times X \to [0, \infty)$ be a function.

Definition 10 We will say that α is *reflexive* if $\alpha(x, x) \geq 1$ for all $x \in X$.

Definition 11 We will say that α is *transitive* if $\alpha(x, z) \geq 1$ for all $x, y, z \in X$ such that $\alpha(x, y) \geq 1$ and $\alpha(y, z) \geq 1$.

Definition 12 Given two mappings $f, g : X \to X$, we will say that $f(X)$ is an α-*subset of* $g(X)$ if for all $x \in X$, there exists $y \in X$ such that $fx = gy$ and $\alpha(x, y) \geq 1$. In such a case, we will write $f(X) \subseteq_\alpha g(X)$.

Remark 13 If $f(X) \subseteq_\alpha g(X)$ then $f(X) \subseteq g(X)$. If $\alpha(x, y) \geq 1$ for all $x, y \in X$, then $f(X) \subseteq_\alpha g(X)$ if, and only if, $f(X) \subseteq g(X)$.

Definition 14 Given two mappings $f, g : X \to X$, we will say that the pair $\{f, g\}$ is (α, d)-*regular* if

$$\max\{\alpha(x_n, u), \alpha(u, x_n)\} \geq 1 \quad \text{for all } n \in \mathbb{N}$$

provided that $\{x_n\} \subseteq X$ is a sequence such that $\{fx_n\} \to gu$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$.

Theorem 15 *Let $\alpha : X \times X \to [0, \infty)$ be a reflexive, transitive function and let $A, B, S, T : X \to X$ be self-mappings in a metric space (X, d) such that*

(a)   *$\{A, T\}$ and $\{B, S\}$ are weakly compatible;*

(b)   *$T(X) \subseteq_\alpha B(X)$ and $S(X) \subseteq_\alpha A(X)$;*

(c)   *one of $A(X), B(X), S(X),$ and $T(X)$ is complete;*

(d)   *the pairs $\{T, B\}$ and $\{S, A\}$ are (α, d)-regular.*

Also assume that there exist $\phi_1 \in \mathsf{F}_1, \phi_2 \in \mathsf{F}_2, \phi_3 \in \mathsf{F}_3, \phi_4 \in \mathsf{F}_4,$ and $\psi \in \Phi_3$ such that

$$\max\{\alpha(x, y), \alpha(y, x)\} d(Tx, Sy) \leq \psi(M(x, y)) \quad \text{for all } x, y \in X, \tag{16}$$

where

$$M(x, y) = \max\{d(Ax, By), d(Ax, Tx), d(By, Sy), \phi_1(d(Ax, Sy), d(Tx, By)),$$
$$\phi_2(d(Ax, Sy), d(Tx, By), d(Ax, By)), \phi_3(d(Ax, Tx), d(By, Sy), d(Ax, By))$$
$$\phi_4(d(Ax, Sy), d(Tx, By), d(Ax, Tx), d(By, Sy))\}. \tag{17}$$

Then $A, B, S,$ and T have a coincidence point in X.

Proof Let $x_0 \in X$ be an arbitrary point and let $y_1 = Tx_0$. Since $Tx_0 \in T(X) \subseteq_\alpha B(X)$, there exists $x_1 \in X$ such that $y_1 = Tx_0 = Bx_1$ and $\alpha(x_0, x_1) \geq 1$. Let $y_2 = Sx_1$. Since $Sx_1 \in S(X) \subseteq_\alpha$

$A(X)$, there exists $x_2 \in X$ such that $y_2 = Sx_1 = Ax_2$ and $\alpha(x_1, x_2) \geq 1$. Let $y_3 = Tx_2$. Repeating again and again this process, we can find two sequences $\{x_n\}$ and $\{y_n\}$ in X such that, for all $n \in \mathbb{N}_0$,

$$y_{2n+1} = Bx_{2n+1} = Tx_{2n}, \qquad y_{2n+2} = Ax_{2n+2} = Sx_{2n+1}, \quad \text{and} \tag{18}$$

$$\alpha(x_n, x_{n+1}) \geq 1.$$

As α is reflexive and transitive,

$$\alpha(x_n, x_m) \geq 1 \quad \text{for all } n, m \in \mathbb{N} \text{ with } n \leq m.$$

Let define $d_n = d(y_n, y_{n+1})$ for all $n \in \mathbb{N}_0$. We claim that $\lim_{n \to \infty} d_n = 0$. Indeed, by (16) and (18), for all $n \in \mathbb{N}$,

$$d_{2n} = d(y_{2n}, y_{2n+1}) = d(y_{2n+1}, y_{2n}) = d(Tx_{2n}, Sx_{2n-1})$$

$$\leq \max\{\alpha(x_{2n}, x_{2n-1}), \alpha(x_{2n-1}, x_{2n})\} d(Tx_{2n}, Sx_{2n-1}) \leq \psi\big(M(x_{2n}, x_{2n-1})\big). \tag{19}$$

Taking into account the properties that define functions in $\{\mathsf{F}_i\}_{i=1}^4$, we deduce that

$$d(Ax_{2n}, Bx_{2n-1}) = d(y_{2n}, y_{2n-1}) = d(y_{2n-1}, y_{2n}) = d_{2n-1};$$

$$d(Ax_{2n}, Tx_{2n}) = d(y_{2n}, y_{2n+1}) = d_{2n};$$

$$d(Bx_{2n-1}, Sx_{2n-1}) = d(y_{2n-1}, y_{2n}) = d_{2n-1};$$

$$\phi_1\big(d(Ax_{2n}, Sx_{2n-1}), d(Tx_{2n}, Bx_{2n-1})\big)$$

$$= \phi_1\big(d(y_{2n}, y_{2n}), d(y_{2n+1}, y_{2n-1})\big) = \phi_1\big(0, d(y_{2n-1}, y_{2n+1})\big)$$

$$\overset{(\mathsf{F}_{12})}{\leq} \frac{d(y_{2n-1}, y_{2n+1})}{2} \leq \frac{d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})}{2}$$

$$\leq \max\{d(y_{2n+1}, y_{2n}), d(y_{2n}, y_{2n-1})\} = \max\{d_{2n-1}, d_{2n}\};$$

$$\phi_2\big(d(Ax_{2n}, Sx_{2n-1}), d(Tx_{2n}, Bx_{2n-1}), d(Ax_{2n}, Bx_{2n-1})\big)$$

$$= \phi_2\big(d(y_{2n}, y_{2n}), d(y_{2n+1}, y_{2n-1}), d(y_{2n}, y_{2n-1})\big)$$

$$= \phi_2\big(0, d(y_{2n+1}, y_{2n-1}), d(y_{2n}, y_{2n-1})\big) \overset{(\mathsf{F}_{22})}{\leq} \max\{d_{2n-1}, d_{2n}\};$$

$$\phi_3\big(d(Ax_{2n}, Tx_{2n}), d(Bx_{2n-1}, Sx_{2n-1}), d(Ax_{2n}, Bx_{2n-1})\big)$$

$$= \phi_3\big(d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n-1})\big)$$

$$= \phi_3(d_{2n}, d_{2n-1}, d_{2n-1}) \overset{(\mathsf{F}_{32})}{\leq} \max\{d_{2n-1}, d_{2n}\};$$

$$\phi_4\big(d(Ax_{2n}, Sx_{2n-1}), d(Tx_{2n}, Bx_{2n-1}), d(Ax_{2n}, Tx_{2n}), d(Bx_{2n-1}, Sx_{2n-1})\big)$$

$$= \phi_4\big(d(y_{2n}, y_{2n}), d(y_{2n+1}, y_{2n-1}), d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n})\big)$$

$$= \phi_4\big(0, d(y_{2n+1}, y_{2n-1}), d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n})\big)$$

$$\overset{(\mathsf{F}_{42})}{\leq} \max\left\{\frac{d(y_{2n+1}, y_{2n-1})}{2}, d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n})\right\}$$

$$\leq \max\left\{\frac{d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})}{2}, d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n})\right\}$$

$$= \max\{d_{2n-1}, d_{2n}\}.$$

Hence,

$$\begin{aligned}
M(x_{2n}, x_{2n-1}) = \ &\max\big\{d(Ax_{2n}, Bx_{2n-1}), d(Ax_{2n}, Tx_{2n}), d(Bx_{2n-1}, Sx_{2n-1}), \\
&\phi_1\big(d(Ax_{2n}, Sx_{2n-1}), d(Tx_{2n}, Bx_{2n-1})\big), \\
&\phi_2\big(d(Ax_{2n}, Sx_{2n-1}), d(Tx_{2n}, Bx_{2n-1}), d(Ax_{2n}, Bx_{2n-1})\big), \\
&\phi_3\big(d(Ax_{2n}, Tx_{2n}), d(Bx_{2n-1}, Sx_{2n-1}), d(Ax_{2n}, Bx_{2n-1})\big), \\
&\phi_4\big(d(Ax_{2n}, Sx_{2n-1}), d(Tx_{2n}, Bx_{2n-1}), d(Ax_{2n}, Tx_{2n}), d(Bx_{2n-1}, Sx_{2n-1})\big)\big\} \\
= \ &\max\{d_{2n-1}, d_{2n}\}.
\end{aligned}$$

By (19),

$$d_{2n} \leq \psi\big(M(x_{2n}, x_{2n-1})\big) = \psi\big(\max\{d_{2n-1}, d_{2n}\}\big).$$

If $d_{2n-1} < d_{2n}$ for some n, then $d_{2n} > 0$ and we get the contradiction $d_{2n} \leq \psi(\max\{d_{2n-1}, d_{2n}\}) = \psi(d_{2n})$ (recall Lemma 4). As a consequence, we deduce that $d_{2n} \leq d_{2n-1}$ for all $n \in \mathbb{N}$. Similarly, it can be proved that $d_{2n+1} \leq \psi(\{d_{2n}, d_{2n+1}\})$, so $d_{2n+1} \leq d_{2n}$ for all $n \in \mathbb{N}$. Hence,

$$d_{n+1} \leq \psi(d_n) \quad \text{and} \quad d_{n+1} \leq d_n \quad \text{for all } n \in \mathbb{N}.$$

Since $\{d_n\}$ is a non-increasing sequence of non-negative real numbers, it is convergent. Let $L \geq 0$ be its limit. Taking into account that $\psi \in \Phi_3$ and $d_{n+1} \leq \psi(d_n)$ for all $n \in \mathbb{N}$, we deduce that $\{d_n\} \to 0$, that is,

$$\lim_{n \to \infty} d(y_n, y_{n+1}) = 0.$$

Next, let us show that $\{y_n\}$ is a Cauchy sequence in (X, d) reasoning by contradiction. Assume that $\{y_n\}$ is not a Cauchy sequence. By Proposition 5, the sequence $\{y_{2n}\}$ is not a Cauchy sequence. In particular, there exist $\varepsilon_0 > 0$ and two subsequences $\{y_{2n(k)}\}$ and $\{y_{2m(k)}\}$ of $\{y_{2n}\}$ such that

$$2k < 2n(k) < 2m(k), \quad d(y_{2n(k)}, y_{2m(k)-2}) \leq \varepsilon_0 < d(y_{2n(k)}, y_{2m(k)}) \quad \text{for all } k \in \mathbb{N},$$

$$\begin{aligned}
\lim_{n \to \infty} d(y_{2n(k)}, y_{2m(k)}) &= \lim_{n \to \infty} d(y_{2n(k)}, y_{2m(k)-1}) = \lim_{n \to \infty} d(y_{2n(k)}, y_{2m(k)+1}) \\
&= \lim_{n \to \infty} d(y_{2n(k)-1}, y_{2m(k)-1}) = \lim_{n \to \infty} d(y_{2n(k)-1}, y_{2m(k)+1}) = \varepsilon_0. \quad (20)
\end{aligned}$$

By (16) and (18), for all $k \in \mathbb{N}$,

$$\begin{aligned}
d(y_{2m(k)+1}, y_{2n(k)}) &= d(Tx_{2m(k)}, Sx_{2n(k)-1}) \\
&\leq \max\big\{\alpha(x_{2m(k)}, x_{2n(k)-1}), \alpha(x_{2n(k)-1}, x_{2m(k)})\big\} d(Tx_{2m(k)}, Sx_{2n(k)-1}) \\
&\leq \psi\big(M(x_{2m(k)}, x_{2n(k)-1})\big). \quad (21)
\end{aligned}$$

Notice that

$$\lim_{k\to\infty} d(Ax_{2m(k)}, Bx_{2n(k)-1}) = \lim_{k\to\infty} d(y_{2m(k)}, y_{2n(k)-1}) = \varepsilon_0;$$

$$\lim_{k\to\infty} d(Ax_{2m(k)}, Tx_{2m(k)}) = \lim_{k\to\infty} d(y_{2m(k)}, y_{2m(k)+1}) = 0;$$

$$\lim_{k\to\infty} d(Bx_{2n(k)-1}, Sx_{2n(k)-1}) = \lim_{k\to\infty} d(y_{2n(k)-1}, y_{2n(k)}) = 0;$$

$$\lim_{k\to\infty} \phi_1\big(d(Ax_{2m(k)}, Sx_{2n(k)-1}), d(Tx_{2m(k)}, Bx_{2n(k)-1})\big)$$

$$= \lim_{k\to\infty} \phi_1\big(d(y_{2m(k)}, y_{2n(k)}), d(y_{2m(k)+1}, y_{2n(k)-1})\big)$$

$$\overset{(F_{11})}{=} \phi_1\Big(\lim_{k\to\infty} d(y_{2m(k)}, y_{2n(k)}), \lim_{k\to\infty} d(y_{2m(k)+1}, y_{2n(k)-1})\Big) = \phi_1(\varepsilon_0, \varepsilon_0) \overset{(F_{13})}{\leq} \varepsilon_0;$$

$$\lim_{k\to\infty} \phi_2\big(d(Ax_{2m(k)}, Sx_{2n(k)-1}), d(Tx_{2m(k)}, Bx_{2n(k)-1}), d(Ax_{2m(k)}, Bx_{2n(k)-1})\big)$$

$$= \lim_{k\to\infty} \phi_2\big(d(y_{2m(k)}, y_{2n(k)}), d(y_{2m(k)+1}, y_{2n(k)-1}), d(y_{2m(k)}, y_{2n(k)-1})\big)$$

$$\overset{(F_{21})}{=} \phi_2\Big(\lim_{k\to\infty} d(y_{2m(k)}, y_{2n(k)}), \lim_{k\to\infty} d(y_{2m(k)+1}, y_{2n(k)-1}), \lim_{k\to\infty} d(y_{2m(k)}, y_{2n(k)-1})\Big)$$

$$= \phi_2(\varepsilon_0, \varepsilon_0, \varepsilon_0) \overset{(F_{23})}{\leq} \varepsilon_0;$$

$$\lim_{k\to\infty} \phi_3\big(d(Ax_{2m(k)}, Tx_{2m(k)}), d(Bx_{2n(k)-1}, Sx_{2n(k)-1}), d(Ax_{2m(k)}, Bx_{2n(k)-1})\big)$$

$$= \lim_{k\to\infty} \phi_3\big(d(y_{2m(k)}, y_{2m(k)+1}), d(y_{2n(k)-1}, y_{2n(k)}), d(y_{2m(k)}, y_{2n(k)-1})\big)$$

$$\overset{(F_{31})}{=} \phi_3\Big(\lim_{k\to\infty} d(y_{2m(k)}, y_{2m(k)+1}), \lim_{k\to\infty} d(y_{2n(k)-1}, y_{2n(k)}), \lim_{k\to\infty} d(y_{2m(k)}, y_{2n(k)-1})\Big)$$

$$= \phi_3(0, 0, \varepsilon_0) \overset{(F_{33})}{\leq} \varepsilon_0;$$

$$\lim_{k\to\infty} \phi_4\big(d(Ax_{2m(k)}, Sx_{2n(k)-1}), d(Tx_{2m(k)}, Bx_{2n(k)-1}), d(Ax_{2m(k)}, Tx_{2m(k)}),$$

$$d(Bx_{2n(k)-1}, Sx_{2n(k)-1})\big)$$

$$= \lim_{k\to\infty} \phi_4\big(d(y_{2m(k)}, y_{2n(k)}), d(y_{2m(k)+1}, y_{2n(k)-1}), d(y_{2m(k)}, y_{2m(k)+1}), d(y_{2n(k)-1}, y_{2n(k)})\big)$$

$$\overset{(F_{41})}{=} \phi_4\Big(\lim_{k\to\infty} d(y_{2m(k)}, y_{2n(k)}), \lim_{k\to\infty} d(y_{2m(k)+1}, y_{2n(k)-1}),$$

$$\lim_{k\to\infty} d(y_{2m(k)}, y_{2m(k)+1}), \lim_{k\to\infty} d(y_{2n(k)-1}, y_{2n(k)})\Big)$$

$$= \phi_4(\varepsilon_0, \varepsilon_0, 0, 0) \overset{(F_{43})}{\leq} \varepsilon_0.$$

As a result, it follows from (21) and (20) that

$$\lim_{k\to\infty} M(x_{2m(k)}, x_{2n(k)-1})$$

$$= \lim_{k\to\infty} \max\big\{d(Ax_{2m(k)}, Bx_{2n(k)-1}), d(Ax_{2m(k)}, Tx_{2m(k)}), d(Bx_{2n(k)-1}, Sx_{2n(k)-1}),$$

$$\phi_1\big(d(Ax_{2m(k)}, Sx_{2n(k)-1}), d(Tx_{2m(k)}, Bx_{2n(k)-1})\big),$$

$$\phi_2\big(d(Ax_{2m(k)}, Sx_{2n(k)-1}), d(Tx_{2m(k)}, Bx_{2n(k)-1}), d(Ax_{2m(k)}, Bx_{2n(k)-1})\big),$$

$$\phi_3\big(d(Ax_{2m(k)}, Tx_{2m(k)}), d(Bx_{2n(k)-1}, Sx_{2n(k)-1}), d(Ax_{2m(k)}, Bx_{2n(k)-1})\big),$$

$$\phi_4\big(d(Ax_{2m(k)}, Sx_{2n(k)-1}), d(Tx_{2m(k)}, Bx_{2n(k)-1}), d(Ax_{2m(k)}, Tx_{2m(k)}),$$

$$d(Bx_{2n(k)-1}, Sx_{2n(k)-1})\big)\Big\}$$

$$= \varepsilon_0.$$

As ψ is upper semi-continuous, we deduce that

$$\varepsilon_0 = \lim_{n\to\infty} d(y_{2m(k)+1}, y_{2n(k)}) = \limsup_{n\to\infty} d(y_{2m(k)+1}, y_{2n(k)})$$

$$\leq \limsup_{n\to\infty} \psi\big(M(x_{2m(k)}, x_{2n(k)-1})\big) \leq \psi(\varepsilon_0) < \varepsilon_0,$$

which is a contradiction. As a result, $\{y_n\}$ is a Cauchy sequence in (X, d). Then, there exists $z \in X$ such that $\{y_n\} \to z$. In particular,

$$z = \lim_{n\to\infty} y_n = \lim_{n\to\infty} Tx_{2n} = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Sx_{2n-1}.$$

Next we distinguish some cases depending on the complete set. Notice that

$$\{y_{2n+2} = Ax_{2n+2} = Sx_{2n+1}\} \subseteq S(X) \subseteq A(X) \quad \text{and}$$

$$\{y_{2n+1} = Bx_{2n+1} = Tx_{2n}\} \subseteq T(X) \subseteq B(X).$$

Assume that $A(X)$ (or $S(X)$) is complete. Then there exists $u \in X$ such that $z = Au$. To show that $Tu = z$, suppose, by contradiction, that $d(Tu, z) > 0$. As the pair $\{S, A\}$ is (α, d)-regular and $\{y_{2n+2} = Sx_{2n+1}\} \to z = Au$, we have

$$\max\big\{\alpha(u, x_{2n+1}), \alpha(u, x_{2n+1})\big\} \geq 1 \quad \text{for all } n \in \mathbb{N}_0.$$

Therefore, for all $n \in \mathbb{N}$,

$$d(Tu, y_{2n+2}) = d(Tu, Sx_{2n+1}) \leq \max\big\{\alpha(u, x_{2n+1}), \alpha(u, x_{2n+1})\big\} d(Tu, Sx_{2n+1})$$

$$\leq \psi\big(M(u, x_{2n+1})\big). \tag{22}$$

Notice that

$$\lim_{n\to\infty} d(Au, Bx_{2n+1}) = \lim_{n\to\infty} d(z, y_{2n+1}) = 0;$$

$$\lim_{n\to\infty} d(Au, Tu) = d(z, Tu) > 0;$$

$$\lim_{n\to\infty} d(Bx_{2n+1}, Sx_{2n+1}) = d(y_{2n+1}, y_{2n+2}) = 0;$$

$$\lim_{n\to\infty} \phi_1\big(d(Au, Sx_{2n+1}), d(Tu, Bx_{2n+1})\big)$$

$$= \lim_{n\to\infty} \phi_1\big(d(z, y_{2n+2}), d(Tu, y_{2n+1})\big)$$

$$\overset{(F_{11})}{=} \phi_1\Big(\lim_{n\to\infty} d(z, y_{2n+2}), \lim_{n\to\infty} d(Tu, y_{2n+1})\Big)$$

$$= \phi_1\big(0, d(Tu, z)\big) \overset{(F_{12})}{\leq} \frac{d(Tu, z)}{2} < d(Tu, z);$$

$$\lim_{n\to\infty} \phi_2\big(d(Au, Sx_{2n+1}), d(Tu, Bx_{2n+1}), d(Au, Bx_{2n+1})\big)$$

$$= \lim_{n\to\infty} \phi_2\big(d(z, y_{2n+2}), d(Tu, y_{2n+1}), d(z, y_{2n+1})\big)$$

$$\overset{(F_{21})}{=} \phi_2\Big(\lim_{n\to\infty} d(z, y_{2n+2}), \lim_{n\to\infty} d(Tu, y_{2n+1}), \lim_{n\to\infty} d(z, y_{2n+1})\Big)$$

$$= \phi_2\big(0, d(Tu, z), 0\big) \overset{(F_{22})}{\leq} d(Tu, z);$$

$$\lim_{n\to\infty} \phi_3\big(d(Au, Tu), d(Bx_{2n+1}, Sx_{2n+1}), d(Au, Bx_{2n+1})\big)$$

$$= \lim_{n\to\infty} \phi_3\big(d(z, Tu), d(y_{2n+1}, y_{2n+2}), d(z, y_{2n+1})\big)$$

$$\overset{(F_{31})}{=} \phi_3\Big(\lim_{n\to\infty} d(z, Tu), \lim_{n\to\infty} d(y_{2n+1}, y_{2n+2}), \lim_{n\to\infty} d(z, y_{2n+1})\Big)$$

$$= \phi_3\big(d(z, Tu), 0, 0\big) \overset{(F_{32})}{\leq} d(z, Tu);$$

$$\lim_{n\to\infty} \phi_4\big(d(Au, Sx_{2n+1}), d(Tu, Bx_{2n+1}), d(Au, Tu), d(Bx_{2n+1}, Sx_{2n+1})\big)$$

$$= \lim_{n\to\infty} \phi_4\big(d(z, y_{2n+2}), d(Tu, y_{2n+1}), d(z, Tu), d(y_{2n+1}, y_{2n+2})\big)$$

$$\overset{(F_{41})}{=} \phi_4\Big(\lim_{n\to\infty} d(z, y_{2n+2}), \lim_{n\to\infty} d(Tu, y_{2n+1}), \lim_{n\to\infty} d(z, Tu), \lim_{n\to\infty} d(y_{2n+1}, y_{2n+2})\Big)$$

$$= \phi_4\big(0, d(Tu, z), d(z, Tu), 0\big) \overset{(F_{42})}{\leq} d(z, Tu).$$

Since

$$\lim_{n\to\infty} M(u, x_{2n+1})$$

$$= \lim_{n\to\infty} \max\big\{d(Au, Bx_{2n+1}), d(Au, Tu), d(Bx_{2n+1}, Sx_{2n+1}),$$

$$\phi_1\big(d(Au, Sx_{2n+1}), d(Tu, Bx_{2n+1})\big),$$

$$\phi_2\big(d(Au, Sx_{2n+1}), d(Tu, Bx_{2n+1}), d(Au, Bx_{2n+1})\big),$$

$$\phi_3\big(d(Au, Tu), d(Bx_{2n+1}, Sx_{2n+1}), d(Au, Bx_{2n+1})\big),$$

$$\phi_4\big(d(Au, Sx_{2n+1}), d(Tu, Bx_{2n+1}), d(Au, Tu), d(Bx_{2n+1}, Sx_{2n+1})\big)\big\}$$

$$= d(z, Tu) > 0,$$

letting $n \to \infty$ in (22), the upper semi-continuity of ψ yields

$$d(z, Tu) = \limsup_{n\to\infty} d(Tu, y_{2n+2}) \leq \limsup_{n\to\infty} \psi\big(M(u, x_{2n+1})\big)$$

$$\leq \psi\Big(\limsup_{n\to\infty} M(u, x_{2n+1})\Big) = \psi\big(d(z, Tu)\big) < d(z, Tu),$$

which is a contradiction. As a consequence, $Tu = z = Au$. As $T(X) \subseteq_\alpha B(X)$, there exists $w \in X$ such that $Tu = Bw$ and $\alpha(u, w) \geq 1$. Then $Au = Tu = z = Bw$. We claim that $z = Sw$. To prove it, suppose that $d(z, Sw) > 0$. As the pair $\{T, B\}$ is (α, d)-regular and $\{y_{2n+1} = Tx_{2n}\} \to z = Bw$,

$$\max\big\{\alpha(x_{2n}, w), \alpha(w, x_{2n})\big\} \geq 1 \quad \text{for all } n \in \mathbb{N}_0.$$

By (16), for all $n \in \mathbb{N}$,

$$d(y_{2n+1}, Sw) = d(Tx_{2n}, Sw) \leq \max\{\alpha(x_{2n}, w), \alpha(w, x_{2n})\} d(Tx_{2n}, Sw)$$

$$\leq \psi(M(x_{2n}, w)). \tag{23}$$

Notice that

$$\lim_{n \to \infty} d(Ax_{2n}, Bw) = \lim_{n \to \infty} d(y_{2n}, z) = 0;$$

$$\lim_{n \to \infty} d(Ax_{2n}, Tx_{2n}) = \lim_{n \to \infty} d(y_{2n}, y_{2n+1}) = 0;$$

$$\lim_{n \to \infty} d(Bw, Sw) = d(z, Sw) > 0;$$

$$\lim_{n \to \infty} \phi_1(d(Ax_{2n}, Sw), d(Tx_{2n}, Bw))$$

$$= \lim_{n \to \infty} \phi_1(d(y_{2n}, Sw), d(y_{2n+1}, z))$$

$$\stackrel{(F_{11})}{=} \phi_1\left(\lim_{n \to \infty} d(y_{2n}, Sw), \lim_{n \to \infty} d(y_{2n+1}, z)\right) = \phi_1(d(z, Sw), 0) \stackrel{(F_{14})}{\leq} d(z, Sw);$$

$$\lim_{n \to \infty} \phi_2(d(Ax_{2n}, Sw), d(Tx_{2n}, Bw), d(Ax_{2n}, Bw))$$

$$= \lim_{n \to \infty} \phi_2(d(y_{2n}, Sw), d(y_{2n+1}, z), d(y_{2n}, z))$$

$$\stackrel{(F_{21})}{=} \phi_2\left(\lim_{n \to \infty} d(y_{2n}, Sw), \lim_{n \to \infty} d(y_{2n+1}, z), \lim_{n \to \infty} d(y_{2n}, z)\right)$$

$$= \phi_2(d(z, Sw), 0, 0) \stackrel{(F_{24})}{\leq} d(z, Sw);$$

$$\lim_{n \to \infty} \phi_3(d(Ax_{2n}, Tx_{2n}), d(Bw, Sw), d(Ax_{2n}, Bw))$$

$$= \lim_{n \to \infty} \phi_3(d(y_{2n}, y_{2n+1}), d(z, Sw), d(y_{2n}, z))$$

$$\stackrel{(F_{31})}{=} \phi_3\left(\lim_{n \to \infty} d(y_{2n}, y_{2n+1}), \lim_{n \to \infty} d(z, Sw), \lim_{n \to \infty} d(y_{2n}, z)\right)$$

$$= \phi_3(0, d(z, Sw), 0) \stackrel{(F_{34})}{\leq} d(z, Sw);$$

$$\lim_{n \to \infty} \phi_4(d(Ax_{2n}, Sw), d(Tx_{2n}, Bw), d(Ax_{2n}, Tx_{2n}), d(Bw, Sw))$$

$$= \lim_{n \to \infty} \phi_4(d(y_{2n}, Sw), d(y_{2n+1}, z), d(y_{2n}, y_{2n+1}), d(z, Sw))$$

$$\stackrel{(F_{41})}{=} \phi_4\left(\lim_{n \to \infty} d(y_{2n}, Sw), \lim_{n \to \infty} d(y_{2n+1}, z), \lim_{n \to \infty} d(y_{2n}, y_{2n+1}), \lim_{n \to \infty} d(z, Sw)\right)$$

$$= \phi_4(d(z, Sw), 0, 0, d(z, Sw)) \stackrel{(F_{44})}{\leq} d(z, Sw).$$

Hence,

$$\lim_{n \to \infty} M(x_{2n}, w) = \lim_{n \to \infty} \max\{d(Ax_{2n}, Bw), d(Ax_{2n}, Tx_{2n}), d(Bw, Sw),$$

$$\phi_1(d(Ax_{2n}, Sw), d(Tx_{2n}, Bw)),$$

$$\phi_2(d(Ax_{2n}, Sw), d(Tx_{2n}, Bw), d(Ax_{2n}, Bw)),$$

$$\phi_3(d(Ax_{2n}, Tx_{2n}), d(Bw, Sw), d(Ax_{2n}, Bw)),$$

$$\phi_4\big(d(Ax_{2n}, Sw), d(Tx_{2n}, Bw), d(Ax_{2n}, Tx_{2n}), d(Bw, Sw)\big)\big\}$$
$$= d(z, Sw) > 0.$$

Again, letting $n \to \infty$ in (23) and using the upper semi-continuity of ψ, we deduce that

$$d(z, Sw) = \limsup_{n\to\infty} d(y_{2n+1}, Sw) \leq \limsup_{n\to\infty} \psi\big(M(x_{2n}, w)\big)$$
$$\leq \psi\Big(\lim_{n\to\infty} M(x_{2n}, w)\Big) = \psi\big(d(z, Sw)\big) < d(z, Sw),$$

which is a contradiction. Thus, $Sw = z$, which means that $z = Tu = Au = Bw = Sw$. As $\{A, T\}$ and $\{B, S\}$ are weakly compatible,

$$Az = ATu = TAu = Tz \quad \text{and} \quad Bz = BSw = SBw = Sz.$$

Next, let us show that $Tz = Sz$. On the contrary case, suppose that $d(Tz, Sz) > 0$. Hence, by (16),

$$d(Tz, Sz) \leq \max\big\{\alpha(z, z), \alpha(z, z)\big\} d(Tz, Sz) \leq \psi\big(M(z, z)\big). \tag{24}$$

Notice that

$$d(Az, Bz) = d(Tz, Sz) > 0;$$

$$d(Az, Tz) = d(Tz, Tz) = 0;$$

$$d(Bz, Sz) = d(Sz, Sz) = 0;$$

$$\phi_1\big(d(Az, Sz), d(Tz, Bz)\big) = \phi_1\big(d(Tz, Sz), d(Tz, Sz)\big) \overset{(F_{13})}{\leq} d(Tz, Sz);$$

$$\phi_2\big(d(Az, Sz), d(Tz, Bz), d(Az, Bz)\big) = \phi_2\big(d(Tz, Sz), d(Tz, Sz), d(Tz, Sz)\big) \overset{(F_{23})}{\leq} d(Tz, Sz);$$

$$\phi_3\big(d(Az, Tz), d(Bz, Sz), d(Az, Bz)\big) = \phi_3\big(d(Tz, Tz), d(Sz, Sz), d(Tz, Sz)\big)$$
$$= \phi_3\big(0, 0, d(Tz, Sz)\big) \overset{(F_{33})}{\leq} d(Tz, Sz);$$

$$\phi_4\big(d(Az, Sz), d(Tz, Bz), d(Az, Tz), d(Bz, Sz)\big)$$
$$= \phi_4\big(d(Tz, Sz), d(Tz, Sz), d(Tz, Tz), d(Sz, Sz)\big)$$
$$= \phi_4\big(d(Tz, Sz), d(Tz, Sz), 0, 0\big) \overset{(F_{43})}{\leq} d(Tz, Sz).$$

Therefore,

$$M(z, z) = \max\big\{d(Az, Bz), d(Az, Tz), d(Bz, Sz),$$
$$\phi_1\big(d(Az, Sz), d(Tz, Bz)\big), \phi_2\big(d(Az, Sz), d(Tz, Bz), d(Az, Bz)\big),$$
$$\phi_3\big(d(Az, Tz), d(Bz, Sz), d(Az, Bz)\big),$$
$$\phi_4\big(d(Az, Sz), d(Tz, Bz), d(Az, Tz), d(Bz, Sz)\big)\big\}$$
$$= d(Tz, Sz) > 0.$$

Again, it follows from (24) that

$$d(Tz, Sz) \leq \psi\big(M(z, z)\big) = \psi\big(d(Tz, Sz)\big) < d(Tz, Sz),$$

which is a contradiction. As a consequence, we conclude that $Tz = Sz$, which means that

$$Az = Tz = Sz = Bz.$$

In particular, z is a coincidence point of A, B, T, and S. □

In the following result, we describe some sufficient conditions to ensure that the coincidence point is a common fixed point, and it is unique.

Theorem 16 *Under the hypotheses of Theorem* 15, *suppose that* $\max\{\alpha(z, w), \alpha(w, z)\} \geq 1$ *for all coincidence point z of A, B, S, and T, and all* $w \in S^{-1}(\{z\})$. *Then A, B, S, and T have, at least, a common fixed point.*

Furthermore, if we additionally assume that $\max\{\alpha(x, y), \alpha(y, x)\} \geq 1$ *for all distinct common fixed points x and y of A, B, S, and T, then A, B, S, and T have a unique common fixed point.*

Proof In such a case, we can repeat, point by point, the arguments of the proof of Theorem 6 in order to demonstrate the following facts:

- $Tz = z$ as in (13), so z is a common fixed point of A, B, S, and T;
- z is the unique common fixed point of A, B, S, and T, as in (14).

This completes the proof. □

The following result corresponds to the case in which $T = B$ and $S = A$ in Theorem 15.

Corollary 17 *Let* $\alpha : X \times X \to [0, \infty)$ *be a reflexive, transitive function and let* $T, S : X \to X$ *be self-mappings in a metric space* (X, d) *such that*

(a) *the pair* $\{T, S\}$ *is weakly compatible;*

(b) *one of* $T(X)$ *or* $S(X)$ *is complete.*

Also assume that there exist $\phi_1 \in \mathsf{F}_1$, $\phi_2 \in \mathsf{F}_2$, $\phi_3 \in \mathsf{F}_3$, $\phi_4 \in \mathsf{F}_4$, *and* $\psi \in \Phi_3$ *such that*

$$\max\big\{\alpha(x, y), \alpha(y, x)\big\} d(Tx, Sy) \leq \psi\big(M(x, y)\big) \quad \text{for all } x, y \in X,$$

where

$$M(x, y) = \max\big\{d(Sx, Ty), d(Sx, Tx), d(Ty, Sy), \phi_1\big(d(Sx, Sy), d(Tx, Ty)\big),$$
$$\phi_2\big(d(Sx, Sy), d(Tx, Ty), d(Sx, Ty)\big), \phi_3\big(d(Sx, Tx), d(Ty, Sy), d(Sx, Ty)\big)$$
$$\phi_4\big(d(Sx, Sy), d(Tx, Ty), d(Sx, Tx), d(Ty, Sy)\big)\big\}.$$

Then T and S have a coincidence point in X.

Remark 18 Theorem 15 can be particularized to the case in which the metric space (X, d) is endowed with a partial order $\preccurlyeq$ (or even with a reflexive, transitive binary relation) by defining

$$\alpha_\preccurlyeq(x, y) = \begin{cases} 1, & \text{if } x \preccurlyeq y, \\ 0, & \text{otherwise.} \end{cases}$$

In such a case, the contractivity condition (16) turns into the following one:

$$d(Tx, Sy) \leq \psi\big(M(x, y)\big) \quad \text{for all } x, y \in X \text{ such that } x \preccurlyeq y.$$

In [38–40], the authors illustrated their main results with a list of several corollaries by choosing the function α in different ways. We could repeat here their studies but, for short, we only describe the case in which $S = T$, and A and B are the identity mapping I_X in X in Theorem 15.

Corollary 19 *Let $\alpha : X \times X \to [0, \infty)$ be a reflexive, transitive function and let $T : X \to X$ be a self-mapping in a metric space (X, d) such that*

(b) $\alpha(x, Tx) \geq 1$ *for all $x \in X$;*

(c) *X (or $T(X)$) is complete;*

(d) *the pair $\{T, I_X\}$ is (α, d)-regular.*

Also assume that there exist $\phi_1 \in \mathsf{F}_1, \phi_2 \in \mathsf{F}_2, \phi_3 \in \mathsf{F}_3, \phi_4 \in \mathsf{F}_4$, and $\psi \in \Phi_3$ such that

$$\max\big\{\alpha(x, y), \alpha(y, x)\big\} d(Tx, Ty) \leq \psi\big(M(x, y)\big) \quad \text{for all } x, y \in X,$$

where

$$M(x, y) = \max\big\{ d(x, y), d(x, Tx), d(y, Ty), \phi_1\big(d(x, Ty), d(Tx, y)\big),$$
$$\phi_2\big(d(x, Ty), d(Tx, y), d(x, y)\big), \phi_3\big(d(x, Tx), d(y, Ty), d(x, y)\big)$$
$$\phi_4\big(d(x, Ty), d(Tx, y), d(x, Tx), d(y, Ty)\big)\big\}.$$

Then T has, at least, a fixed point in X.

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