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# On monotone nonexpansive mappings in $L_1([0,1])$

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**Abstract**

Let the set $C \subset L_1([0,1])$ be nonempty, convex and compact for the convergence almost everywhere and $T : C \to C$ be a monotone nonexpansive mapping. In this paper, we study the behavior of the Krasnoselskii-Ishikawa iteration sequence $\{f_n\}$ defined by $f_{n+1} = \lambda f_n + (1-\lambda)T(f_n)$, $n = 1, 2, \ldots$, $\lambda \in (0,1)$. Then we prove a fixed point theorem for these mappings. Our result is new and was never investigated.

**MSC:** Primary 46B20; 47E10

**Keywords:** convergence almost everywhere; fixed point; Ishikawa iteration; Krasnoselskii iteration; Lebesgue measure; monotone mapping; nonexpansive mapping

## 1 Introduction

Banach's contraction principle [1] is remarkable in its simplicity, yet it is perhaps the most widely applied fixed point theorem in all of analysis. This is because the contractive condition on the mapping is simple and easy to test in a complete metric space. The principle itself finds almost canonical applications in the theory of differential and integral equations. Over the years, many mathematicians successfully extended this fundamental theorem. Recently, a version of this theorem was given in partially ordered metric spaces [2, 3] and in metric spaces with a graph [4].

In this work, we discuss the case of monotone nonexpansive mappings defined in $L_1([0,1])$. Nonexpansive mappings are those which have Lipschitz constant equal to 1. The fixed point theory for such mappings is very rich [5–9] and has many applications in nonlinear functional analysis [10]. It is worth mentioning that the work presented here is new and has never been carried out.

For more on metric fixed point theory, the reader may consult the book of Khamsi and Kirk [11].

## 2 Preliminaries

Consider the Banach space $L_1([0,1])$ of real valued functions defined on $[0,1]$ with absolute value Lebesgue integrable, *i.e.*, $\int_0^1 |f(x)|\,dx < +\infty$. As usual, we have $f = 0$ if and only if the set $\{x \in [0,1]; f(x) = 0\}$ has Lebesgue measure 0. In this case, we say $f = 0$ almost everywhere. An element of $L_1([0,1])$ is therefore seen as a class of functions. The norm of any $f \in L_1([0,1])$ is given by

$$\|f\| = \int_0^1 \left|f(x)\right| dx.$$

Throughout this paper, we will write $L_1$ instead of $L_1([0,1])$. Recall that $f \leq g$ (called comparable) if and only if $f(x) \leq g(x)$ almost everywhere, for any $f, g \in L_1$. It is well known that $L_1$ is a Banach lattice (see [12]). We adopt the convention $f \geq g$ if and only if $g \leq f$. Note that order intervals are closed for convergence almost everywhere and convex. Recall that an order interval is a subset of the form

$$[f, \to) = \{g \in L_1; f \leq g\} \quad \text{or} \quad (\leftarrow, f] = \{g \in L_1; g \leq f\},$$

for any $f \in L_1$. As a direct consequence of this, the subset

$$[f, g] = \{h \in L_1; f \leq h \leq g\} = [f, \to) \cap (\leftarrow, g]$$

is closed and convex, for any $f, g \in L_1$.

Next we give the definition of monotone mappings.

**Definition 2.1** Let $C$ be a nonempty subset of $L_1$. A map $T : C \to L_1$ is said to be
(a) monotone if $T(f) \leq T(g)$ whenever $f \leq g$;
(b) monotone $K$-Lipschitzian, $K \in \mathbb{R}^+$, if $T$ is monotone and

$$\left\| T(g) - T(f) \right\| \leq K \|g - f\|,$$

whenever $f \leq g$. If $K = 1$, then $T$ is said to be a monotone nonexpansive mapping.

A point $f \in C$ is said to be a fixed point of $T$ if $T(f) = f$. The set of fixed points of $T$ is denoted by $\mathrm{Fix}(T)$.

**Remark 2.1** It is not difficult to show that a monotone nonexpansive mapping may not be continuous. Therefore it is quite difficult to expect any nice behavior that will imply the existence of a fixed point for this class of mappings.

The following lemma will be crucial to prove the main result of this paper.

**Lemma 2.1** [13] *If $\{f_n\}$ is a sequence of $L^p$-uniformly bounded functions on a measure space, and $f_n \to f$ almost everywhere, then*

$$\liminf_{n \to \infty} \|f_n\|^p = \liminf_{n \to \infty} \|f_n - f\|^p + \|f\|^p,$$

*for all $p \in (0, \infty)$.*

In particular, this result holds when $p = 1$.

## 3 Iteration process for monotone nonexpansive mappings

Let us first recall the definition of the Krasnoselskii iteration [14] (see also [6, 15, 16]).

**Definition 3.1** Let $C$ be a nonempty convex subset of $L_1$. Let $T : C \to C$ be a monotone mapping. Fix $\lambda \in (0,1)$ and $f_1 \in C$. The Krasnoselskii iteration sequence $\{f_n\} \subset C$ is defined by

$$f_{n+1} = \lambda f_n + (1-\lambda)T(f_n), \quad n \geq 1. \tag{KIS}$$

We have the following technical lemma.

**Lemma 3.1** *Let $C$ be a nonempty convex subset of $L_1$. Let $T: C \to C$ be a monotone mapping. Fix $\lambda \in (0,1)$ and $f_1 \in C$. Assume that $f_1 \preceq T(f_1)$. The iteration sequence $\{f_n\}$ defined by* (KIS), *which starts at $f_1$, satisfies*

$$f_n \preceq f_{n+1} \preceq T(f_n) \preceq T(f_{n+1}), \tag{KI}$$

*for any $n \geq 1$. Moreover, if $\{f_n\}$ has two subsequences which converge almost everywhere to $g$ and $h$, then we must have $g = h$.*

*Proof* First note that if $f \preceq g$ holds, then we have $f \preceq \lambda f + (1-\lambda)g \preceq g$ since order intervals are convex. Therefore it is just enough to prove $f_n \preceq T(f_n)$, for any $n \geq 1$. By assumption, we have $f_1 \preceq T(f_1)$. Assume that $f_n \preceq T(f_n)$, for $n \geq 1$. Then we have $f_n \preceq \lambda f_n + (1-\lambda)T(f_n)$, *i.e.*, $f_n \preceq f_{n+1} \preceq T(f_n)$. Since $T$ is monotone, we get $T(f_n) \preceq T(f_{n+1})$, which implies $f_{n+1} \preceq T(f_{n+1})$. By induction, we conclude that the inequalities (KI) hold for any $n \geq 1$. Next let $\{f_{\phi(n)}\}$ be a subsequence of $\{f_n\}$ which converges almost everywhere to $g$. Fix $k \geq 1$. Then the order interval $[f_k, \to)$ contains the sequence $\{f_{\phi(n)}\}$ except maybe for finitely many elements. Since order intervals are almost everywhere closed and convex, we conclude that $g \in [f_k, \to)$, *i.e.*, $f_k \preceq g$ for any $k \geq 1$. Consequently, if $\{f_n\}$ has another subsequence which converges almost everywhere to $h$, then we must have $h = g$. Indeed, since $f_n \preceq g$, for any $n \geq 1$, we get $h \preceq g$. Similarly, we have $g \preceq h$, which implies $h = g$. □

**Remark 3.1** Note that under the assumptions of Lemma 3.1, if we assume $T(f_1) \preceq f_1$, then we have

$$T(f_{n+1}) \preceq T(f_n) \preceq f_{n+1} \preceq f_n,$$

for any $n \geq 1$. The conclusion on convergence almost everywhere limits of $\{f_n\}$ also holds.

In the general theory of nonexpansive mappings, the iteration sequence defined by (KIS) provides an approximate fixed point sequence of $T$, *i.e.*, $\lim_{n\to+\infty} \|f_n - T(f_n)\| = 0$ (see *e.g.* [6, 17]). It is amazing that this result holds for monotone nonexpansive mappings as well.

**Theorem 3.1** *Let $C \subset L_1$ be nonempty, convex and compact for the convergence almost everywhere. Let $T: C \to C$ be a monotone nonexpansive mapping. Assume there exists $f_1 \in C$ such that $f_1$ and $T(f_1)$ are comparable. Then $\{f_n\}$, defined by* (KIS), *converges almost everywhere to some $f \in C$ and* $\lim_{n\to\infty} \|f_n - T(f_n)\| = 0$. *Moreover, $f$ and $f_n$ are comparable for any $n \geq 1$.*

*Proof* Without loss of generality, we may assume $f_1 \preceq T(f_1)$. As $C$ is compact for the convergence almost everywhere, so there exists a subsequence of $\{f_n\}$ which converges almost everywhere to $f \in C$. Lemma 3.1 implies that $\{f_n\}$ converges almost everywhere to $f$ and

$f_n \leq f$, for any $n \geq 1$. For the second part of this theorem, we will need the following inequality proved in [5, 18]:

$$(1+n\lambda)\big\|T(f_i)-f_i\big\| \leq \big\|T(f_{i+n})-f_i\big\| + (1-\lambda)^{-n}\big(\big\|T(f_i)-f_i\big\| - \big\|T(f_{i+n})-f_{i+n}\big\|\big), \qquad \text{(GK)}$$

for any $i, n \geq 1$. We note that $\{\|f_n - T(f_n)\|\}$ is decreasing. Indeed, we have $f_{n+1} - f_n = (1-\lambda)(T(f_n) - f_n)$, for any $n \geq 0$. Therefore $\{\|f_n - T(f_n)\|\}$ is decreasing if and only if $\{\|f_{n+1} - f_n\|\}$ is decreasing which follows from the inequality

$$\|f_{n+2} - f_{n+1}\| \leq \lambda\|f_{n+1} - f_n\| + (1-\lambda)\big\|T(f_{n+1}) - T(f_n)\big\| \leq \|f_{n+1} - f_n\|,$$

for any $n \geq 0$. Set $\lim_{n\to+\infty} \|f_n - T(f_n)\| = R$. In general, the sequence $\{f_n\}$ has no reason to be bounded even though it converges almost everywhere. But in this case, it is bounded. Indeed, using properties of the partial order and the convergence almost everywhere, we know that $f_1 \leq f_n \leq f$, for any $n \geq 1$. Hence $f_n$ and $f$ are comparable, for any $n \geq 1$. Moreover, we have $0 \leq f_n - f_1 \leq f - f_1$, which implies $\|f_n - f_1\| \leq \|f - f_1\|$, for any $n \geq 1$. Hence

$$\begin{aligned}\big\|T(f_{i+n}) - f_i\big\| &\leq \big\|T(f_{i+n}) - T(f_1)\big\| + \big\|T(f_1) - T(f_i)\big\| + \big\|T(f_i) - f_i\big\| \\ &\leq \|f_{i+n} - f_1\| + \|f_1 - f_i\| + \big\|T(f_1) - f_1\big\| \\ &\leq \|f - f_1\| + \|f - f_1\| + \big\|T(f_1) - f_1\big\|,\end{aligned}$$

for any $i, n \geq 1$. Next we let $i \to +\infty$ in the inequality (GK) to obtain

$$(1+n\lambda)R \leq 2\|f - f_1\| + \big\|T(f_1) - f_1\big\|,$$

where we used the fact $\lim_{i\to+\infty}(\|T(f_i) - f_i\| - \|T(f_{i+n}) - f_{i+n}\|) = R - R = 0$, for any $n \geq 1$. Obviously, this will imply $R = 0$, *i.e.*,

$$\lim_{n\to+\infty} \big\|f_n - T(f_n)\big\| = 0. \qquad \square$$

Next we state the main result of this paper.

**Theorem 3.2** *Let $C \subset L_1$ be nonempty, convex and compact for the convergence almost everywhere. Let $T : C \to C$ be a monotone nonexpansive mapping. Assume there exists $f_1 \in C$ such that $f_1$ and $T(f_1)$ are comparable. Then $\{f_n\}$, defined by* (KIS), *converges almost everywhere to some $f \in C$ which is a fixed point of $T$, i.e., $T(f) = f$. Moreover, $f$ and $f_1$ are comparable.*

*Proof* Without loss of generality, we may assume that $f_1 \leq T(f_1)$. Theorem 3.1 implies that $\{f_n\}$ converges almost everywhere to some $f \in C$ where $f_n$ and $f$ are comparable, for any $n \geq 1$. Since $\{f_n\}$ is uniformly bounded, Lemma 2.1 implies

$$\liminf_{n\to\infty}\big\|f_n - T(f)\big\| = \liminf_{n\to\infty}\|f_n - f\| + \big\|f - T(f)\big\|.$$

Theorem 3.1 implies $\lim_{n\to+\infty} \|f_n - T(f_n)\| = 0$. Therefore we get

$$\liminf_{n\to\infty}\big\|T(f_n) - T(f)\big\| = \liminf_{n\to\infty}\|f_n - f\| + \big\|f - T(f)\big\|.$$

On the other hand, we know that each $f_n$ is comparable with $f$, for each $n \geq 1$, so we have

$$\liminf_{n\to\infty} \|f_n - f\| + \|f - T(f)\| = \liminf_{n\to\infty} \|T(f_n) - T(f)\| \leq \liminf_{n\to\infty} \|f_n - f\|,$$

which obviously implies $\|f - T(f)\| = 0$ or $T(f) = f$. □

This result is a generalization of the original existence theorem in [19, 20] for nonexpansive mappings which are not monotone. As we said earlier, monotone nonexpansive mappings are not necessarily continuous. Therefore this class of mappings is bigger and is used to answer questions about the existence of positive or negative solutions in some applications considered in [2, 3].

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors contributed equally to the writing of this paper. All authors read and approved the manuscript.

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