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The class of (α, ψ)-type contractions in b-metric spaces and fixed point theorems

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Abstract

We study the existence and uniqueness of fixed points for self-operators defined in a b-metric space and belonging to the class of (α, ψ)-type contraction mappings. The obtained results generalize and unify several existing fixed point theorems in the literature.

MSC: 47H10

Keywords: α-ψ-type contraction; b-metric space; fixed point; existence; uniqueness

1 Introduction and preliminaries

Very recently, we studied in [1] the existence and uniqueness of fixed points for self-operators defined in a metric space and belonging to the class of (α, ψ)-type contraction mappings (see [2–5] for some works in this direction). We proved that the class of α-ψ-type contractions includes large classes of contraction-type operators, whose fixed points can be obtained by means of the Picard iteration. The aim of this paper is to extend the obtained results in [1] to self-operators defined in a b-metric space.

We start by recalling the following definition.

Definition 1.1 ([6]) Let X be a nonempty set. A mapping $d : X \times X \to [0,\infty)$ is called b-metric if there exists a real number $b \geq 1$ such that for every $x, y, z \in X$, we have

(i) $d(x,y) = 0$ if and only if $x = y$;
(ii) $d(x,y) = d(y,x)$;
(iii) $d(x,z) \leq b[d(x,y) + d(y,z)]$.

In this case, the pair (X,d) is called a b-metric space.

There exist many examples in the literature (see [6–8]) showing that the class of b-metrics is effectively larger than that of metric spaces.

The notions of convergence, compactness, closedness and completeness in b-metric spaces are given in the same way as in metric spaces. For works on fixed point theory in b-metric spaces, we refer to [9–12] and the references therein.

Definition 1.2 ([13]) Let $\psi : [0,\infty) \to [0,\infty)$ be a given function. We say that ψ is a comparison function if it is increasing and $\psi^n(t) \to 0$, $n \to \infty$, for any $t \geq 0$, where ψ^n is the nth iterate of ψ.

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In [13, 14], several results regarding comparison functions can be found. Among these we recall the following.

Lemma 1.3 *If $\psi : [0,\infty) \to [0,\infty)$ is a comparison function, then*

(i) *each iterate ψ^k of ψ, $k \geq 1$, is also a comparison function*;

(ii) *ψ is continuous at zero*;

(iii) *$\psi(t) < t$ for any $t > 0$*;

(iv) *$\psi(0) = 0$.*

The following concept was introduced in [15].

Definition 1.4 Let $b \geq 1$ be a real number. A mapping $\psi : [0,\infty) \to [0,\infty)$ is called a b-comparison function if

(i) ψ is monotone increasing;

(ii) there exist $k_0 \in \mathbb{N}$, $a \in (0,1)$ and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ such that

$$b^{k+1}\psi^{k+1}(t) \leq ab^k\psi^k(t) + v_k$$

for $k \geq k_0$ and any $t \geq 0$.

The following lemma has been proved.

Lemma 1.5 ([15, 16]) *Let $\psi : [0,\infty) \to [0,\infty)$ be a b-comparison function. Then*

(i) *the series $\sum_{k=0}^{\infty} b^k\psi^k(t)$ converges for any $t \geq 0$*;

(ii) *the function $s_b : [0,\infty) \to [0,\infty)$ defined by*

$$s_b(t) = \sum_{k=0}^{\infty} b^k\psi^k(t), \quad t \geq 0$$

is increasing and continuous at 0.

Lemma 1.6 ([17]) *Any b-comparison function is a comparison function.*

Throughout this paper, for $b \geq 1$, we denote by Ψ_b the set of b-comparison functions.

Definition 1.7 Let (X,d) be a b-metric space with constant $b \geq 1$, and let $T : X \to X$ be a given mapping. We say that T is an α-ψ contraction if there exist a b-comparison function $\psi \in \Psi_b$ and a function $\alpha : X \times X \to \mathbb{R}$ such that

$$\alpha(x,y)d(Tx,Ty) \leq \psi\big(d(x,y)\big) \quad \text{for all } x,y \in X. \tag{1.1}$$

2 Main results

Let $T : X \to X$ be a given mapping. We denote by $\mathrm{Fix}(T)$ the set of its fixed points; that is,

$$\mathrm{Fix}(T) = \{x \in X : x = Tx\}.$$

For $b \geq 1$ and $\psi \in \Psi_b$, let Σ_ψ^b be the set defined by

$$\Sigma_\psi^b = \{\sigma \in (0,\infty) : \sigma\psi \in \Psi_b\}.$$

We have the following result.

Proposition 2.1 *Let (X,d) be a b-metric space with constant $b \geq 1$, and let $T : X \to X$ be a given mapping. Suppose that there exist $\alpha : X \times X \to \mathbb{R}$ and $\psi \in \Psi_b$ such that T is an α-ψ contraction. Suppose that there exists $\sigma \in \Sigma_\psi^b$ and for some positive integer p, there exists a finite sequence $\{\xi_i\}_{i=0}^p \subset X$ such that*

$$\xi_0 = x_0, \qquad \xi_p = Tx_0, \qquad \alpha(T^n\xi_i, T^n\xi_{i+1}) \geq \sigma^{-1}, \quad n \in \mathbb{N}, i = 0,\ldots,p-1, x_0 \in X. \tag{2.1}$$

Then $\{T^n x_0\}$ is a Cauchy sequence in (X,d).

Proof Let $\varphi = \sigma\psi$. By the definition of Σ_ψ^b, we have $\varphi \in \Psi_b$. Let $\{\xi_i\}_{i=0}^p$ be a finite sequence in X satisfying (2.1). Consider the sequence $\{x_n\}_{n\in\mathbb{N}}$ in X defined by $x_{n+1} = Tx_n$, $n \in \mathbb{N}$. We claim that

$$d(T^r\xi_i, T^r\xi_{i+1}) \leq \varphi^r(d(\xi_i, \xi_{i+1})), \quad r \in \mathbb{N}, i = 0,\ldots,p-1. \tag{2.2}$$

Let $i \in \{0,1,\ldots,p-1\}$. From (2.1), we have

$$\sigma^{-1}d(T\xi_i, T\xi_{i+1}) \leq \alpha(\xi_i,\xi_{i+1})d(T\xi_i, T\xi_{i+1}) \leq \psi(d(\xi_i,\xi_{i+1})),$$

which implies that

$$d(T\xi_i, T\xi_{i+1}) \leq \varphi(d(\xi_i,\xi_{i+1})). \tag{2.3}$$

Again, we have

$$\sigma^{-1}d(T^2\xi_i, T^2\xi_{i+1}) \leq \alpha(T\xi_i, T\xi_{i+1})d(T(T\xi_i), T(T\xi_{i+1})) \leq \psi(d(T\xi_i, T\xi_{i+1})),$$

which implies that

$$d(T^2\xi_i, T^2\xi_{i+1}) \leq \varphi(d(T\xi_i, T\xi_{i+1})). \tag{2.4}$$

Since φ is an increasing function (from Lemma 1.6), from (2.3) and (2.4), we obtain

$$d(T^2\xi_i, T^2\xi_{i+1}) \leq \varphi^2(d(\xi_i,\xi_{i+1})).$$

Continuing this process, by induction we obtain (2.2).

Now, using the property (iii) of a b-metric and (2.2), for every $n \in \mathbb{N}$, we have

$$\begin{aligned} d(x_n, x_{n+1}) &= d(T^n x_0, T^{n+1}x_0) \\ &\leq bd(T^n\xi_0, T^n\xi_1) + b^2 d(T^n\xi_1, T^n\xi_2) + \cdots + b^p d(T^n\xi_{p-1}, T^n\xi_p) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=0}^{p-1} b^{i+1} d\big(T^n \xi_i, T^n \xi_{i+1}\big) \\
&\le \sum_{i=0}^{p-1} b^{i+1} \varphi^n\big(d(\xi_i, \xi_{i+1})\big).
\end{aligned}
$$

Thus we proved that

$$
d(x_n, x_{n+1}) \le \sum_{i=0}^{p-1} b^{i+1} \varphi^n\big(d(\xi_i, \xi_{i+1})\big), \quad n \in \mathbb{N},
$$

which implies that for $q \ge 1$,

$$
\begin{aligned}
d(x_n, x_{n+q}) &\le \sum_{j=n}^{n+q-1} b^{j-n+1} d(x_j, x_{j+1}) \\
&\le \sum_{j=n}^{n+q-1} b^{j-n+1} \sum_{i=0}^{p-1} b^{i+1} \varphi^j\big(d(\xi_i, \xi_{i+1})\big) \\
&= \frac{1}{b^{n-1}} \sum_{i=0}^{p-1} b^{i+1} \sum_{j=n}^{n+q-1} b^j \varphi^j\big(d(\xi_i, \xi_{i+1})\big) \\
&\le \frac{1}{b^{n-1}} \sum_{i=0}^{p-1} b^{i+1} \sum_{j=n}^{\infty} b^j \varphi^j\big(d(\xi_i, \xi_{i+1})\big).
\end{aligned}
$$

Since $b \ge 1$, using Lemma 1.5(i), we obtain

$$
\frac{1}{b^{n-1}} \sum_{i=0}^{p-1} b^{i+1} \sum_{j=n}^{\infty} b^j \varphi^j\big(d(\xi_i, \xi_{i+1})\big) \to 0 \quad \text{as } n \to \infty.
$$

This proves that $\{x_n\}$ is a Cauchy sequence in the b-metric space (X, d). □

Our first main result is the following fixed point theorem which requires the continuity of the mapping T.

Theorem 2.2 *Let (X, d) be a complete b-metric space with constant $b \ge 1$, and let $T : X \to X$ be a given mapping. Suppose that there exist $\alpha : X \times X \to \mathbb{R}$ and $\psi \in \Psi_b$ such that T is an α-ψ contraction. Suppose also that* (2.1) *is satisfied. Then $\{T^n x_0\}$ converges to some $x^* \in X$. Moreover, if T is continuous, then $x^* \in \operatorname{Fix}(T)$.*

Proof From Proposition 2.1, we know that $\{T^n x_0\}$ is a Cauchy sequence. Since (X, d) is a complete b-metric space, there exists $x^* \in X$ such that

$$
\lim_{n\to\infty} d\big(T^n x_0, x^*\big) = 0.
$$

The continuity of T yields

$$
\lim_{n\to\infty} d\big(T^{n+1} x_0, Tx^*\big) = 0.
$$

By the uniqueness of the limit, we obtain $x^* = Tx^*$, that is, $x^* \in \operatorname{Fix}(T)$. □

In the next theorem, we omit the continuity assumption of T.

Theorem 2.3 *Let (X,d) be a complete b-metric space with constant $b \geq 1$, and let $T : X \to X$ be a given mapping. Suppose that there exist $\alpha : X \times X \to \mathbb{R}$ and $\psi \in \Psi_b$ such that T is an α-ψ contraction. Suppose also that* (2.1) *is satisfied. Then $\{T^n x_0\}$ converges to some $x^* \in X$. Moreover, if there exists a subsequence $\{T^{\gamma(n)} x_0\}$ of $\{T^n x_0\}$ such that*

$$\max\bigl\{\alpha\bigl(T^{\gamma(n)}x_0, x^*\bigr), \alpha\bigl(x^*, T^{\gamma(n)}x_0\bigr)\bigr\} \geq \ell \in (0,\infty), \quad n \text{ large enough},$$

then $x^* \in \operatorname{Fix}(T)$.

Proof From Proposition 2.1 and the completeness of the b-metric space (X,d), we know that $\{T^n x_0\}$ converges to some $x^* \in X$.

Suppose now that there exists a subsequence $\{T^{\gamma(n)} x_0\}$ of $\{T^n x_0\}$ such that

$$\max\bigl\{\alpha\bigl(T^{\gamma(n)}x_0, x^*\bigr), \alpha\bigl(x^*, T^{\gamma(n)}x_0\bigr)\bigr\} \geq \ell \in (0,\infty), \quad n \text{ large enough}. \tag{2.5}$$

Since T is an α-ψ contraction, we have

$$\alpha\bigl(T^{\gamma(n)}x_0, x^*\bigr) d\bigl(T^{\gamma(n)+1}x_0, Tx^*\bigr) \leq \psi\bigl(d\bigl(T^{\gamma(n)}x_0, x^*\bigr)\bigr), \quad n \in \mathbb{N}$$

and

$$\alpha\bigl(x^*, T^{\gamma(n)}x_0\bigr) d\bigl(T^{\gamma(n)+1}x_0, Tx^*\bigr) \leq \psi\bigl(d\bigl(T^{\gamma(n)}x_0, x^*\bigr)\bigr), \quad n \in \mathbb{N}.$$

Thus we have

$$\max\bigl\{\alpha\bigl(T^{\gamma(n)}x_0, x^*\bigr), \alpha\bigl(x^*, T^{\gamma(n)}x_0\bigr)\bigr\} d\bigl(T^{\gamma(n)+1}x_0, Tx^*\bigr) \leq \psi\bigl(d\bigl(T^{\gamma(n)}x_0, x^*\bigr)\bigr), \quad n \in \mathbb{N}.$$

From (2.5), we get

$$\ell d\bigl(T^{\gamma(n)+1}x_0, Tx^*\bigr) \leq \psi\bigl(d\bigl(T^{\gamma(n)}x_0, x^*\bigr)\bigr), \quad n \text{ large enough}. \tag{2.6}$$

On the other hand, using the property (iii) of a b-metric, we get

$$d\bigl(T^{\gamma(n)+1}x_0, Tx^*\bigr) \geq \frac{1}{b} d\bigl(x^*, Tx^*\bigr) - d\bigl(x^*, T^{\gamma(n)+1}x_0\bigr), \quad n \in \mathbb{N}. \tag{2.7}$$

Now, (2.6) and (2.7) yield

$$\ell\left(\frac{1}{b} d\bigl(x^*, Tx^*\bigr) - d\bigl(x^*, T^{\gamma(n)+1}x_0\bigr)\right) \leq \psi\bigl(d\bigl(T^{\gamma(n)}x_0, x^*\bigr)\bigr), \quad n \text{ large enough}.$$

Letting $n \to \infty$ in the above inequality, using Lemma 1.6 and Lemma 1.3(ii) and (iv), we obtain

$$0 \leq \frac{\ell}{b} d\bigl(x^*, Tx^*\bigr) \leq \psi(0) = 0,$$

which implies that $d(x^*, Tx^*) = 0$, that is, $x^* \in \operatorname{Fix}(T)$. □

We provide now a sufficient condition for the uniqueness of the fixed point.

Theorem 2.4 *Let (X,d) be a b-metric space with constant $b \geq 1$, and let $T : X \to X$ be a given mapping. Suppose that there exist $\alpha : X \times X \to \mathbb{R}$ and $\psi \in \Psi_b$ such that T is an α-ψ contraction. Suppose also that*

(i) $\mathrm{Fix}(T) \neq \emptyset$;

(ii) *for every pair $(x,y) \in \mathrm{Fix}(T) \times \mathrm{Fix}(T)$ with $x \neq y$, if $\alpha(x,y) < 1$, then there exists $\eta \in \Sigma_\psi^b$ and for some positive integer q, there is a finite sequence $\{\zeta_i(x,y)\}_{i=0}^q \subset X$ such that*

$$\zeta_0(x,y) = x, \qquad \zeta_q(x,y) = y, \qquad \alpha\big(T^n\zeta_i(x,y), T^n\zeta_{i+1}(x,y)\big) \geq \eta^{-1}$$

for $n \in \mathbb{N}$ and $i = 0, \dots, q-1$.

Then T has a unique fixed point.

Proof Let $\varphi = \eta\psi \in \Psi_b$. Suppose that $u, v \in X$ are two fixed points of T such that $d(u,v) > 0$. We consider two cases.

Case 1: $\alpha(u,v) \geq 1$. Since T is an α-ψ contraction, we have

$$d(u,v) \leq \alpha(u,v)d(Tu,Tv) \leq \psi\big(d(u,v)\big).$$

On the other hand, from Lemma 1.6 and Lemma 1.3(iii), we have

$$\psi\big(d(u,v)\big) < d(u,v).$$

The two above inequalities yield a contradiction.

Case 2: $\alpha(u,v) < 1$. By assumption, there exists a finite sequence $\{\zeta_i(u,v)\}_{i=0}^q$ in X such that

$$\zeta_0(u,v) = u, \qquad \zeta_q(u,v) = v, \qquad \alpha\big(T^n\zeta_i(u,v), T^n\zeta_{i+1}(u,v)\big) \geq \eta^{-1}$$

for $n \in \mathbb{N}$ and $i = 0, \dots, q-1$. As in the proof of Proposition 2.1, we can establish that

$$d\big(T^r\zeta_i(u,v), T^r\zeta_{i+1}(u,v)\big) \leq \varphi^r\big(d\big(\zeta_i(u,v), \zeta_{i+1}(u,v)\big)\big), \quad r \in \mathbb{N}, i = 0, \dots, q-1. \tag{2.8}$$

On the other hand, we have

$$\begin{aligned} d(u,v) &= d\big(T^n u, T^n v\big) \\ &\leq \sum_{i=0}^{q-1} b^{i+1} d\big(T^n\zeta_i(u,v), T^n\zeta_{i+1}(u,v)\big) \\ &\leq \sum_{i=0}^{q-1} b^{i+1} \varphi^n\big(d\big(\zeta_i(u,v), \zeta_{i+1}(u,v)\big)\big) \to 0 \quad \text{as } n \to \infty \text{ (by Lemma 1.6).} \end{aligned}$$

Then $u = v$, which is a contradiction. □

3 Particular cases

In this section, we deduce from our main theorems several fixed point theorems in b-metric spaces.

3.1 The class of ψ-type contractions in b-metric spaces

Definition 3.1 Let (X,d) be a b-metric space with constant $b \geq 1$. A mapping $T : X \to X$ is said to be a ψ-contraction if there exists $\psi \in \Psi_b$ such that

$$d(Tx,Ty) \leq \psi\big(d(x,y)\big) \quad \text{for all } x,y \in X. \tag{3.1}$$

Theorem 3.2 *Let (X,d) be a b-metric space with constant $b \geq 1$, and let $T : X \to X$ be a given mapping. Suppose that there exists $\psi \in \Psi_b$ such that T is a ψ-contraction. Then there exists $\alpha : X \times X \to \mathbb{R}$ such that T is an α-ψ contraction.*

Proof Consider the function $\alpha : X \times X \to \mathbb{R}$ defined by

$$\alpha(x,y) = 1 \quad \text{for all } x,y \in X. \tag{3.2}$$

Clearly, from (3.1), T is an α-ψ contraction. □

Corollary 3.3 ([17]) *Let (X,d) be a complete b-metric space with constant $b \geq 1$, and let $T : X \to X$ be a given mapping. If T is a ψ-contraction for some $\psi \in \Psi_b$, then T has a unique fixed point. Moreover, for any $x_0 \in X$, the Picard sequence $\{T^n x_0\}$ converges to this fixed point.*

Proof From Lemma 1.6, we have

$$d(Tx,Ty) \leq d(x,y) \quad \text{for all } x,y \in X,$$

which implies that T is a continuous mapping. From Theorem 3.2, T is an α-ψ contraction, where α is defined by (3.2). Clearly, for any $x_0 \in X$, (2.1) is satisfied with $p = 1$ and $\sigma = 1$. By Theorem 2.2, $\{T^n x_0\}$ converges to a fixed point of T. The uniqueness follows immediately from (3.2) and Theorem 2.4. □

Corollary 3.4 *Let (X,d) be a complete b-metric space with constant $b \geq 1$, and let $T : X \to X$ be a given mapping. Suppose that*

$$d(Tx,Ty) \leq kd(x,y) \quad \textit{for all } x,y \in X$$

for some constant $k \in (0,1/b)$. Then T has a unique fixed point. Moreover, for any $x_0 \in X$, the Picard sequence $\{T^n x_0\}$ converges to this fixed point.

Proof It is an immediate consequence of Corollary 3.3 with $\psi(t) = kt$. □

3.2 The class of rational-type contractions in b-metric spaces

3.2.1 Dass-Gupta-type contraction in b-metric spaces

Definition 3.5 Let (X,d) be a b-metric space with constant $b \geq 1$. A mapping $T : X \to X$ is said to be a Dass-Gupta contraction if there exist constants $\lambda, \mu \geq 0$ with $\lambda b + \mu < 1$ such that

$$d(Tx,Ty) \le \mu d(y,Ty)\frac{1+d(x,Tx)}{1+d(x,y)} + \lambda d(x,y) \quad \text{for all } x,y \in X. \tag{3.3}$$

Theorem 3.6 *Let (X,d) be a b-metric space with constant $b \ge 1$, and let $T : X \to X$ be a given mapping. Suppose that T is a Dass-Gupta contraction. Then there exist $\psi \in \Psi_b$ and $\alpha : X \times X \to \mathbb{R}$ such that T is an α-ψ contraction.*

Proof From (3.3), for all $x, y \in X$, we have

$$d(Tx,Ty) - \mu d(y,Ty)\frac{1+d(x,Tx)}{1+d(x,y)} \le \lambda d(x,y),$$

which yields

$$\left(1-\mu\frac{d(y,Ty)(1+d(x,Tx))}{(1+d(x,y))d(Tx,Ty)}\right)d(Tx,Ty) \le \lambda d(x,y), \quad x,y \in X, Tx \ne Ty. \tag{3.4}$$

Consider the functions $\psi : [0,\infty) \to [0,\infty)$ and $\alpha : X \times X \to \mathbb{R}$ defined by

$$\psi(t) = \lambda t, \quad t \ge 0 \tag{3.5}$$

and

$$\alpha(x,y) = \begin{cases} 1-\mu\frac{d(y,Ty)(1+d(x,Tx))}{(1+d(x,y))d(Tx,Ty)}, & \text{if } Tx \ne Ty, \\ 0, & \text{otherwise.} \end{cases} \tag{3.6}$$

Since $0 \le \lambda b < 1$, then $\psi \in \Psi_b$. On the other hand, from (3.4) we have

$$\alpha(x,y)d(Tx,Ty) \le \psi\big(d(x,y)\big) \quad \text{for all } x,y \in X.$$

Then T is an α-ψ contraction. □

Corollary 3.7 *Let (X,d) be a complete b-metric space with constant $b \ge 1$, and let $T : X \to X$ be a given mapping. If T is a Dass-Gupta contraction with parameters $\lambda, \mu \ge 0$ such that $\lambda b + \mu < 1$, then T has a unique fixed point. Moreover, for any $x_0 \in X$, the Picard sequence $\{T^n x_0\}$ converges to this fixed point.*

Proof Let x_0 be an arbitrary point in X. If for some $r \in \mathbb{N}$, $T^r x_0 = T^{r+1}x_0$, then $T^r x_0$ will be a fixed point of T. So we can suppose that $T^r x_0 \ne T^{r+1}x_0$ for all $r \in \mathbb{N}$. From (3.6), for all $n \in \mathbb{N}$, we have

$$\begin{aligned} \alpha\big(T^n x_0, T^{n+1}x_0\big) &= 1-\mu\frac{d(T^{n+1}x_0,T^{n+2}x_0)(1+d(T^n x_0,T^{n+1}x_0))}{(1+d(T^n x_0,T^{n+1}x_0))d(T^{n+1}x_0,T^{n+2}x_0)} \\ &= 1-\mu > 0. \end{aligned}$$

On the other hand, from (3.5) we have

$$(1-\mu)^{-1}\psi(t) = \frac{\lambda}{1-\mu}t, \quad t \ge 0.$$

From the condition $\lambda b + \mu < 1$, clearly we have $(1-\mu)^{-1}\psi \in \Psi_b$, which is equivalent to $(1-\mu)^{-1} \in \Sigma_\psi^b$. Then (2.1) is satisfied with $p = 1$ and $\sigma = (1-\mu)^{-1}$. From the first part of Theorem 2.3, the sequence $\{T^n x_0\}$ converges to some $x^* \in X$.

Suppose that x^* is not a fixed point of T, that is, $d(x^*, Tx^*) > 0$. Then

$$T^{n+1}x_0 \neq Tx^*, \quad n \text{ large enough.}$$

From (3.6), we have

$$\alpha\bigl(x^*, T^n x_0\bigr) = 1 - \mu \frac{d(T^n x_0, T^{n+1}x_0)(1 + d(x^*, Tx^*))}{(1 + d(T^n x_0, x^*))d(T^{n+1}x_0, Tx^*)}, \quad n \text{ large enough.}$$

On the other hand, using the property (iii) of a b-metric, we have

$$d\bigl(T^{n+1}x_0, Tx^*\bigr) \geq \frac{1}{b} d\bigl(x^*, Tx^*\bigr) - d\bigl(x^*, T^{n+1}x_0\bigr) > 0, \quad n \text{ large enough.}$$

Thus we have

$$\alpha\bigl(x^*, T^n x_0\bigr) \geq 1 - \mu \frac{d(T^n x_0, T^{n+1}x_0)(1 + d(x^*, Tx^*))}{(1 + d(T^n x_0, x^*))(\frac{1}{b} d(x^*, Tx^*) - d(x^*, T^{n+1}x_0))}, \quad n \text{ large enough.}$$

Since

$$\lim_{n\to\infty} 1 - \mu \frac{d(T^n x_0, T^{n+1}x_0)(1 + d(x^*, Tx^*))}{(1 + d(T^n x_0, x^*))(\frac{1}{b} d(x^*, Tx^*) - d(x^*, T^{n+1}x_0))} = 1,$$

we have

$$\alpha\bigl(x^*, T^n x_0\bigr) > \frac{1}{2}, \quad n \text{ large enough.}$$

By Theorem 2.3, we deduce that $x^* \in \operatorname{Fix}(T)$, which is a contradiction. Thus $\operatorname{Fix}(T) \neq \emptyset$.

For the uniqueness, observe that for every pair $(x, y) \in \operatorname{Fix}(T) \times \operatorname{Fix}(T)$ with $x \neq y$, we have $\alpha(x, y) = 1$. By Theorem 2.4, x^* is the unique fixed point of T. □

If $b = 1$, Corollary 3.7 recovers the Dass-Gupta fixed point theorem [18].

3.2.2 Jaggi-type contraction in b-metric spaces

Definition 3.8 Let (X, d) be a b-metric space with constant $b \geq 1$. A mapping $T : X \to X$ is said to be a Jaggi contraction if there exist constants $\lambda, \mu \geq 0$ with $\lambda b + \mu < 1$ such that

$$d(Tx, Ty) \leq \mu \frac{d(x, Tx)d(y, Ty)}{d(x, y)} + \lambda d(x, y) \quad \text{for all } x, y \in X, x \neq y. \tag{3.7}$$

Theorem 3.9 *Let (X, d) be a b-metric space with constant $b \geq 1$, and let $T : X \to X$ be a given mapping. Suppose that T is a Jaggi contraction. Then there exist $\psi \in \Psi_b$ and $\alpha : X \times X \to \mathbb{R}$ such that T is an α-ψ contraction.*

Proof From (3.7), for all $x, y \in X$ with $x \neq y$, we have

$$d(Tx, Ty) - \mu \frac{d(x, Tx)d(y, Ty)}{d(x, y)} \leq \lambda d(x, y),$$

which yields

$$\left(1-\mu\frac{d(x,Tx)d(y,Ty)}{d(x,y)d(Tx,Ty)}\right)d(Tx,Ty)\leq\lambda d(x,y),\quad x,y\in X, Tx\neq Ty. \tag{3.8}$$

Consider the functions $\psi:[0,\infty)\to[0,\infty)$ and $\alpha:X\times X\to\mathbb{R}$ defined by

$$\psi(t)=\lambda t,\quad t\geq 0 \tag{3.9}$$

and

$$\alpha(x,y)=\begin{cases}1-\mu\frac{d(x,Tx)d(y,Ty)}{d(x,y)d(Tx,Ty)}, & \text{if } Tx\neq Ty,\\ 0, & \text{otherwise.}\end{cases} \tag{3.10}$$

Since $\lambda b<1$, we have $\psi\in\Psi_b$. From (3.8), we have

$$\alpha(x,y)d(Tx,Ty)\leq\psi\big(d(x,y)\big)\quad\text{for all } x,y\in X.$$

Then T is an α-ψ contraction. □

Corollary 3.10 *Let (X,d) be a complete b-metric space with constant $b\geq 1$, and let $T:X\to X$ be a continuous mapping. If T is a Jaggi contraction with parameters $\lambda,\mu\geq 0$ such that $\lambda b+\mu<1$, then T has a unique fixed point. Moreover, for any $x_0\in X$, the Picard sequence $\{T^nx_0\}$ converges to this fixed point.*

Proof Let x_0 be an arbitrary point in X. Without loss of generality, we can suppose that $T^rx_0\neq T^{r+1}x_0$ for all $r\in\mathbb{N}$. From (3.10), for all $n\in\mathbb{N}$, we have

$$\alpha\big(T^nx_0,T^{n+1}x_0\big)=1-\mu\frac{d(T^nx_0,T^{n+1}x_0)d(T^{n+1}x_0,T^{n+2}x_0)}{d(T^nx_0,T^{n+1}x_0)d(T^{n+1}x_0,T^{n+2}x_0)}=1-\mu>0.$$

On the other hand, from (3.9), for all $t\geq 0$, we have

$$(1-\mu)^{-1}\psi(t)=\frac{\lambda}{1-\mu}t.$$

Since $\lambda b+\mu<1$, we have $(1-\mu)^{-1}\psi\in\Psi_b$, that is, $(1-\mu)^{-1}\in\Sigma_\psi^b$. Then (2.1) is satisfied with $p=1$ and $\sigma=(1-\mu)^{-1}$. By the first part of Theorem 2.2, $\{T^nx_0\}$ converges to some $x^*\in X$. Since T is continuous, by the second part of Theorem 2.2, x^* is a fixed point of T. Moreover, for every pair $(x,y)\in\mathrm{Fix}(T)\times\mathrm{Fix}(T)$ with $x\neq y$, we have $\alpha(x,y)=1$. Then, by Theorem 2.4, x^* is the unique fixed point of T. □

If $b=1$, Corollary 3.10 recovers the Jaggi fixed point theorem [19].

3.3 The class of Berinde-type mappings in *b*-metric spaces

Definition 3.11 Let (X,d) be a b-metric space with constant $b\geq 1$. A mapping $T:X\to X$ is said to be a Berinde-type contraction if there exist $\lambda\in(0,1/b)$ and $L\geq 0$ such that

$$d(Tx,Ty)\leq\lambda d(x,y)+Ld(y,Tx)\quad\text{for all } x,y\in X. \tag{3.11}$$

Theorem 3.12 *Let (X,d) be a b-metric space with constant $b \geq 1$, and let $T : X \to X$ be a given mapping. If T is a Berinde-type contraction, then there exist $\alpha : X \times X \to \mathbb{R}$ and $\psi \in \Psi_b$ such that T is an α-ψ contraction.*

Proof From (3.11), we have

$$d(Tx,Ty) - Ld(y,Tx) \leq \lambda d(x,y) \quad \text{for all } x,y \in X,$$

which yields

$$\left(1 - L\frac{d(y,Tx)}{d(Tx,Ty)}\right)d(Tx,Ty) \leq \lambda d(x,y), \quad x,y \in X, Tx \neq Ty. \tag{3.12}$$

Consider the functions $\psi : [0,\infty) \to [0,\infty)$ and $\alpha : X \times X \to \mathbb{R}$ defined by

$$\psi(t) = \lambda t, \quad t \geq 0$$

and

$$\alpha(x,y) = \begin{cases} 1 - L\frac{d(y,Tx)}{d(Tx,Ty)}, & \text{if } Tx \neq Ty, \\ 0, & \text{otherwise.} \end{cases} \tag{3.13}$$

Since $\lambda b < 1$, then $\psi \in \Psi_b$. From (3.12), we have

$$\alpha(x,y)d(Tx,Ty) \leq \psi\big(d(x,y)\big) \quad \text{for all } x,y \in X.$$

Then T is an α-ψ contraction. □

Corollary 3.13 *Let (X,d) be a complete b-metric space with constant $b \geq 1$, and let $T : X \to X$ be a given mapping. If T is a Berinde-type contraction with parameters $\lambda, L \geq 0$ such that $0 < \lambda b < 1$, then for any $x_0 \in X$, the Picard sequence $\{T^n x_0\}$ converges to a fixed point of T.*

Proof Let x_0 be an arbitrary point in X. Without loss of generality, we can suppose that $T^r x_0 \neq T^{r+1} x_0$ for all $r \in \mathbb{N}$. From (3.13), for all $n \in \mathbb{N}$, we have

$$\alpha\big(T^n x_0, T^{n+1} x_0\big) = 1 - L\frac{d(T^{n+1}x_0, T^{n+1}x_0)}{d(T^{n+1}x_0, T^{n+2}x_0)} = 1.$$

Then (2.1) holds with $\sigma = 1$ and $p = 1$. From the first part of Theorem 2.3, the sequence $\{T^n x_0\}$ converges to some $x^* \in X$.

Suppose now that x^* is not a fixed point of T, that is, $d(x^*, Tx^*) > 0$. Then

$$T^{n+1}x_0 \neq Tx^*, \quad n \text{ large enough.}$$

From (3.13), we have

$$\alpha\big(T^n x_0, x^*\big) = 1 - L\frac{d(x^*, T^{n+1}x_0)}{d(T^{n+1}x_0, Tx^*)}, \quad n \text{ large enough.}$$

Using the property (iii) of a b-metric, we have

$$d\bigl(T^{n+1}x_0, Tx^*\bigr) \geq \frac{1}{b}d\bigl(x^*, Tx^*\bigr) - d\bigl(x^*, T^{n+1}x_0\bigr) > 0, \quad n \text{ large enough.}$$

Thus we have

$$\alpha\bigl(T^n x_0, x^*\bigr) \geq 1 - L\frac{d(x^*, T^{n+1}x_0)}{\frac{1}{b}d(x^*, Tx^*) - d(x^*, T^{n+1}x_0)}, \quad n \text{ large enough.}$$

Since

$$\lim_{n\to\infty} 1 - L\frac{d(x^*, T^{n+1}x_0)}{\frac{1}{b}d(x^*, Tx^*) - d(x^*, T^{n+1}x_0)} = 1,$$

then

$$\alpha\bigl(T^n x_0, x^*\bigr) > \frac{1}{2}, \quad n \text{ large enough.}$$

By Theorem 2.3, we deduce that $x^* \in \operatorname{Fix}(T)$, which is a contradiction.

Thus x^* is a fixed point of T. □

If $b = 1$, Corollary 3.13 recovers the Berinde fixed point theorem [20].

Note that a Berinde mapping need not have a unique fixed point (see [21], Example 2.11).

Corollary 3.14 *Let (X, d) be a complete b-metric space with constant $b \geq 1$, and let $T : X \to X$ be a given mapping. Suppose that there exists a constant $k \in (0, 1/b(b+1))$ such that*

$$d(Tx, Ty) \leq k\bigl(d(x, Tx) + d(y, Ty)\bigr) \quad \textit{for all } x, y \in X. \tag{3.14}$$

Then, for any $x_0 \in X$, the Picard sequence $\{T^n x_0\}$ converges to a fixed point of T.

Proof At first, observe that from (3.14), for all $x, y \in X$, we have

$$d(Tx, Ty) \leq \lambda d(x, y) + Ld(y, Tx),$$

where

$$\lambda = \frac{kb}{1 - kb} \quad \text{and} \quad L = \frac{2kb}{1 - kb}.$$

With the condition $k \in (0, 1/b(b+1))$, we have $0 < \lambda < 1/b$ and $L \geq 0$. Then T is a Berinde-type contraction. From Corollary 3.13, if $x_0 \in X$, then $\{T^n x_0\}$ converges to a fixed point of T. □

If $b = 1$, Corollary 3.14 recovers the Kannan fixed point theorem [22].

Corollary 3.15 *Let (X,d) be a complete b-metric space with constant $b \geq 1$, and let $T: X \to X$ be a given mapping. Suppose that there exists a constant $k \in (0,1/2b^2)$ such that*

$$d(Tx,Ty) \leq k\big(d(x,Ty)+d(y,Tx)\big) \quad \text{for all } x,y \in X. \tag{3.15}$$

Then, for any $x_0 \in X$, the Picard sequence $\{T^n x_0\}$ converges to a fixed point of T.

Proof From (3.15), we have

$$d(Tx,Ty) \leq \lambda d(x,y) + Ld(y,Tx),$$

where

$$\lambda = \frac{kb}{1-kb^2} \quad \text{and} \quad L = \frac{k(b^2+1)}{1-kb^2}.$$

With the condition $k \in (0,1/2b^2)$, we have $0 < \lambda < 1/b$ and $L \geq 0$. Then T is a Berinde-type contraction. From Corollary 3.13, if $x_0 \in X$, then $\{T^n x_0\}$ converges to a fixed point of T. □

If $b = 1$, Corollary 3.15 recovers the Chatterjee fixed point theorem [23].

3.4 Ćirić-type mappings in *b*-metric spaces

Definition 3.16 Let (X,d) be a b-metric space with constant $b \geq 1$. A mapping $T: X \to X$ is said to be a Ćirić-type mapping if there exists $\lambda \in (0,1/b)$ such that for all $x,y \in X$, we have

$$\min\big\{d(Tx,Ty), d(x,Tx), d(y,Ty)\big\} - \min\big\{d(x,Ty), d(y,Tx)\big\} \leq \lambda d(x,y). \tag{3.16}$$

Theorem 3.17 *Let (X,d) be a b-metric space with constant $b \geq 1$, and let $T: X \to X$ be a given mapping. If T is a Ćirić-type mapping with parameter $\lambda \in (0,1/b)$, then there exist $\alpha: X \times X \to \mathbb{R}$ and $\psi \in \Psi_b$ such that T is an α-ψ contraction.*

Proof Consider the functions $\psi: [0,\infty) \to [0,\infty)$ and $\alpha: X \times X \to \mathbb{R}$ defined by

$$\psi(t) = \lambda t, \quad t \geq 0 \tag{3.17}$$

and

$$\alpha(x,y) = \begin{cases} \min\{1, \frac{d(x,Tx)}{d(Tx,Ty)}, \frac{d(y,Ty)}{d(Tx,Ty)}\} - \min\{\frac{d(x,Ty)}{d(Tx,Ty)}, \frac{d(y,Tx)}{d(Tx,Ty)}\}, & \text{if } Tx \neq Ty, \\ 0, & \text{otherwise.} \end{cases} \tag{3.18}$$

From (3.16), we have

$$\alpha(x,y)d(Tx,Ty) \leq \psi\big(d(x,y)\big) \quad \text{for all } x,y \in X, \tag{3.19}$$

which implies that T is an α-ψ contraction. □

Corollary 3.18 *Let (X,d) be a complete b-metric space with constant $b \geq 1$, and let $T : X \to X$ be a continuous mapping. If T is a Ćirić-type mapping with parameter $\lambda \in (0,1/b)$, then for any $x_0 \in X$, the Picard sequence $\{T^n x_0\}$ converges to a fixed point of T.*

Proof Let $x_0 \in X$ be an arbitrary point. Without loss of generality, we can suppose that $T^r x_0 \neq T^{r+1} x_0$ for all $r \in \mathbb{N}$. From (3.18), for all $n \in \mathbb{N}$, we have

$$\begin{aligned}\alpha\bigl(T^n x_0, T^{n+1} x_0\bigr) &= \min\left\{1, \frac{d(T^n x_0, T^{n+1} x_0)}{d(T^{n+1} x_0, T^{n+2} x_0)}, \frac{d(T^{n+1} x_0, T^{n+2} x_0)}{d(T^{n+1} x_0, T^{n+2} x_0)}\right\} \\ &\quad - \min\left\{\frac{d(T^n x_0, T^{n+2} x_0)}{d(T^{n+1} x_0, T^{n+2} x_0)}, \frac{d(T^{n+1} x_0, T^{n+1} x_0)}{d(T^{n+1} x_0, T^{n+2} x_0)}\right\} \\ &= \min\left\{1, \frac{d(T^n x_0, T^{n+1} x_0)}{d(T^{n+1} x_0, T^{n+2} x_0)}\right\}.\end{aligned}$$

Suppose that for some $n \in \mathbb{N}$, we have

$$\alpha\bigl(T^n x_0, T^{n+1} x_0\bigr) = \frac{d(T^n x_0, T^{n+1} x_0)}{d(T^{n+1} x_0, T^{n+2} x_0)}.$$

In this case, from (3.17) and (3.19), we have

$$d\bigl(T^n x_0, T^{n+1} x_0\bigr) \leq \lambda d\bigl(T^n x_0, T^{n+1} x_0\bigr).$$

This implies (from the assumption $T^r x_0 \neq T^{r+1} x_0$ for all $r \in \mathbb{N}$) that $\lambda \geq 1$, which is a contradiction. Then

$$\alpha\bigl(T^n x_0, T^{n+1} x_0\bigr) = 1 \quad \text{for all } n \in \mathbb{N}.$$

Then (2.1) is satisfied with $p = 1$ and $\sigma = 1$. By Theorem 2.3, we deduce that the sequence $\{T^n x_0\}$ converges to a fixed point of T. □

If $b = 1$, Corollary 3.18 recovers Ćirić's fixed point theorem [24].

3.5 Edelstein fixed point theorem in *b*-metric spaces

Another consequence of our main results is the following generalized version of Edelstein fixed point theorem [25] in b-metric spaces.

Corollary 3.19 *Let (X,d) be a complete b-metric space with constant $b \geq 1$, and ε-chainable for some $\varepsilon > 0$; i.e., given $x, y \in X$, there exist a positive integer N and a sequence $\{x_i\}_{i=0}^N \subset X$ such that*

$$x_0 = x, \qquad x_N = y, \qquad d(x_i, x_{i+1}) < \varepsilon \quad \textit{for } i = 0, \ldots, N-1. \tag{3.20}$$

Let $T : X \to X$ be a given mapping such that

$$x, y \in X, \quad d(x,y) < \varepsilon \quad \Longrightarrow \quad d(Tx, Ty) \leq \psi\bigl(d(x,y)\bigr) \tag{3.21}$$

for some $\psi \in \Psi_b$. Then T has a unique fixed point.

Proof It is clear from (3.21) that the mapping T is continuous. Now, consider the function $\alpha : X \times X \to \mathbb{R}$ defined by

$$\alpha(x,y) = \begin{cases} 1, & \text{if } d(x,y) < \varepsilon, \\ 0, & \text{otherwise}. \end{cases} \tag{3.22}$$

From (3.21), we have

$$\alpha(x,y)d(Tx,Ty) \le \psi\big(d(x,y)\big) \quad \text{for all } x,y \in X.$$

Let $x_0 \in X$. For $x = x_0$ and $y = Tx_0$, from (3.20) and (3.22), for some positive integer p, there exists a finite sequence $\{\xi_i\}_{i=0}^p \subset X$ such that

$$x_0 = \xi_0, \qquad \xi_p = Tx_0, \qquad \alpha(\xi_i, \xi_{i+1}) \ge 1 \quad \text{for } i = 0, \ldots, p-1.$$

Now, let $i \in \{0, \ldots, p-1\}$ be fixed. From (3.22) and (3.21), we have

$$\begin{aligned} \alpha(\xi_i, \xi_{i+1}) \ge 1 \quad &\Longrightarrow \quad d(\xi_i, \xi_{i+1}) < \varepsilon \\ &\Longrightarrow \quad d(T\xi_i, T\xi_{i+1}) \le \psi\big(d(\xi_i, \xi_{i+1})\big) \le d(\xi_i, \xi_{i+1}) < \varepsilon \\ &\Longrightarrow \quad \alpha(T\xi_i, T\xi_{i+1}) \ge 1. \end{aligned}$$

Again,

$$\begin{aligned} \alpha(T\xi_i, T\xi_{i+1}) \ge 1 \quad &\Longrightarrow \quad d(T\xi_i, T\xi_{i+1}) < \varepsilon \\ &\Longrightarrow \quad d\big(T^2\xi_i, T^2\xi_{i+1}\big) \le \psi\big(d(T\xi_i, T\xi_{i+1})\big) \le d(T\xi_i, T\xi_{i+1}) < \varepsilon \\ &\Longrightarrow \quad \alpha\big(T^2\xi_i, T^2\xi_{i+1}\big) \ge 1. \end{aligned}$$

By induction, we obtain

$$\alpha\big(T^n\xi_i, T^{n+1}\xi_{i+1}\big) \ge 1 \quad \text{for all } n \in \mathbb{N}.$$

Then (2.1) is satisfied with $\sigma = 1$. From Theorem 2.2, the sequence $\{T^n x_0\}$ converges to a fixed point of T. Using a similar argument, we can see that condition (ii) of Theorem 2.4 is satisfied, which implies that T has a unique fixed point. □

3.6 Contractive mapping theorems in *b*-metric spaces with a partial order

Let (X,d) be a b-metric space with constant $b \ge 1$, and let $\preceq$ be a partial order on X. We denote

$$\Delta = \big\{(x,y) \in X \times X : x \preceq y \text{ or } y \preceq x\big\}.$$

Corollary 3.20 *Let $T : X \to X$ be a given mapping. Suppose that there exists $\psi \in \Psi_b$ such that*

$$d(Tx,Ty) \le \psi\big(d(x,y)\big) \quad \textit{for all } (x,y) \in \Delta. \tag{3.23}$$

Suppose also that

(i) T *is continuous*;

(ii) *for some positive integer* p, *there exists a finite sequence* $\{\xi_i\}_{i=0}^{p} \subset X$ *such that*

$$\xi_0 = x_0, \qquad \xi_p = Tx_0, \qquad \left(T^n\xi_i, T^n\xi_{i+1}\right) \in \Delta, \quad n \in \mathbb{N}, i = 0, \ldots, p-1. \tag{3.24}$$

Then $\{T^n x_0\}$ *converges to a fixed point of* T.

Proof Consider the function $\alpha : X \times X \to \mathbb{R}$ defined by

$$\alpha(x,y) = \begin{cases} 1, & \text{if } (x,y) \in \Delta, \\ 0, & \text{otherwise}. \end{cases} \tag{3.25}$$

From (3.23), we have

$$\alpha(x,y)d(Tx,Ty) \leq \psi\left(d(x,y)\right) \quad \text{for all } x,y \in X.$$

Then the result follows from Theorem 2.2 with $\sigma = 1$. □

Corollary 3.21 *Let* $T : X \to X$ *be a given mapping. Suppose that*

(i) *there exists* $\psi \in \Psi_b$ *such that* (3.23) *holds*;

(ii) *condition* (3.24) *holds*.

Then $\{T^n x_0\}$ *converges to some* $x^* \in X$. *Moreover, if*

(iii) *there exist a subsequence* $\{T^{\gamma(n)}x_0\}$ *of* $\{T^n x_0\}$ *and* $N \in \mathbb{N}$ *such that*

$$\left(T^{\gamma(n)}x_0, x^*\right) \in \Delta, \quad n \geq N,$$

then x^* *is a fixed point of* T.

Proof We continue to use the same function α defined by (3.25). From the first part of Theorem 2.3, the sequence $\{T^n x_0\}$ converges to some $x^* \in X$. From (iii) and (3.25), we have

$$\alpha\left(T^{\gamma(n)}x_0, x^*\right) = 1, \quad n \geq N.$$

By the second part of Theorem 2.3 (with $\ell = 1$), we deduce that x^* is a fixed point of T. □

The next result follows from Theorem 2.4 with $\eta = 1$.

Corollary 3.22 *Let* $T : X \to X$ *be a given mapping. Suppose that*

(i) *there exists* $\psi \in \Psi_b$ *such that* (3.23) *holds*;

(ii) $\mathrm{Fix}(T) \neq \emptyset$;

(iii) *for every pair* $(x,y) \in \mathrm{Fix}(T) \times \mathrm{Fix}(T)$ *with* $x \neq y$, *if* $(x,y) \notin \Delta$, *there exist a positive integer* q *and a finite sequence* $\{\zeta_i(x,y)\}_{i=0}^{q} \subset X$ *such that*

$$\zeta_0(x,y) = x, \qquad \zeta_q(x,y) = y, \qquad \left(T^n\zeta_i(x,y), T^n\zeta_{i+1}(x,y)\right) \in \Delta$$

for $n \in \mathbb{N}$ *and* $i = 0, \ldots, q-1$.

Then T *has a unique fixed point*.

Observe that in our results we do not suppose that T is monotone or T preserves order as it is supposed in many papers (see [26–28] and others).

Competing interests
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